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FURTHER REMARKS ON ROBUST ESTIMATION IN DEPENDENT SITUATIONS[1]

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Results of an earlier paper giving first order optimal robust M-estimators of a location parameter in certain dependent situations are extended to the case where the dependency can be modeled by a symmetric form of a $(2k + 1)$st order moving average scheme. It is also shown that the results can not be extended to finding second order optimal M-estimators. That is, no M-estimators can be optimal to the second order unless they explicitly adapt themselves to the assumed model for dependency.

1. Extensions of earlier results. In [2] the author assumed a special form of a symmetric third order moving average scheme for errors of measurement, and expanded the asymptotic variance of estimators in powers of ρ (the coefficient in the moving average scheme). The first order terms of this expansion were then used to find first order optimal M-estimators, and various estimators were compared in terms of maximum asymptotic variance over the class of ε-contaminated normals under the moving average model. These results can be directly extended to the following generalization of the moving average model: let $\mathbf{S}$ be an $n \times n$ symmetric circulant matrix with first row $(1\alpha_1\alpha_2 \cdots \alpha_k 0 \cdots 0 \alpha_k \cdots \alpha_1)$ where $(\alpha_1, \cdots, \alpha_k)$ are such that S is invertible. Let $\mathbf{e} = (1 \quad 1 \cdots 1)'$ and assume that the observation vector $\mathbf{X} = (X_1, \cdots, X_n)'$ satisfies

$$\mathbf{X} = \theta\mathbf{e} + \mathbf{SY} \tag{1.1}$$

where θ is an unknown location parameter and $\mathbf{Y} = (Y_1, \cdots, Y_n)'$ with $(Y_1, \cdots, Y_n)$ i.i.d. with continuous symmetric cdf G. This is essentially a $(2k + 1)$st order moving average scheme with the first k and last k observations modified. As in [2], if α_i are known, $d\mathbf{S}^{-1}\mathbf{X} = \theta e + d\mathbf{Y}$ where $d = 1 + 2\beta$, and

$$\beta = \Sigma_{i=1}^k \alpha_i \tag{1.2}$$

(since $\mathbf{Se} = \mathbf{de}$); and Theorem 1.1 of [2] gives the result that, for any location invariant asymptotically normal sequence of estimators of θ, the asymptotic variance is bounded below by $d^2\sigma_0^2$ where σ_0^2 is the inverse Fisher information for G.

Now as in [2], suppose a sequence of estimators, $\hat{\theta}_n$, is given by $\hat{\theta}_n = T(F_n) + o_p(n^{-\frac{1}{2}})$ where F_n is the empirical cdf and T is a functional on the space of all cdfs with influence curve $I_F(x)$ (an odd function of x). Further, suppose

$$n^{\frac{1}{2}}(T(F_n) - T(F)) \rightarrow_D \mathfrak{N}(0, \sigma^2)$$

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where

$$\sigma^2 = EI_F^2(X) + 2\Sigma_{i=2}^{\infty} EI_F(X_i)I_F(X_1) \tag{1.3}$$

and F is the marginal distribution of X_i. The following generalization of Theorem (2.1) of [2] can now be stated:

THEOREM 1.1. *Under model* (1.1) *above, assume F and T are such that*

$$|I_F(x)| \leq C_1 \quad \text{and} \quad \|I_F(x) - I_G(x)\| \leq C_2\|F - G\|$$

where $\|\cdot\|$ *denotes the "sup" norm, and assume further that* $EY^2 \leq C_3$, *and G has a characteristic function* $\phi_Y(u)$ *such that* $\int u^2|\phi_Y(u)|du < +\infty$. *Then with* σ^2 *given by* (1.3) *and* β *by* (1.2),

$$\sigma^2 = EI_G^2(Y) - 4\beta EYI_G(Y)\int I_G(x)g'(x)dx + \mathcal{O}(\beta_2) \tag{1.4}$$

where g is the density for G and $\beta_2 = \Sigma_{i=1}^k \alpha_i^2$.

PROOF. Following the proof of Theorem 2.1 of [2], the joint characteristic functions and densities of the pairs (X_1, X_i) can be expanded to obtain

$$EI_F^2(X) = EI_F^2(Y) + \mathcal{O}(\beta_2)$$

$$EI_F(X_1)I_F(X_i) = -2\alpha_i EYI_F(Y)\int I_F(x)g'(x)dx + \mathcal{O}(\beta_2) \quad \text{for } i = 2, 3, \cdots, k,$$

$$EI_F(X_1)I_F(X_i) = \mathcal{O}(\beta_2) \text{ for } i = k+1, \cdots, 2k+1,$$

$$EI_F(X_1)I_F(X_i) = 0 \qquad \text{for } i > 2k+1.$$

Furthermore, as in [2], $\|F - G\| = \mathcal{O}(\beta_2)$, and the result follows. □

Furthermore, the following results (equations (2.8) and (2.11) in [2]) also generalize:

(i) If (1.4) holds and $I_G(x)$ is absolutely continuous,

$$\sigma^2 = EI_G^2(Y) + 4\beta EYI_G(Y)EI_G'(Y) + \mathcal{O}(\beta_2). \tag{1.5}$$

(ii) If an M-estimator has kernel, ψ, (assumed to be an odd function) which is bounded and absolutely continuous with ψ' bounded, then

$$\sigma^2 = E\psi^2(Y)/(E\psi'(Y))^2 + 4\beta EY\psi(Y)/E\psi'(Y) + \mathcal{O}(\beta_2). \tag{1.6}$$

Lastly the optimality theorem (3.1) of [2] also generalizes to model (1.1), and the M-estimator (equation 3.11) presented in [2] is still optimal to order $\mathcal{O}(\beta_2)$.

REMARK. The results of this section can be extended to the more usual moving average model where the matrix $\mathbf{S}$ in (1.1) is replaced by $\tilde{\mathbf{S}}$, a symmetric band matrix with diagonal and superdiagonal elements $(1\alpha_1\alpha_2 \cdots \alpha_k 0\, 0 \cdots 0)$. In fact, under model (1.1) the marginal joint distribution of $(X_{k+1}, X_{k+2}, \cdots X_{n-k})$ is given by the usual moving average model. Thus, the following lemma shows that the asymptotic distribution of M-estimators is the same under both models.

LEMMA 1.2. *Let ψ be the kernel of an M-estimator and suppose ψ satisfies the hypotheses of Theorem* (A.4) *of* [2]. *Let $\hat{\theta}_n$ and $\tilde{\theta}_n$ be roots respectively of*

$$\Sigma_{i=1}^{n}\psi(X_i - \hat{\theta}_n) = 0 \qquad \textit{and} \quad \Sigma_{i=k+1}^{n-k}\psi(X_i - \tilde{\theta}_n) = 0.$$

For definiteness, choose the root nearest a consistent estimator (*say the median*) *if there is more than one root*; *and define $\hat{\theta}_n$ or $\tilde{\theta}_n$ to be zero if there is no root. Then* $(\hat{\theta}_n - \tilde{\theta}_n) = \mathcal{O}_p(1/n)$.

PROOF. From Theorem (A.4) of [2] (see equation (A.1)), for any $\varepsilon > 0$ there is a continuously differentiable approximation $\tilde{\psi}$ to ψ such that $\|\psi - \tilde{\psi}\| \leqslant \varepsilon$ and

$$\hat{\theta}_n - \tilde{\theta}_n = -\frac{1}{n}\Sigma_{i=1}^{n}\tilde{\psi}(X_i - \tilde{\theta}_n)/B_n(\varepsilon)$$

where $B_n(\varepsilon) \to_P b > 0$ *uniformly* in ε. Thus, if $\varepsilon = 1/n$, $B_n(1/n) \to_P b$. But

$$\begin{aligned}|\frac{1}{n}\Sigma_{i=1}^{n}\tilde{\psi}(X_i - \tilde{\theta}_n)| &\leqslant \varepsilon + |\frac{1}{n}\Sigma_{i=1}^{n}\psi(X_i - \tilde{\theta}_n)| \\ &\leqslant \frac{1}{n} + |\frac{1}{n}(\Sigma_{i=1}^{k} + \Sigma_{i=n-k+1}^{n})\psi(X_i - \tilde{\theta}_n) + \Sigma_{i=k+1}^{n-k}\psi(X_i - \tilde{\theta}_n)| \\ &\leqslant \frac{1}{n} + \frac{2k \sup|\psi(x)|}{n},\end{aligned}$$

and the result follows. □

Thus, Theorem 1.1 and the other extensions presented earlier hold under model (1.1) and under the usual moving average model. However, the optimality result of [2] does not directly extend since the lower bound $(d^2\sigma_0^2)$ for the asymptotic variance of any sequence of estimators (given by Theorem 1.1 of [2]) requires that $\mathbf{S}$ be a circulant matrix. Thus, to extend the optimality result, either the result of Stone [3] used in Theorem 1.1 of [2] would have to be extended, or the class of estimators would have to be restricted to those for which appropriate asymptotic normality could be proven. It should be noted that Koul [1] has obtained such results for a wide variety of M-, L-, and R-estimators under strong mixing conditions (and, thus, under the moving average model).

2. Impossibility of finding second order optimal estimators. Since higher order expansions from the asymptotic variance are given in [2] (Theorem 5.1), one might hope to find higher order corrections to the first order optimal M-estimator derived there. In particular, one might hope to find an M-estimator which is optimal to order $o(\rho^2)$ as in Theorem 3.1 of [2] (where $k = 1$ and $\rho = \alpha_1$). The following result shows that this hope is illusory, and that higher order optimality would require estimators which explicitly adapt themselves to the assumed model for dependence.

In particular, consider the class of ε-contaminated normal distributions. Let g_0 be the density of Huber's least favorable distribution, G_0, which may be specified by defining

$$\psi_0(x) = -\frac{d}{dx}\log g_0(x) = x \qquad |x| \leqslant k$$
$$= k \operatorname{sgn} x \quad |x| > k$$

where $k = k(\varepsilon)$ satisfies equation (3.9) of [2]. We will first extend the expansion of Theorem 5.1 of [2] to include g_0 (for which the conditions of Theorem 5.1 do not hold); and then show that under G_0 essentially no collection of M-estimators can have asymptotic variance achieving the lower bound given by Theorem 1.1 of [2] to order $\mathcal{O}(\rho^2)$ (for ρ small). Note that since any influence curve can be used to define an M-estimator, the result of Theorem 2.2 directly extends to any collection of estimators whose influence curves satisfy the conditions of Lemma 2.1 below and whose asymptotic variance can be expanded to order ρ^2 as in Theorem 5.1 [2].

LEMMA 2.1. *Let ψ be an odd function satisfying the hypotheses of Theorem A.4 of* [2]. *Assume model* (1.1) *holds with $k = 1$, $\alpha_1 = \rho$ and $G = G_0$; and let $V(\psi)$ denote the asymptotic variance of the M-estimator with kernel ψ given by Theorem A.4 of* [2]. *Then, there is a constant $b > 0$ depending only on G_0 such that, with $Y \sim G_0$,*

$$V(\psi) = f_1(\psi) + \rho^2 EY^2 f_2(\psi) + b\{1 + E(\psi^4(Y) + \psi'^4(Y))\}o(\rho^2) \tag{2.1}$$

$$f_1(\psi) = \frac{E\psi^2(Y)}{(E\psi'(Y))^2} + 4\rho\frac{EY\psi(Y)}{E\psi'(Y)} \tag{2.2}$$

$$f_2(\psi) = 2 + \frac{\int\psi^2(x)g_0''(x)dx}{(E\psi'(Y))^2} - 2\frac{E\psi^2(Y)\int\psi'(x)g_0''(x)dx}{(E\psi'(Y))^3} \tag{2.3}$$

and where $o(\rho^2)$ is completely independent of ψ.

PROOF. From Theorem A.4 of [2],

$$V(\psi) = (E\psi^2(X_1) + 2E\psi(X_1)\psi(X_2) + 2E\psi(X_1)\psi(X_3))/(E\psi'(X_1))^2 \tag{2.4}$$

where X_i are defined by (1.1). Consider the term $E\psi(X_1)\psi(X_2)$. Transforming from (Y_0, Y_1, Y_2, Y_3) to (Y_0, X_1, X_2, Y_3) we find

$$f(y_0, x_1, x_2, y_3) = g_0(y_0)g_0(x_1 + \delta_1)g_0(x_2 + \delta_2)g_0(y_3)\cdot(1 - \rho^2)^{-1} \tag{2.5}$$

where

$$\delta_1 = \frac{1}{1-\rho^2}[\rho^2(x_1 + y_3) - \rho(x_2 + y_0)]$$

$$\delta_2 = \frac{1}{1-\rho^2}[\rho^2(x_2 + y_0) - \rho(x_1 + y_3)].$$

Since g_0' is absolutely continuous, we have the Taylor expansion (with integral remainder):

$$g_0(x_1 + \delta_1) = g_0(x_1) + \delta_1 g_0'(x_1) + \frac{1}{2}\delta_1^2 g_0''(x_1) + e_1(\rho) \tag{2.6}$$

where

$$e_1(\rho) = \int_{x_1}^{x_1+\delta_1}(x_1 + \delta_1 - u)g_0''(u)du - \frac{1}{2}\delta_1^2 g_0''(x_1).$$

Since $g_0''(x_1)$ is continuous except at $\pm k$, it is easy to see that as $\rho \to 0$, $e_1(\rho)/\rho^2 \to 0$ almost everywhere in (y_0, x_1, x_2, y_3). To apply the dominated convergence theorem $e_1(\rho)/\rho^2$ is bounded as follows: let $A = |Y_0| + |X_1| + |X_2| + |Y_3|$, and let b denote a generic constant. Note that

$$|g_0'(x)| \leqslant kg_0(x) \quad |g_0''(x)| \leqslant (1 + k^2)g_0(x). \tag{2.7}$$

Also, if $\rho \leqslant \frac{1}{2}$ (say), $|\delta_1|/\rho \leqslant bA$. Thus, for $\tilde{x}$ between x_1 and $x_1 + \delta_1$,

$$\begin{aligned} |e_1(\rho)|/\rho^2 &\leqslant b\frac{|\delta_1|}{\rho^2}\left|\int_{x_1}^{x_1+\delta_1} g_0(u)du\right| + bA^2 g_0(x_1), \\ &\leqslant bA^2 g_0(x_1)\left\{1 + \frac{g_0(\tilde{x})}{g_0(x_1)}\right\} \\ &= bA^2 g_0(x_1)\left\{1 + e^{l_0(\tilde{x}) - l_0(x_1)}\right\} \\ &\leqslant bA^2 g_0(x_1)(1 + e^{k|\delta_1|}) \\ &\leqslant bA^2 g_0(x_1)(1 + e^{b\rho A}) \end{aligned} \tag{2.8}$$

where $l_0(x) = \log g_0(x)$ and the fact that $|l_0'(x)| \leqslant k$ has been used. (Note that the bound in (2.8) is integrable with respect to $g_0(y_0)g_0(x_2)g_0(y_3)dy_0dx_1dx_2dy_3$ for ρ small enough.)

Now expanding $g_0(x_2 + \delta_2)$ as in (2.6) (with error term $e_2(\rho)$) and using (2.5), $E\psi(X_1)\psi(X_2)$ can be expanded as a polynomial in ρ of 4th degree plus integrals of error terms of the following form:

$$D = \int\int\int\int|\psi(x_1)\psi(x_2)e_1(\rho)g_0(y_0)(\delta_2^j/j!)g_0^{(j)}(x_2)g_0(y_3)|dy_0dx_1dx_2dy_3.$$

Using (2.7) and Cauchy-Schwarz

$$\begin{aligned} |D| &\leqslant bE\psi^2(Y)\rho^2\left\{\int\int\int\int\frac{A^{2j}}{\rho^4}e_1^2(\rho)g_0(y_0)g_0(x_2)g_0(y_3)dy_0dx_1dx_2dy_3\right\}^{\frac{1}{2}} \\ &\leqslant b(1 + E\psi^4(Y))o(\rho^2) \end{aligned}$$

where the last inequality follows using the dominated convergence theorem and (2.8) (for ρ small enough). Terms of the expansion of $E\psi(X_1)\psi(X_2)$ in ρ^3 and ρ^4 similarly lead to error terms with the same bound as for D. Thus, calculating the other terms of the expansion (as in Theorem 5.1 of [2]) gives

$$E\psi(X_1)\psi(X_2) = 2\rho EY\psi(Y)\cdot E\psi'(Y) + b(1 + E\psi^4(Y))o(\rho^2).$$

In a similar manner,

$$E\psi^2(X_1) = E\psi^2(Y) + \rho^2 EY^2 \int \psi^2(x) g_0''(x)dx + b(1 + E\psi^4(Y))o(\rho^2)$$

$$E\psi'(X_1) = E\psi'(Y) + \rho^2 EY^2 \int \psi'(x) g_0''(x)dx + b(1 + E\psi'^4(Y))o(\rho^2)$$

$$E\psi(X_1)\psi(X_3) = \rho^2 EY^2 (E\psi'(Y))^2 + b(1 + E\psi^4(Y))o(\rho^2).$$

The lemma follows using (2.4). □

THEOREM 2.1. *Let* $\{\psi_\rho : \rho > 0\}$ *be a collection of odd, bounded, absolutely continuous functions with* $E\psi_\rho'(Y) > 0$ *(where* $Y \sim G_0$*) and such that for some* $B > 0$

$$E\{\psi_\rho^4(Y) + \psi_\rho'^4(Y)\} < B \quad \text{for all } \rho. \tag{2.9}$$

Let $v(\rho) = V(\psi_\rho)$ *denote the asymptotic variance of the M-estimator corresponding to* ψ_ρ *under model* (1.1) *with* $k = 1$, $\alpha_1 = \rho$, *and* $G = G_0$; *and let* $v_0 = v(0)$ *denote the inverse Fisher information for* G_0. *Then there is* $b > 0$ *and* $\rho_0 > 0$ *such that for* $0 \leqslant \rho \leqslant \rho_0$

$$v(\rho) - (1 + 2\rho)^2 v_0 \geqslant b\rho^2. \tag{2.10}$$

Note that from Theorem 1.1 of [2], $(1 + 2\rho)^2 v_0$ is the smallest possible asymptotic variance for any sequence of invariant estimators; and this bound may be achieved by applying the M-estimator for ψ_0 to the coordinates of $\mathbf{S}^{-1}\mathbf{X}$ (with $\mathbf{S}$ as in (1.1)).

PROOF. The proof is divided into three parts. Let $Y \sim G_0$ throughout.

(i) Claim: if (2.10) does not hold, there is a subsequence $\{\rho_n\}$ of values of ρ tending to zero such that as $n \to \infty$

$$\frac{1}{\rho_n^2}\left(v(\rho_n) - (1 + 2\rho_n)^2 v_0\right) \to 0 \tag{2.11}$$

and, with $f_2(\psi)$ given by (2.3),

$$f_2(\psi_{\rho_n}) \to c > 4. \tag{2.12}$$

To prove (2.12), without loss of generality, let ψ_ρ and ψ_0 be normalized so that $E\psi_\rho'(Y) = E\psi_0'(Y) = 1$. Given ψ_ρ, define t_ρ and h_ρ so that $\psi_\rho(x) = \psi_0(x) + t_\rho h_\rho(x)$, where h_ρ is normalized so that $Eh_\rho^2(Y) = 1$.

From Lemma 2.1 (and the normalization of ψ_ρ' and ψ_0'),

$$\begin{aligned} v(\rho) &= E\psi_\rho^2(Y) + \mathcal{O}(\rho) \\ &= E\left(\psi_0(Y) + t_\rho h_\rho(Y)\right)^2 + \mathcal{O}(\rho) \\ &= v_0 + 2t_\rho E\psi_0(Y)h_\rho(Y) + t_\rho^2 Eh_\rho^2(Y) + \mathcal{O}(\rho). \end{aligned} \tag{2.13}$$

Since $\psi_0(x)$ is proportional to $-g_0'(x)/g_0(x)$,

$$\begin{aligned} E\psi_0(Y)h_\rho(Y) \propto \int h_\rho(x)g_0'(x)dx &= -\int h_\rho'(x)g_0(x)dx \\ &= -E\frac{1}{t_\rho}(\psi_\rho'(Y) - \psi_0'(Y)) = 0. \end{aligned}$$

Therefore, $v(\rho) = v_0 + t_\rho^2 + \mathcal{O}(\rho)$.

Hence, if (2.10) does not hold, there is a subsequence $\rho_n \to 0$ such that (2.11) holds and $t_{\rho_n} \to 0$. Furthermore, from (2.13)

$$E\psi_{\rho_n}^2(Y) \to E\psi_0^2(Y).$$

To prove convergence of $f_2(\psi_\rho)$ (see (2.3)) it remains to consider $\int \psi_\rho'(x) g_0''(x)dx$ and $\int \psi_\rho^2(x) g_0''(x)dx$. For the latter, from (2.7), $|\psi_\rho^2(x) g_0''(x)| \leqslant (1 + k^2)\psi_\rho^2(x) g_0(x)$; and the extended dominated convergence theorem shows that $\int \psi_{\rho_n}^2(x) g_0''(x)dx \to \int \psi_0^2(x) g_0''(x)dx$. Using integration by parts,

$$\begin{aligned}\int \psi_\rho'(x) g_0''(x)dx &= 2\psi_\rho(k) g_0''(k^-) - 2\int_0^k \psi_\rho(x) g_0''(x)dx \\ &\quad - 2\psi_\rho(k) g_0''(k^+) - 2\int_k^\infty \psi_\rho(x) g_0''(x)dx.\end{aligned}$$

Since $t_{\rho_n} \to 0$, $\psi_{\rho_n}(k) \to \psi_0(k)$; and since $|\psi_\rho(x) g_0''(x)| \leqslant (k + k^3)(1 + \psi_\rho^2(x)) g_0(x)$, the integrals converge by the extended dominated convergence theorem. Therefore, if (2.10) does not hold, $f_2(\psi_{\rho_n}) \to f_2(\psi_0)$. Direct calculation shows that (under the normalization $E\psi_0'(Y) = 1$),

$$E\psi_0^2(Y) = 1/D \qquad \int \psi_0'(x) g_0''(x)dx = -2(1-\varepsilon)k\phi(k)/D$$

$$\int \psi_0^2(x) g_0''(x)dx = 2(1-\varepsilon)[2\Phi(k) - 1 - 2k\phi(k)]/D^2$$

where $D = (1-\varepsilon)(2\Phi(k) - 1)$. Computing (2.3), $f_2(\psi_0) = 2 + 2/D > 4$, and the claim follows.

(ii) We now show that $f_1(\psi) \geqslant v_0(1 + 2\rho)^2 - 4\rho^2 EY^2$ for any ψ. To do this, minimize $f_1(\psi)$ (2.2) over ψ subject to $E\psi'(Y) = 1$. By the Lagrange multiplier method it suffices to assume $E\psi'(Y) = 1$ and minimize

$$\tilde{f}(\psi) = E\psi^2(Y) + 4\rho EY\psi(Y) + \lambda \int \psi(x) g_0'(x)dx.$$

Since $\tilde{f}$ is convex, standard variational techniques (setting directional derivatives to zero) show that the minimizing ψ^* satisfies

$$\psi^*(x) = -\frac{1}{2}\lambda g_0'(x)/g_0(x) - 2\rho x.$$

From the condition,

$$1 = E\psi^{*\prime}(Y) = -\frac{1}{2}\lambda E\frac{d^2}{dx^2}\log g_0(Y) - 2\rho = \frac{1}{2}\lambda/v_0 - 2\rho.$$

Hence, $\frac{1}{2}\lambda = v_0(1 + 2\rho)$ and $\psi^*(x) = -v_0(1 + 2\rho)(d/dx)\log g_0(x) - 2\rho x$ (which agrees with the result in [2]). Direct calculation now shows that $f_1(\psi^*) = v_0(1 + 2\rho)^2 - 4\rho^2 EY^2$, and part (ii) follows.

(iii) From part (i), if (2.10) does not hold there is $\rho_n \to 0$ such that (2.11) holds, $f_2(\psi_{\rho_n}) \to c > 4$, and (from (2.9)) the error term in $v(\rho_n)$ as given by (2.1) is $o(\rho_n^2)$. Combining this with part (ii), as $\rho_n \to 0$,

$$\begin{aligned}\lim \frac{1}{\rho_n^2}\left(v(\rho_n) - (1 + 2\rho_n)^2 v_0\right) &= \lim \frac{1}{\rho_n^2}\left(f_1(\psi_{\rho_n}) - (1 + 2\rho_n)^2 v_0\right) + cEY^2 \\ &\geqslant -4EY^2 + cEY^2 > 0\end{aligned}$$

(since $c > 4$). This contradicts (2.11) and thus establishes the theorem. ∎

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