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IDENTIFICATION AND SELECTION PROCEDURES BASED ON TESTS

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Let $X_1, \cdots, X_k$ be independent random variables with distributions $Q_1, \cdots, Q_k$ defined on a common range space $(\mathcal{X}, \mathcal{G})$. At the beginning it is assumed that the Q_i are known but not the pairing with the X_i, and the goal is to identify the X_i which comes from Q_1.

First it is shown that every procedure based on a total ordering in $\mathcal{X}$ can be viewed as being based on (the p-value of) a test for deciding between $H_0 : \{Q_1\}$ versus $H_1 : \{Q_2, \cdots, Q_k\}$. Then the class of procedures based on tests is studied in detail. It is demonstrated how typical properties of tests φ (powerfulness, unbiasedness, consistency etc.) transfer to the corresponding procedures S_φ. The next step is to get free of the assumption that $Q_1, \cdots, Q_k$ are known, thereby passing over from identification to selection procedures.

Throughout this paper the objective is to compare procedures (not to establish specified ones), and as one main result it is shown that the asymptotic relative efficiency (Pitman) of one test φ with respect to a second test ψ and of S_φ with respect to S_ψ are identical.

1. Introduction. Let $X_1, \cdots, X_k$ be independent random variables defined on a probability space $(\Omega, \mathcal{F})$ carrying a probability measure P, and $Q_1, \cdots, Q_k$ denote the probability distributions of $X_1, \cdots, X_k$ induced on their range space, $(\mathcal{X}, \mathcal{G})$, say. At first it is assumed that all the Q_i are known, but not the pairing with the X_i, and our goal is to identify the X_i (for simplicity we assume that there is only one) which comes from Q_1. Because we are interested only in identification procedures which are invariant under permutations of the observables $X_1, \cdots, X_k$, we assume without loss of generality that X_i has distribution Q_i, $i = 1, \cdots, k$.

In this section we shall see that every procedure based on a total ordering in $\mathcal{X}$ can be viewed as being based on (the p-value of) a test for deciding between $H_0 : \{Q_1\}$ versus $H_1 : \{Q_2, \cdots, Q_k\}$. Thus it seems justified to study the class of procedures based on tests in more detail. This is done in Section 2. In Section 3 an attempt is made to get free of the assumption that the Q_i are known, thereby passing over from identification to selection procedures. It will be shown how typical properties of tests φ (especially optimality) transfer to the corresponding selection procedures S_φ. Especially in Section 4 it turns out that the asymptotic relative efficiency (Pitman) of one test φ w.r.t. a second test ψ and of S_φ w.r.t. S_ψ are identical. Similar results for A.R.E. (Bahadur) can be found in Gaynor (1976), where, instead of tests, estimators are under concern. Reconsidering subset selec-

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tion procedures from our point of view, one arrives at results which eventually will be published elsewhere.

The emphasis of this paper lies in *comparison* of procedures, using techniques and results which can be found in classical (Neyman and Pearson) testing theory. To begin with, let "$<_s$" be an order relation in $\mathcal{X}$ with following properties:

(a) For $x, y \in \mathcal{X}$ exactly one of the relations $x <_s y$, $x >_s y$ or $x =_s y$ holds.

(b) For $x, y, z \in \mathcal{X}$ $x \leqslant_s y, y \leqslant_s z$ implies $x \leqslant_s z$. Both $x <_s y, y \leqslant_s z$ and $x \leqslant_s y$, $y <_s z$ imply $x <_s z$.

(c) For $x \in \mathcal{X} \{y \in \mathcal{X} | y =_s x\}$ and $\{y \in \mathcal{X} | y <_s x\}$ belong to $\mathcal{G}$, and $F_i(x) = Q_i\{y \in \mathcal{X} | y \leqslant_s x\}$, $x \in \mathcal{X}$, $i = 1, \cdots, k$, are measurable mappings from $(\mathcal{X}, \mathcal{G})$ to $([0, 1], \mathcal{B}_1)$, $\mathcal{B}_1$ denoting the Borel sets in [0, 1].

DEFINITION 1. The procedure S_s which selects $i \in \{1, \cdots, k\}$ if $X_i >_s X_j$, $j \neq i$, and splits ties (if any) at random, is called the identification procedure based on "$<_s$".

REMARK 1. In applications the X_i typically are samples from k specified populations, and "$<_s$" is given by $x <_s y$ iff $s(x) < s(y)$, $x, y \in \mathcal{X}$, where $s : \mathcal{X} \to \mathbb{R}$ is a suitable real-valued statistic.

For example, s may be the sample mean, variance or generalized variance in case of given normal populations, when the problem is to identify the population with the largest corresponding parameter. Most of the procedures proposed elsewhere fit into this framework. However, one important exception should be pointed out clearly: nonparametric procedures based on *joint* ranks (see Lee and Dudewicz (1974) for an overview) are not based on order relations satisfying (a)-(c).

Now if "$<_{F_1}$" is the order relation given by $x <_{F_1} y$ iff $F_1(x) < F_1(y)$, $x, y \in \mathcal{X}$, then it has properties (a)-(c). We will show in the sequel, that S_{F_1} is as good as S_s w.r.t. probability of correct selection, and that S_{F_1} is equivalent to selecting for the largest p-value of a test ψ for $H_0 : \{Q_1\}$ versus $H_1 : \{Q_2, \cdots, Q_k\}$, given in the proof of Theorem 1.

REMARK 2. Introducing random variables $U_1, \cdots, U_k$, independently uniformly distributed in [0, 1] and independent of $X_1, \cdots, X_k$, and defining $x^* = (x, u) <_{s^*} (y, v) = y^*$ iff $x <_s y$ or $x =_s y$, $u < v$, $(x, u), (y, v) \in \mathcal{X}^* = \mathcal{X} \times [0, 1]$, the $X_i^* = (X_i, U_i)$ are clearly nonatomic: $P\{X_i^* =_{s^*} x^*\} = 0$, $x^* \in \mathcal{X}^*$, $i = 1, \cdots, k$. Thus because of $P\{X_i^* =_{s^*} X_j^*\} = 0$, $i \neq j$, S_{s^*} represents a nonrandomized version of S_s in a suitable enlarged sample space.

Though we could assume now, without loss of generality, that the X_i are nonatomic, we prefer to remain in the previous case. The reason is that tests usually are defined on $\mathcal{X}$ and it is more convenient to formulate our results in the familiar language of classical (Neyman and Pearson) testing theory.

LEMMA 1. *For all $x \in \mathcal{X}$ we have*

$$P\{X_1 >_s x, F_1(X_1) = F_1(x)\} = 0. \tag{1.1}$$

If X_1 *is nonatomic* w.r.t. "$=_s$", *then* (1.1) *remains valid if* $>_s$ *is replaced by* $<_s$, *and therefore* $F_1(X_1)$ *is nonatomic, too.*

PROOF. Let $A(x) = \{y \in \mathcal{X} | y >_s x, F_1(y) = F_1(x)\}, x \in \mathcal{X}$.

$$\begin{aligned} Q_1(A(x))^2 &\geqslant 2Q_1 \times Q_1(A(x) \times A(x) \cap \{(y, z)| y <_s z\}) \\ &= 2\int_{A(x)} Q_1(A(x) \cap \{z | z >_s y\}) dQ_1(y) \\ &= 2\int_{A(x)} Q_1(A(x) \cap \{z | z >_s x\}) dQ_1(y) \\ &= 2\int_{A(x)} Q_1(A(x)) dQ_1(y) = 2Q_1(A(x))^2, \qquad x \in \mathcal{X}, \end{aligned}$$

and therefore $Q_1(A(x)) = 0$ for each $x \in \mathcal{X}$. If X_1 is nonatomic, we start with $B(x) = \{y \in \mathcal{X} | y <_s x, F_1(y) = F_1(x)\}$, and get $Q_1(B(x))^2 \geqslant 2Q_1 \times Q_1(B(x) \times B(x) \cap \{(y, z)| y \geqslant_s z\})$. Proceeding analogously we finally get $Q_1(B(x)) = 0$ for each $x \in \mathcal{X}$.

LEMMA 2. *If* S_s *and* S_{F_1} *split ties according to the same randomization scheme, then*

$$P(\{S_s = 1\} \setminus \{S_{F_1} = 1\}) = 0. \tag{1.2}$$

PROOF. Let Δ denote symmetric differences and $j \in \{2, \cdots, k\}$. By $\{F_1(X_1) > F_1(X_j)\} \subseteq \{X_1 >_s X_j\} \subseteq \{F_1(X_1) \geqslant F_1(X_j)\}$ and Lemma 1 we have $P(\{X_1 >_s X_j\} \Delta \{F_1(X_1) > F_1(X_j)\}) = 0$, which together with $\{X_1 =_s X_j\} \subseteq \{F_1(X_1) = F_1(X_j)\}$ implies (1.2).

REMARK 3. It may happen that $P\{X_1 <_s X_j, F_1(X_1) = F_1(X_j)\} > 0$ and therefore $P(\{S_{F_1} = 1\} \setminus \{S_s = 1\}) > 0$ occurs. But if the X_i are nonatomic w.r.t. "$=_s$", then Lemma 1 guarantees $P(\{S_s = 1\} \Delta \{S_{F_1} = 1\}) = 0$.

REMARK 4. In concrete situations—i.e., situations where $Q_1, \cdots, Q_k$ are given explicitly—one usually looks for a suitable statistic s, such that $F_1(x) = P\{X_1 \leqslant_s x\} \geqslant P\{X_i \leqslant_s x\} = F_i(x), x \in \mathcal{X}, i \geqslant 2$, holds (cf. Remark 1), and then takes S_s resp. S_{F_1} as a reasonable procedure.

The main idea of this paper is as follows: let X be an (auxiliary) random variable defined on $(\Omega, \mathfrak{F})$ and Q be its distribution induced on the range space $(\mathcal{X}, \mathcal{G})$, and consider the testing problem $H_0 : Q = Q_1$ versus $H_1 : Q \in \{Q_2, \cdots, Q_k\}$. If for each $\alpha \in [0, 1]$ φ_α is a "good" test at the level α, then the following procedure S_φ seems to be reasonable: "select $j \in \{1, \cdots, k\}$, if X_j is the last of $X_1, \cdots, X_k$ which becomes significant under $\{\varphi_\alpha\}_{\alpha \in [0, 1]}$ when α increases from 0 to 1, and split ties (if any) at random".

DEFINITION 2. A test φ is a family $\{\varphi_\alpha\}_{\alpha \in [0, 1]}$ of measurable mappings $\varphi_\alpha : (\mathcal{X}, \mathcal{G}) \to ([0, 1], \mathfrak{B}_1), \alpha \in [0, 1]$. It is called *monotone* (m.), if for each $x \in \mathcal{X}$ $\varphi_\alpha(x)$ is monotone nondecreasing in $\alpha \in [0, 1]$, and it is called *standardized* w.r.t. Q_1(s.(Q_1)), if $E\varphi_\alpha(X_1) = \alpha, \alpha \in [0, 1]$.

REMARK 5. If φ is an m.s. (Q_1)-test for any H_0 versus H_1, X a random variable and U independently of X uniformly distributed in [0, 1], then the usual way of reaching decisions is to reject H_0 iff $U \leqslant \varphi_\alpha(X)$, $\alpha \in [0, 1]$ being the predetermined level. This suggests defining the p-value of a test in the following manner:

DEFINITION 3. Let φ be a monotone test. The function $p_\varphi : \mathfrak{X} \times [0, 1] \to [0, 1]$ given by $p_\varphi(x, u) = \inf\{\alpha | u \leqslant \varphi_\alpha(x)\}$, $x \in \mathfrak{X}$, $u \in [0, 1]$, is called the p-value of (x, u) w.r.t. φ. If φ is nonrandomized (i.e., if $\varphi_\alpha : (\mathfrak{X}, \mathcal{G}) \to \{0, 1\}$, $\alpha \in [0, 1]$) this reduces to $p_\varphi : \mathfrak{X} \to [0, 1]$ with $p_\varphi(x) = \inf\{\alpha | \varphi_\alpha(x) = 1\}$, $x \in \mathfrak{X}$.

Now we come back to $X_1, \cdots, X_k$ and let $U_1, \cdots, U_k$ be independent uniformly in [0, 1] distributed random variables, independent of $X_1, \cdots, X_k$, too.

DEFINITION 4. Let φ be a monotone test. The procedure S_φ which selects $i \in \{1, \cdots, k\}$ if $p_\varphi(X_i, U_i) > p_\varphi(X_j, U_j)$, $i \neq j$, and splits ties (if any) at random, is called the identification procedure based on test φ.

Now we state the main result of this section, using procedure S_{s^*} (cf. Remark 2) for convenience.

THEOREM 1. *There exists an* m.s. (Q_1)*-test* ψ *with*

$$P(\{S_{s^*}(X_1, U_1, \cdots, X_k, U_k) = 1\} \setminus \{S_\psi(X_1, U_1, \cdots, X_k, U_k) = 1\}) = 0. \tag{1.3}$$

PROOF. Let for $\alpha \in [0, 1]$ $\psi_\alpha(x) = 1$ if $x \notin$ support (Q_1) and if $x \in$ support (Q_1),

$$\begin{aligned} \psi_\alpha(x) &= 1 \qquad \text{iff} \quad F_1(x) < c(\alpha), \\ &= k(\alpha) \qquad\quad F_1(x) = c(\alpha) \\ &= 0 \qquad\qquad F_1(x) > c(\alpha) \end{aligned}$$

where $E\psi_\alpha(X_1) = P\{F_1(X_1) < c(\alpha)\} + k(\alpha)P\{F_1(X_1) = c(\alpha)\} = \alpha$ and $k(\alpha) = 1$ if $P\{F_1(X_1) = c(\alpha)\} = 0$. Clearly $\psi = \{\psi_\alpha\}_{\alpha \in [0, 1]}$ is an m.s. (Q_1)-test. Since for $u \in [0, 1]$ and $x \in$ support (Q_1) we have

$$p_\psi(x, u) = P\{F_1(X_1) < F_1(x)\} + uP\{F_1(X_1) = F_1(x)\},$$

$$\begin{aligned} &\{S_\psi(X_1, U_1, \cdots, X_k, U_k) = 1\} \\ &\quad = \bigcup_{I \subseteq \{2, \cdots, k\}} \{F_1(X_1) \\ &\quad = F_1(X_i), U_1 > U_i, i \in I, F_1(X_1) > F_1(X_j), j \notin I\} \\ &\quad = \{S_{F^*_1}(X_1, U_1, \cdots, X_k, U_k) = 1\}, \end{aligned}$$

which together with Lemma 2 implies (1.3). In view of this result we restrict our further considerations to identification procedures based on m.s. (Q_1)-tests.

2. Identification procedures based on tests. In this section we still maintain the assumptions, stated at the beginning of Section 1. But the objects of interest now are tests and no longer total orderings: the general class of all tests defined on $\mathfrak{X}$ including the randomized ones.

Now each m.s. (Q_1)-test φ can be modified to $\tilde{\varphi}$ such that

(2.1) $\tilde{\varphi}_\alpha(x)$ is right-continuous in $\alpha \in [0, 1]$ for all $x \in \mathfrak{X}$, and

(2.2) $\tilde{\varphi}_\alpha(x) = 1, \alpha \in [0, 1],$ if x does not belong to support (Q_1).

In order to arrive at a concise formula in (2.7), we choose support $(Q_1) = \{x | f_1(x) > 0\}$, where f_1 is the Radon-Nikodym derivative of Q_1 w.r.t. $Q = Q_1 + \cdots + Q_k$, which clearly dominates the Q_i. (By this we avoid the existence of Q_i-atoms, $i \geqslant 2$, in support (Q_1), which are not Q_1-atoms simultaneously.). On the other hand, it should be pointed out that the more important formula (2.6) holds true for every choice of support (Q_1).

Since $\tilde{\varphi}$ still is m.s. (Q_1) and $\varphi_\alpha(x) \leqslant \tilde{\varphi}_\alpha(x), \alpha \in [0, 1], x \in \mathfrak{X}$, holds, and therefore identification procedure $S_{\tilde{\varphi}}$ based on $\tilde{\varphi}$ is as good as S_φ based on φ (which we shall see very soon), we restrict our further attention to tests satisfying (2.1) and (2.2). Besides we remark that test ψ appearing in Theorem 1 has these properties already, if support (Q_1) is chosen properly.

Important for the following is the fact that for every *m.*-test φ satisfying (2.1) we have for every random variable X

(2.3) $$P\{p_\varphi(X, U) \leqslant \alpha\} = E\varphi_\alpha(X), \qquad \alpha \in [0, 1],$$

if U is independently of X uniformly distributed in [0, 1]. This follows from the monotonicity and (2.1), since then for all $x \in \mathfrak{X}$, $u, \alpha \in [0, 1]$, $p_\varphi(x, u) \leqslant \alpha$ is equivalent to $u \leqslant \varphi_\alpha(x)$.

Still having in view $X_1, U_1, \cdots, X_k, U_k$ as defined in Section 1, we first state

LEMMA 3. *For every* m.s. (Q_1)-*test* φ *and* $i \in \{1, \cdots, k\}$,

(2.4) $E\varphi_\alpha(X_i)$ *is a continuous function of* $\alpha \in [0, 1]$, *and*

(2.5) $E\varphi_1(X_i) = 1.$

PROOF. Let $G_\alpha = \{x \in \mathfrak{X} | \varphi_{\alpha-}(x) < \varphi_{\alpha+}(x)\}$, $\alpha \in [0, 1]$. Since φ is s. (Q_1), we have for each $\alpha \in [0, 1]$

$$E(\varphi_{\alpha+}(X_1) - \varphi_{\alpha-}(X_1)) = E\varphi_{\alpha+}(X_1) - E\varphi_{\alpha-}(X_1) = 0,$$

and thus by monotonicity of φ $Q_1(G_\alpha) = 0$ holds. By standard arguments taking $Q = Q_1 + \cdots + Q_k$ $Q(G_\alpha \cap \text{support}(Q_1)) = 0$ and finally $Q_i(G_\alpha \cap \text{support}(Q_1)) = 0$, $i = 1, \cdots, k$, follow, if support (Q_1) is chosen as indicated above. Since by (2.2) we have for $i = 1, \cdots, k$

$$E\varphi_{\alpha+}(X_i) - E\varphi_{\alpha-}(X_i) \leqslant Q_i(G_\alpha \cap \text{support}(Q_1)) = 0,$$

(2.4) is proved. And since φ is s. (Q_1), (2.5) follows by (2.2).

In view of the next theorem it should be pointed out that (2.3), together with (2.4), says that for every m.s. (Q_1)-test φ and $i \in \{1, \cdots, k\} p_\varphi(X_i, U_i)$ has a continuous distribution and especially that $p_\varphi(X_1, U_1)$ is uniformly distributed in [0, 1]. (Thus beyond $U_1, \cdots, U_k$ no further randomization is needed for S_φ.)

Now we state our main result, which shows that for m.s. (Q_1)-tests φ the distribution of S_φ depends on φ only through its power function, and this in a simple and impressive manner:

THEOREM 2. *For each* m.s. (Q_1)-*test* φ

$$P\{S_\varphi = 1\} = \textstyle\int_0^1 \prod_{j=2}^k E\varphi_\alpha(X_j)d\alpha \tag{2.6}$$

and for $i \geqslant 2$

$$P\{S_\varphi = i\} = \textstyle\int_0^1 \alpha \prod_{j=2;\, j\neq i}^k E\varphi_\alpha(X_j)dE\varphi_\alpha(X_i), \tag{2.7}$$

where integration is w.r.t. α.

PROOF. Let $i \in \{1, \cdots, k\}$. By (2.3) and Lemma 3 we have

$$P\{S_\varphi(X_1, U_1, \cdots, X_k, U_k) = i\} = P\{p_\varphi(X_i, U_i) > p_\varphi(X_j, U_j), j \neq i\},$$

as we pointed out above. The right hand side equals to

$$\begin{aligned}\textstyle\int_0^1 P\{p_\varphi(X_j, U_j) < \alpha, j \neq i \mid p_\varphi(X_i, U_i) = \alpha\} P\{p_\varphi(X_i, U_i) \in d\alpha\} \\ = \textstyle\int_0^1 \prod_{j=1;\, j\neq i}^k P\{p_\varphi(X_j, U_j) < \alpha\} P\{p_\varphi(X_i, U_i) \in d\alpha\} \\ = \textstyle\int_0^1 \prod_{j=1;\, j\neq i}^k E\varphi_\alpha(X_j)dE\varphi_\alpha(X_i),\end{aligned}$$

which in turn is equal to the right hand side of (2.6) for $i = 1$ and of (2.7) for $i \in \{2, \cdots, k\}$.

The following statements are immediate consequences of Theorem 2:

(I) *Sufficiency.* If $T : (\mathcal{X}, \mathcal{G}) \to (\mathcal{X}', \mathcal{G}')$ is a sufficient statistic for $Q_1, \cdots, Q_k$, then we can confine ourselves to procedures based on tests which depend on $x \in \mathcal{X}$ only through $T(x)$.

(II) *Power relations.* If φ and ψ are m.s. (Q_1)-tests with $E\psi_\alpha(X_j) \leqslant E\varphi_\alpha(X_j)$, $\alpha \in [0, 1]$, $j \geqslant 2$, then

$$P\{S_\psi = 1\} \leqslant P\{S_\varphi = 1\}. \tag{2.8}$$

In plain words: the better the test the better the identification procedure. And besides we remark that (in indifference zone formulations) maximin-tests induce maximin-procedures.

(III) *Unbiasedness.* If φ is a m.s. (Q_1)-test which is unbiased (i.e., $E\varphi_\alpha(X_j) \geqslant \alpha$, $\alpha \in [0, 1]$, $j \geqslant 2$), then S_φ is unbiased:

$$P\{S_\varphi = 1\} \geqslant k^{-1}. \tag{2.9}$$

Unfortunately the converse statement does not hold true! Thus the important question (cf. Section 3) remains open, whether each unbiased procedure based on a total ordering can be equalled or beaten (w.r.t. probability of correct identification) by a procedure based on an unbiased test.

(IV) *Monotonicity.* If φ is an m.s. (Q_1)-test and $i, j \in \{1, \cdots, k\}$,

$$E\varphi_\alpha(X_i) \leqslant E\varphi_\alpha(X_j), \qquad \alpha \in [0, 1] \tag{2.10}$$

implies

$$P\{S_\varphi = j\} \leqslant P\{S_\varphi = i\}, \tag{2.11}$$

(cf. Gupta and Nagel (1971)). Though the proof is straightforward, we will sketch it briefly, because on page 133 of Lee and Dudewicz (1974), this (in another context) was stated as an open problem: we start with $P\{S_\varphi = j\}$, integrate it by parts and apply (2.10). Then we proceed with the outcoming result analogously and finally arrive at $P\{S_\varphi = i\}$ as an upper bound.

(V) *Consistency.* Let $X_1^{(n)}, \cdots, X_k^{(n)}, n \in \mathbb{N} = \{1, 2, \cdots\}$ be independent random variables defined on $(\Omega, \mathfrak{F})$ with range space $(\mathfrak{X}, \mathfrak{G})$ and distributions $Q_1^{(n)}, \cdots, Q_k^{(n)}$, $n \in \mathbb{N}$, let $U_1, \cdots, U_k$ be as before and let $\varphi^{(n)}$ for each $n \in \mathbb{N}$ be an m.s. (Q_1)-test. If $\{\varphi^{(n)}\}_{n \in \mathbb{N}}$ is consistent for H_1 in the usual sense, then

$$\lim_{n\to\infty} P\left\{S_{\varphi(n)}\left(X_1^{(n)}, U_1, \cdots, X_k^{(n)}, U_k\right) = 1\right\} = 1. \tag{2.12}$$

(VI) *Other topics.* Finally let us mention that other typical properties of tests such as invariance, local and asymptotic most powerfulness in view of Theorem 2 can be treated analogously. In Section 4 Pitman's asymptotic relative efficiency will be studied in detail.

3. Selection procedures based on tests. In this section an attempt is made to get rid of the assumption that $Q_1, \cdots, Q_k$ are known, thereby passing over from identification to selection procedures. For the rest of this paper, let $\{Q_\vartheta\}_{\vartheta \in \theta}$, $\theta \subseteq \mathbb{R}$ (or $\theta \subseteq \mathbb{R}^m$ when nuisance parameters are involved), be a given family of distributions with densities $\{f_\vartheta\}_{\vartheta \in \theta}$ w.r.t. a σ-finite measure μ defined on $(\mathfrak{X}, \mathfrak{G})$. For $Q_1, \cdots, Q_k$, the distributions of $X_1, \cdots, X_k$, we assume that $Q_i = Q_{\vartheta_i}$ holds for certain unknown $\vartheta_i \in \theta$, $i = 1, \cdots, k$, and our goal is now to select the (for simplicity) unique population with the largest parameter $\max\{\vartheta_1, \cdots, \vartheta_k\} = \vartheta_0$, say. (The smallest parameter problem can be treated analogously). Without loss of generality we assume that $\vartheta_1 = \vartheta_0$ holds, and for simplicity we use now the symbol s. (ϑ) instead of s. (Q_ϑ).

DEFINITION 5. Let φ be a monotone test. Then S_φ, as defined in Definition 4, now is called the selection procedure based on test φ. Such a procedure is called uniformly best, if its probability of correct selection maximizes the probabilities of correct selection of all procedures based on monotone tests in all possible parameter situations.

In analogy to development of testing theory a first step is to look at families $\{f_\vartheta\}_{\vartheta \in \theta}$ with monotone likelihood ratios, thereby arriving at a well-known result (cf. Lehmann (1966)):

COROLLARY 1. *Let $T : \mathfrak{X} \to \mathbb{R}$ be a (sufficient) statistic and for $\vartheta_1 < \vartheta_2$, $\vartheta_1, \vartheta_2 \in \theta \subseteq \mathbb{R}$, $f_{\vartheta_2}(x)/f_{\vartheta_1}(x)$ be a nondecreasing function of $T(x)$, $x \in \mathfrak{X}$. Then the procedure which selects the population with the largest T-value, and splits ties (if any) at random, is uniformly best.*

PROOF. Let us first assume that $\vartheta_1 > \vartheta_2, \cdots, \vartheta_k$ are fixed. If ψ is any monotone test, S_ψ is based on the order relation generated by its p-value p_ψ, which is defined on $\mathcal{X}^* = \mathcal{X} \times [0, 1]$. (Note that beyond $U_1, \cdots, U_k$ an additional randomization scheme may be necessary to establish S_ψ). Let $Q^*_{\vartheta_i} = Q_{\vartheta_i} \times W_i$, W_i being the uniform distribution on $[0, 1]$, $i = 1, \cdots, k$. Then in view of Theorem 1 (applied to the new setup $(\mathcal{X}^*, Q^*_{\vartheta_i})$, $i = 1, \cdots, k$) we can assume that ψ is an m.s. $(Q^*_{\vartheta_i})$-test, defined on $\mathcal{X}^*$.

The U.M.P. level α test for $\bar{H}_0 : \vartheta = \vartheta_1$ versus $\bar{H}_1 : \vartheta < \vartheta_1$ on the other hand is given by

$$\varphi_\alpha(x, u) = 1, \quad = \gamma(\alpha), \quad = 0 \qquad \text{iff} \quad T(x) \begin{matrix} < \\ = \\ > \end{matrix} \eta(\alpha),$$

$x \in \mathcal{X}$, $u \in [0, 1]$ and $E\varphi_\alpha(X_1) = \alpha$, $\alpha \in [0, 1]$, which clearly does not depend on the values $u \in [0, 1]$. By Theorem 2 we get

$$P\{S_\psi = 1\} \leqslant P\{S_\varphi = 1\} \qquad \text{for the fixed} \quad \vartheta_1, \cdots, \vartheta_k. \tag{3.1}$$

Now S_φ selects according to the largest T-value and therefore is independent of $Q_{\vartheta_1}, \cdots, Q_{\vartheta_k}$. Thus (3.1) holds in all cases where $\vartheta_1 > \vartheta_2, \cdots, \vartheta_k$, and the proof is completed.

REMARK 6. If $\mathcal{X} = \mathbb{R}^n$, $n \in \mathbb{N}$, it can be seen easily, that the optimal procedure given in Corollary 1 is most economical in the sense that no other procedure based on a monotone test can reach the same probability of correct selection with a sample size smaller than n. This is because U.M.P. (likelihood ratio) tests are most economical in an analogous sense. In the "indifference zone"-formulation a similar result was derived by Hall (1959) without the assumption of invariance. But on the other hand, location and scale parameters only are admitted there.

EXAMPLE 1. Let $X_i = (X_{i1}, \cdots, X_{in})$, $i = 1, \cdots, k$, where X_{ij}, $i = 1, \cdots, k$, $j = 1, \cdots, n$ are independently normal distributed random variables with means ϑ_i, $i = 1, \cdots, k$, and common known variance $\sigma^2 > 0$, and let $\bar{X}_i = (X_{i1} + \cdots + X_{in})/n$, $i = 1, \cdots, k$. Then the uniformly best procedure S_φ selects $i \in \{1, \cdots, k\}$ iff $\bar{X}_i > \bar{X}_r$, $r \neq i$. Since S_φ does not depend on σ^2, it is even uniformly best if we admit a two-dimensional parameter space where σ^2 acts as nuisance parameter, and σ^2 needs neither to be known nor to be estimated.

In the "indifference zone"-formulation, Bechhofer (1954) has shown that this procedure is optimal, too. On the other hand, S_φ is a special case of Gupta's means procedure ($\bar{X}_i > \bar{X}_r - d$, $r \neq i$) for $d = 0$, which was proposed by Gupta (1956) in the "subset selection"-formulation.

Unfortunately, in many situations U.M.P.-tests do not exist and only U.M.P.-unbiased tests are available. Then we can only say that the corresponding "optimal" procedure beats all others based on unbiased tests. Especially in view of (2.3) all procedures are beaten which are based on orderings induced by statistics $T : \mathcal{X} \to \mathbb{R}$ being stochastically nondecreasing in ϑ. Thus this class seems to be not too narrow. And if the open question in (III) could be answered positively, this class could be replaced by the larger class consisting of all unbiased procedures based on total orderings.

Typically such situations occur when two-sided testing problems or multiparameter exponential families are involved. And in many situations—somewhat disappointing—the reasonable procedure based on an U.M.P.-unbiased test may depend on ϑ_0. The following examples are given for illustration.

EXAMPLE 2. Let the X_i be defined as in Example 1 but our goal is now to select the population with the largest $|\vartheta_i|$, ϑ_0, say. For ϑ_0 fixed the U.M.P.-unbiased s. (ϑ_0)-test ψ for $\tilde{H}_0 : |\vartheta| = \vartheta_0$ versus $H_1 : |\vartheta| < \vartheta_0$ rejects small values of $|\bar{x}|$ and therefore leads to procedure S_ψ which selects the population with the largest $|\bar{X}_i|$. Since S_ψ does not depend on ϑ_0, it is uniformly best among all procedures based on unbiased m.s. (ϑ_0)-tests, $\vartheta_0 \in \mathbb{R}$ (but it is not uniformly best in general). In Rizvi (1963) this procedure (beside others) is studied in detail.

EXAMPLE 3. Let the X_i be as before with the only difference that the variances $\sigma_i^2 > 0$ now depend on $i \in \{1, \cdots, k\}$ and are unknown. If our goal is the same as in Example 1 and if $\vartheta_0 = \max_i \vartheta_i$ is known, procedure $S_{\hat{\varphi}}$ based on the U.M.P.-unbiased s. (ϑ_0)-test (for a suitable θ) selects the population with the largest t-statistic

$$(n(n-1))^{\frac{1}{2}}(\bar{X}_i - \vartheta_0)\left(\Sigma_{r=1}^k (X_{ir} - \bar{X}_i)^2\right)^{-\frac{1}{2}}.$$

Since $S_{\hat{\varphi}}$ depends on ϑ_0 we conclude that in case of unknown ϑ_0 there is no procedure that beats all others based on unbiased m.s. (ϑ_0)-tests, $\vartheta_0 \in \mathbb{R}$. This is true even in the case of known but different σ_i^2, $i = 1, \cdots, k$.

If one replaces the unknown ϑ_0 by the estimator $\max_i \bar{X}_i$, then $S_{\hat{\varphi}}$ reduces to the procedure in Example 1. If one takes another (better) estimator, then one may be led to a procedure which is no longer based on a monotone test (or total ordering).

REMARK 7. If in case of monotone likelihood ratios one wants to find a confidence interval for $\vartheta_0 = \max_i \vartheta_i$ simultaneously with selecting, it seems natural to take (for a fixed confidence coefficient $\beta \in [0, 1]$) the confidence interval given by the U.M.P.-unbiased two-sided s. (ϑ_0)-test for a suitable $\theta = [\vartheta_*, \vartheta^*]$ with $\vartheta_* < \vartheta_0 < \vartheta^*$. As can be seen immediately there exist $\alpha_1(\beta), \alpha_2(\beta) \in [0, 1]$ such that for "CS" denoting correct selection and "CD" denoting correct decision (i.e., ϑ_0 being covered by the confidence interval)

$$P\{CS, CD\} = \int_{\alpha_1(\beta)}^{\alpha_2(\beta)} \Pi_{j=2}^k E\varphi_\alpha(X_j)\,d\alpha, \tag{3.2}$$

where φ is the s. (ϑ_0)-version of the best one-sided test.

EXAMPLE 4. Let the X_i be as in Example 1 and our goal be now to find a confidence interval of fixed length L simultaneously with selecting for $\vartheta_0 = \max_i \vartheta_i$. If one selects the population with the largest $\bar{X}_i$ and takes $\{\vartheta \,||\max_i \bar{X}_i - \vartheta| \leqslant L/2\}$ as the confidence interval, then (3.2) can be expressed as follows: let

$\alpha_1(L) = 1 - \Phi(n^{\frac{1}{2}}L/2\delta)$, Φ denoting the cdf of $N(0, 1)$. Then (3.2) becomes

$$(3.3) \qquad P\{CS, CD\} = \int_{\alpha_1(L)}^{1-\alpha_1(L)} \prod_{j=2}^{k} E\varphi_\alpha(X_j)d\alpha,$$

where φ is now the s. (ϑ_0)-version of the Gauss-test. Formulas (3.2) and (3.3) may serve as a basis for comparisons of competing procedures (by stepping through interesting values of ϑ_0). The problem of establishing procedures of this type in the "indifference zone"-approach (i.e., finding least favorable configurations) under assumption of M.L.R. is treated in Rizvi and Saxena (1974). Improvements of the confidence interval (L remaining fixed), taking into account the bias of estimator $\max_i \overline{X}_i$ w.r.t. $\max_i \vartheta_i$ are given in Dudewicz and Tong (1971) and in Alam, Saxena and Tong (1973). They result in some modifications of the boundaries of the integral in (3.3) in an obvious manner.

4. Asymptotic relative efficiency. For convenience we shall treat only the one-sample case, for after this has been carried out it should be evident how to apply the method to other cases.

Let for $n \in \mathbb{N}$, $X^{(n)} = (X_1^{(n)}, \cdots, X_n^{(n)})$ be samples, i.e., $X_1^{(n)}, \cdots, X_n^{(n)}$ be i.i.d. random variables with distributions $Q\vartheta_{(n)}$, $\vartheta^{(n)} \in \theta \subseteq \mathbb{R}$ which for convenience is indicated also by suffixes at the probabilities and expectations now. Let $\varphi^{(n)}$ and $\psi^{(n)}$, $n \in \mathbb{N}$, be consistent m.s. (ϑ_0)-tests for some $\vartheta_0 \in \theta$ with the following properties:

$$(4.1) \qquad \text{For } \vartheta^{(n)} = \vartheta_0 + \eta n^{-\frac{1}{2}} + o(n^{-\frac{1}{2}}), \qquad \eta > 0, n \in \mathbb{N}, 0 < \alpha < 1$$

$$\lim_{n\to\infty} E\vartheta^{(n)}\varphi_\alpha^{(n)}(X^{(n)}) = 1 - \Phi(u_\alpha - \eta\delta_\varphi), \qquad \delta_\varphi \in \mathbb{R},$$

where $u_\alpha = \Phi^{-1}(1 - \alpha)$. The same for ψ with δ_φ replaced by δ_ψ. Then the asymptotic relative efficiency (Pitman) of φ w.r.t. ψ at ϑ_0 as is well known is given by

$$(4.2) \qquad \text{A.R.E.}\,(\varphi, \psi) = (\delta_\varphi/\delta_\psi)^2.$$

Now let for $n \in \mathbb{N}$ $X_1^{(n)}, \cdots, X_k^{(n)}$ be independent samples with distributions $Q_{\vartheta_i^{(n)}}$, $\vartheta_i^{(n)} \in \theta$, $i = 1, \cdots, k$, and $U_1, \cdots, U_k$ as before.

Definition 6. If there exists a mapping $n^* : \mathbb{N} \to \mathbb{N}$ such that for all $\boldsymbol{\vartheta}^{(n)} = (\vartheta_1^{(n)}, \cdots, \vartheta_k^{(n)})$ with

$$(4.3) \qquad \vartheta_1^{(n)} = \vartheta_0 \text{ and } \vartheta_j^{(n)} = \vartheta_0 + \eta_j n^{-\frac{1}{2}} + o(n^{-\frac{1}{2}}),$$

$$\eta_j > 0, j \in \{2, \cdots, k\} \text{ and } n \in \mathbb{N}$$

$$\lim_{n\to\infty} P_{\boldsymbol{\vartheta}(n)}\{S_{\varphi(n)}(X_1^{(n)}, U_1, \cdots, X_k^{(n)}, U_k) = 1\}$$

$$= \lim_{n\to\infty} P_{\boldsymbol{\vartheta}(n)}\{S_{\psi(n^*(n))}(X_1^{(n^*(n))}, U_1, \cdots, X_k^{(n^*(n))}, U_k) = 1\}$$

and if for all $n^* : \mathbb{N} \to \mathbb{N}$ satisfying (4.3) $\lim_{n\to\infty} n^*(n)/n = e$, say, then we call e the asymptotic relative efficiency of S_φ w.r.t. S_ψ (A.R.E. (S_φ, S_ψ)) at ϑ_0.

Now we can state the main result of this section:

THEOREM 3. *Let* $\varphi^{(n)}$ *and* $\psi^{(n)}$, $n \in \mathbb{N}$, *be consistent* m.s. (ϑ_0)-*tests satisfying* (4.1). *Then*

$$\text{A.R.E.}\,(S_\varphi, S_\psi) = \text{A.R.E.}\,(\varphi, \psi). \tag{4.4}$$

PROOF. Under the assumptions given above we have

$$\begin{aligned}(4.5)\quad \lim_{n\to\infty} P_{\vartheta(n)}\{S_{\varphi(n)}(X_1^{(n)}, U_1, \cdots, X_k^{(n)}, U_k) = 1\} \\ = \lim_{n\to\infty} \textstyle\int_0^1 \prod_{j=2}^k E_{\vartheta_j(n)}\varphi_\alpha^{(n)}(X_j^{(n)})d\alpha \\ = \textstyle\int_0^1 \prod_{j=2}^k \left[\lim_{n\to\infty} E_{\vartheta_j(n)}\varphi_\alpha^{(n)}(X_j^{(n)})\right]d\alpha \\ = \textstyle\int_0^1 \prod_{j=2}^k [1 - \Phi(u_\alpha - \eta_j\delta_\varphi)]\,d\alpha.\end{aligned}$$

For n^* we can assume $n^*(n) = an + o(n)$, $n \in \mathbb{N}$, for some $a \in \mathbb{R}$, because otherwise $E_{\vartheta_j(n)}\psi_\alpha^{(n^*(n))}(X_j^{(n^*(n))})$ either tend to α for $j = 2, \cdots, k$ or to 1 and (4.3) cannot be fulfilled since $\varphi^{(n)}$ satisfies (4.1). Thus we have

$$\begin{aligned}(4.6)\quad \lim_{n\to\infty} P_{\vartheta(n)}\{S_{\psi(n^*(n))}(X_1^{(n^*(n))}, U_1, \cdots, X_k^{(n^*(n))}, U_k) = 1\} \\ = \textstyle\int_0^1 \prod_{j=2}^k\left[1 - \Phi\left(u_\alpha - a^{\frac{1}{2}}\eta_j\delta_\psi\right)\right]d\alpha,\end{aligned}$$

and equality of (4.5) and (4.6) holds if and only if $a = (\delta_\varphi/\delta_\psi)^2$. Thus A.R.E.$(S_\varphi, S_\psi) = (\delta_\varphi/\delta_\psi)^2 =$ A.R.E.(φ, ψ).

In the "indifference zone"-approach similar results can be obtained: let $P^* \in (1/k, 1]$ be fixed, $X_1^{(n)}, \cdots, X_k^{(n)}$ as before but let the distributions be restricted now by

$$\begin{aligned}(4.7)\quad \vartheta_1^{(n)} = \vartheta_0, \qquad \vartheta_j^{(n)} \geqslant \vartheta_0 + \delta^{(m)}, \\ n, m \in \mathbb{N},\ \{\delta^{(m)}\}_{m\in\mathbb{N}} \text{ fixed},\ j = 2, \cdots, k.\end{aligned}$$

For a sequence of m.s. (ϑ_0)-tests $\varphi^{(n)}$, $n \in \mathbb{N}$, we define for each $m \in \mathbb{N}$ $N(\delta^{(m)}, P^*, S_\varphi)$ to be the smallest integer N satisfying

$$\inf_{\vartheta_j^{(N)} \geqslant \vartheta_0 + \delta^{(m)};\, j\geqslant 2} P\{S_{\varphi(N)}(X_1^{(N)}, U_1, \cdots, X_k^{(N)}, U_k) = 1\} \geqslant P^*. \tag{4.8}$$

If $E_{\vartheta j(n)}\varphi_\alpha^{(n)}(X_j^{(n)})$ is nondecreasing in $\vartheta_j^{(n)} \geqslant \vartheta_0$, $n \in \mathbb{N}, j \geqslant 2$, then (4.8) reduces to

$$\textstyle\int_0^1\left[E_{\vartheta_0+\delta^{(m)}}\varphi_\alpha^{(N)}(X_2^{(N)})\right]^{k-1} d\alpha \geqslant P^*. \tag{4.9}$$

Now if A.R.E.$(S_\varphi, S_\psi|$indifference zone) for such tests is defined in the usual manner as the limit

$$\lim_{m\to\infty} N(\delta^{(m)}, P^*, S_\psi)/N(\delta^{(m)}, P^*, S_\varphi)$$

subject to the condition that both S_φ and S_ψ meet P^* asymptotically, then $\delta^{(m)} = \eta m^{-\frac{1}{2}} + o(m^{-\frac{1}{2}})$ for some $\eta > 0$, $m \in \mathbb{N}$, if the N's of φ and ψ are of order m, and therefore we get A.R.E.$(S_\varphi, S_\psi|$indifference zone$) =$ A.R.E.(φ, ψ).

EXAMPLE 5. Let $X_i = (X_{i1}, \cdots, X_{in})$, $i = 1, \cdots, k$, be independent samples of size $n \in \mathbb{N}$ of symmetric distributions in $\mathbb{R}$ with cumulative distribution functions $F_i(t) = F(t + \vartheta_i)$, $t \in \mathbb{R}$, $\vartheta_i \in \theta \subseteq \mathbb{R}$, $i = 1, \cdots, k$, F being no further specified. To find the population with the largest location parameter ϑ_0, Ghosh (1973) proposed a procedure which selects according to the largest Hodges-Lehmann estimator $\hat{\vartheta}(X_i)$, derived from a one-sample signed rank statistic $h(X_i)$. This procedure is equivalent to S_ψ based on the following m.s. (ϑ_0)-test ψ:

$$(4.10) \qquad \psi_\alpha(x) = 1, \quad = \gamma(\alpha), \quad = 0 \qquad \text{iff} \quad \hat{\vartheta}(x) \gtreqless c_F(\alpha), \qquad x \in \mathbb{R}^n,$$

with $E_{\vartheta_0}\psi_\alpha(X_1) = \alpha$, $\alpha \in [0, 1]$.

This because by the location invariance of $\hat{\vartheta}$ procedure S_ψ does not depend on ϑ_0. But it should be pointed out clearly, that ψ in fact is a parametric test depending on F. Now in Hodges and Lehmann (1963) it is shown that the A.R.E.(Pitman) of two tests based on h_i (of the type given above) equals the A.R.E. (in the sense of reciprocal ratio of asymptotic variances) of the Hodges-Lehmann estimators $\hat{\vartheta}_i$ derived from h_i, $i = 1, 2$. And by the asymptotic normality of such estimators the A.R.E.(Pitman) of tests ψ_1 and ψ_2 based on $\hat{\vartheta}_1$ respective $\hat{\vartheta}_2$ according to (4.10) adopts the same value. (Since to the author's knowledge this class (4.10) of parametric location tests is nowhere proposed in literature till now, this may be an interesting result.) Finally in view of our results given above we conclude that A.R.E.(S_{ψ_1}, S_{ψ_2}|indifference zone) adopts this value, too. In Ghosh (1973) this result can be found beside others.

EXAMPLE 6. To give an example for the two-sample case, let for $n \in \mathbb{N}$, $X_i^{(2n)} = (Y_{i1}^{(n)}, Z_{i1}^{(n)}, \cdots, Y_{in}^{(n)}, Z_{in}^{(n)})$, $i = 1, \cdots, k$, be independent samples of size n from bivariate populations π_i, $i = 1, \cdots, k$, the parameter ϑ_i of interest being a measure of association (i.e., rank correlation—or product moment correlation coefficient) between the Y's and the Z's in population π_i, $i = 1, \cdots, k$.

For $\vartheta_0 = 0$ and $\theta = [0, 1]$ the two competing procedures S_τ based on Kendall's tau and S_r based on the sample product moment correlation coefficient r in the normal case have A.R.E.(S_τ, S_r) $= (3/\pi)^2$, since the A.R.E. of the corresponding tests of independence adopts this value (cf. Lehmann (1975), page 316). In Govindarajulu and Gore (1971) this result was derived (beside others) by asymptotic considerations.

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