The Annals of Statistics
1979, Vol. 7, No. 1, 108–115

THE ASYMPTOTIC DISTRIBUTION OF THE SUPREMUM OF THE STANDARDIZED EMPIRICAL DISTRIBUTION FUNCTION ON SUBINTERVALS

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It is well known that the limit distribution of the supremum of the empirical distribution function F_n centered at its expectation F and standardized by division by its standard deviation is degenerate, if the supremum is taken on too large regions $\varepsilon_n < F(u) < \delta_n$. So it is natural to look for sequences of linear transformations, so that for given sequences of sup-regions $(\varepsilon_n, \delta_n)$ the limit of the transformed sup-statistics is nondegenerate.

In this paper a partial answer is given to this problem, including the case $\varepsilon_n \equiv 0$, $\delta_n \equiv 1$. The results are also valid for the Studentized version of the above statistic, and the corresponding two-sided statistics are treated, too.

1. Introduction. Let $X_1, X_2, \cdots$ be independent random variables (irv's) with continuous distribution function (df) F. Let $F_n(u) = n^{-1}\sum_{i=1}^n \mathbf{1}_{\{X_i \leqslant u\}}$ be the associated empirical distribution function and Φ an arbitrary positive function on the open unit interval. Anderson and Darling (1952) investigated the limit distribution of the generalized Kolmogorov statistic

$$K_{n,\Phi} = \sup_{0<F(u)<1} n^{\frac{1}{2}}|F_n(u) - F(u)|\Phi[F(u)].$$

As the variance of $n^{\frac{1}{2}}(F_n(u) - F(u))$ is $F(u)(1 - F(u))$, they assumed that the weight function $\Phi_0(u) = [u(1-u)]^{-\frac{1}{2}}$ "in some sense" assigns to each point u equal weights. But according to Theorem 2 of Čibisov (1966) $K_{n,\Phi_0} \to_{\mathbf{P}} \infty$ holds, so that in a very strong sense Φ_0 cannot yield equal weights. By giving the limit distribution of the supremum of the standardized empirical distribution function on subintervals, our main result will show how slow the divergence of K_{n,Φ_0} is and which $F(u)$-values are responsible. For the construction of confidence contours for F, it is more convenient to substitute $\Phi_0(F(u))$ in the definition of K_{n,Φ_0} by $\Phi_0(F_n(u))$, cf. Eicker (1978). We will see that for the resulting "Studentized" versions of the standardized Kolmogorov-Smirnov statistics the limits remain the same, even if subintervals are taken.

But since a comparison with the studies of Steck (1971) and Noé (1972) has shown that the rate of convergence of our generalized K-S statistics to the extreme value distribution E is very slow, we would not encourage anyone to use the confidence intervals based on the asymptotic analysis.

Received March 1976; revised October 1977.

[1]This article was written with the support of the Deutsche Forschungsgemeinschaft.

AMS 1970 *subject classifications.* Primary 62E20; Secondary 60F05.

Key words and phrases. Standardized empirical distribution function, normalized sample quantile process, extreme value distribution, boundary crossing of empirical process, Poisson process, Ornstein-Uhlenbeck process, normalized Brownian bridge process, goodness of fit test, tail estimation.

2. Main results. We will use the following definitions:

$$V_n(u) = n^{\frac{1}{2}}(F_n(u) - F(u))(F(u)(1 - F(u)))^{-\frac{1}{2}} \qquad 0 < F(u) < 1$$

$$\hat{V}_n(u) = n^{\frac{1}{2}}(F_n(u) - F(u))(F_n(u)(1 - F_n(u)))^{-\frac{1}{2}} \quad \text{for} \quad 0 < F_n(u) < 1,$$
$$= 0 \quad \text{for} \quad F_n(u) \in \{0, 1\}$$

$$V_n(\varepsilon, \delta) = \sup_{\varepsilon < F(u) < \delta} V_n(u), \qquad W_n(\varepsilon, \delta) = \sup_{\varepsilon < F(u) < \delta} |V_n(u)|,$$

$$\hat{V}_n(\varepsilon, \delta) = \sup_{\varepsilon < F(u) < \delta} \hat{V}_n(u), \; \hat{W}_n(\varepsilon, \delta) = \sup_{\varepsilon < F(u) < \delta} |\hat{V}_n(u)| (0 \leqslant \varepsilon \leqslant \delta \leqslant 1,$$
suprema over empty sets are defined as $-\infty$)

$$a(x) = (2 \log x)^{\frac{1}{2}}, \qquad b(x) = 2 \log x + 2^{-1} \log_2 x - 2^{-1} \log \pi, \qquad a_n = a(\log n),$$
$$b_n = b(\log n)(x > e, \log_2 x = \log(\log x) \text{ etc.})$$
$$t_x(t) = (t + b(x))/a(x),$$
$$E(t) = \exp[-\exp(-t)] \qquad t \in \mathbb{R}$$
$$\mu_n = (\log n)^3/n \qquad n \geqslant 3$$
$$f_n(u) = (\mu_n \vee u) \wedge (1 - \mu_n) \qquad (u \in [0, 1], \quad a \wedge b = \min(a, b),$$
$$a \vee b = \max(a, b))$$
$$\rho(\varepsilon, \delta) = 2^{-1} \log[\delta(1 - \varepsilon)/(\varepsilon(1 - \delta))] \qquad 0 < \varepsilon \leqslant \delta < 1$$

$\rho_n = \rho_n(\varepsilon_n, \delta_n) = \rho(f_n(\varepsilon_n), f_n(\delta_n))$ $(\{\varepsilon_n\}, \{\delta_n\}$ given sequences from the unit interval).

THEOREM. *If* $\lim_{n\to\infty} \rho_n(\varepsilon_n, \delta_n)/\log n = c$, *then*

$$\lim_{n\to\infty} \mathbb{P}(V_n(\varepsilon_n, \delta_n) < t_{\log n}(t)) = [E(t)]^c \tag{1}$$

$$\lim_{n\to\infty} \mathbb{P}(W_n(\varepsilon_n, \delta_n) < t_{\log n}(t)) = [E(t)]^{2c} \tag{2}$$

$$\lim_{n\to\infty} \mathbb{P}(\hat{V}_n(\varepsilon_n, \delta_n) < t_{\log n}(t)) = [E(t)]^c \tag{3}$$

$$\lim_{n\to\infty} \mathbb{P}(\hat{W}_n(\varepsilon_n, \delta_n) < t_{\log n}(t)) = [E(t)]^{2c}. \tag{4}$$

REMARKS.

1. Due to the definition of ρ_n, c can only lie in [0, 1].
2. A simple calculation shows that for $c > 0$, (1)–(4) is equivalent to

$$\lim_{n\to\infty} \mathbb{P}(V_n(\varepsilon_n, \delta_n) < t_{\rho_n}(t)) = E(t) \tag{5}$$

$$\lim_{n\to\infty} \mathbb{P}(W_n(\varepsilon_n, \delta_n) < t_{\rho_n}(t)) = E^2(t) \tag{6}$$

$$\lim_{n\to\infty} \mathbb{P}(\hat{V}_n(\varepsilon_n, \delta_n) < t_{\rho_n}(t)) = E(t) \tag{7}$$

$$\lim_{n\to\infty} \mathbb{P}(\hat{W}_n(\varepsilon_n, \delta_n) < t_{\rho_n}(t)) = E^2(t). \tag{8}$$

3. Analogously to the proof of the theorem one shows that the above statements remain true when $F_n(u) - F(u)$ in the definitions of $V_n(u)$ and $\hat{V}_n(u)$ is replaced by $F(u) - F_n(u)$.

EXAMPLE 1. For $\varepsilon_n \equiv 0$, $\delta_n \equiv 1$, (1) leads to

$$\lim_{n\to\infty} \mathbb{P}\Big(\sup_{0<F(u)<1} n^{\frac{1}{2}}(F_n(u) - F(u))(F(u)(1 - F(u)))^{-\frac{1}{2}} < \big(t + 2\log_2 n + \tfrac{1}{2}\log_3 n - \tfrac{1}{2}\log \pi\big)(2\log_2 n)^{-\frac{1}{2}}\Big) = \exp[-\exp(-t)].$$

This and the corresponding version of (2) was already given in Jaeschke (1975), while (3) and (4) for this special case was found by Eicker (1978).

EXAMPLE 2. For $0 < \alpha$, $\beta < \infty$; $\varepsilon_n = n^{-\alpha(1+o(1))}$ and $\delta_n = 1 - n^{-\beta(1+o(1))}$, we have

$$\lim_{n\to\infty} \rho_n/\log n = ((1 \wedge \alpha) + (1 \wedge \beta))/2 = c.$$

EXAMPLE 3. While, e.g., in the case $\limsup_{n\to\infty}(\log \delta_n)/\log n \leqslant -1$ $(0 \leqslant \varepsilon_n \leqslant \delta_n \leqslant 1)$ we only have the degenerate statements (1)–(4) with $c = 0$, if $\varepsilon_n \wedge (1 - \delta_n) \geqslant \mu_n$ and $\rho_n \to \infty$, (5)–(8) remain true due to Lemmas 1, 2 and 5 below, then leading to nondegenerate limits even if $\lim_{n\to\infty} \rho_n/\log n$ does not exist. The following corollaries to the theorem are stated only for V_n, but they are also true for W_n, $\hat{V}_n$ and $\hat{W}_n$.

COROLLARY 1. *If* $\liminf_{n\to\infty} \rho_n/\log n > 0$, *then*

$$(2\log_2 n)^{-\frac{1}{2}} V_n(\varepsilon_n, \delta_n) \to_{\mathbf{P}} 1. \tag{9}$$

COROLLARY 2. $\limsup_{n\to\infty}(\log \varepsilon_n)/\log n \leqslant -1$ iff

$$a_n[V_n(0, \varepsilon_n) \vee V_n(1 - \varepsilon_n, 1)] - b_n \to_{\mathbf{P}} -\infty.$$

COROLLARY 3. *Let* $\varepsilon_n \leqslant \frac{1}{2}$ *for all* $n \in \mathbb{N}$. *Then* $\lim_{n\to\infty}(\log \varepsilon_n)/\log n = 0$ iff

$$a_n V_n(\varepsilon_n, 1 - \varepsilon_n) - b_n \to_{\mathbf{P}} -\infty.$$

PROOF OF THE THEOREM. While in [6] and [7] the special case $\varepsilon_n \equiv 0$ and $\delta_n \equiv 1$ has been reduced to the maximum of standardized partial sums $\max_{k=1,\dots,n} k^{-\frac{1}{2}}\sum_{i=1}^{k} Y_i$ with limit distribution given by Darling and Erdös (1956), in the following the invariance principle of Komlós, Major and Tusnády (1975) will be applied, using the corresponding results for the standardized Brownian bridge process and some modified results of [6] and [7]. As F is continuous, we may from now on assume F to be $U(0, 1)$, the uniform distribution over the unit interval. First we investigate the standardized Brownian bridge and state

LEMMA 1. *Let* $0 < \varepsilon_n \leqslant \delta_n < 1$, $\tau_n = \rho(\varepsilon_n, \delta_n) \to \infty$ *and* B *a Brownian bridge* (B.B.), i.e., *a Gauss process with* $\mathbb{E}(B(u)) = 0$, $\mathbb{E}(B(u)B(v)) = u(1 - v)$ *for* $0 \leqslant u \leqslant$

$v \leqslant 1$. *Then for* $t \in \mathbb{R}$

(10) $$P_n = \mathbb{P}\Big(\sup_{\varepsilon_n < u < \delta_n} B(u)(u(1-u))^{-\frac{1}{2}} < t_{\tau_n}(t)\Big) \to E(t)$$

and

(11) $$\mathbb{P}\Big(\sup_{\varepsilon_n < u < \delta_n} |B(u)|(u(1-u))^{-\frac{1}{2}} < t_{\tau_n}(t)\Big) \to E^2(t).$$

PROOF. It is well known that there is an Uhlenbeck process U, i.e., a stationary Gauss-Markov process with mean 0 and $E(U(u)U(v)) = \exp[-|u-v|]$ $(u, v \in \mathbb{R})$, so that $B(u) = (u(1-u))^{\frac{1}{2}} \times U(2^{-1}\log(u/(1-u)))$ $(0 < u < 1)$. Therefore $P_n = \mathbb{P}(\sup_{0<u<\tau_n} U(u) < t_{\tau_n}(t)) \to E(t)$, where the last statement follows from Lemma 3.4 and Lemma 3.10 of Darling and Erdös (1956). Similarly, (11) follows. □

LEMMA 2. *Let* $\varepsilon_n \wedge (1-\delta_n) \geqslant \mu_n = (\log n)^3/n$. *Then on a rich enough probability space there is a sequence of* B.B.'s $\{B_n\}$, *so that* (*with* $-\infty + \infty = 0$)

(12) $$V_n(\varepsilon_n, \delta_n) - \sup_{\varepsilon_n < u < \delta_n} B_n(u)(u(1-u))^{-\frac{1}{2}} = o\Big((\log_2 n)^{-\frac{1}{2}}\Big)\big[= o(1/a(\tau_n))\big] \quad \text{a.s.}$$

PROOF. According to Theorem 3 of Komlós, Major and Tusnády (1975), on a rich enough probability space there are B.B.'s B_n, such that

$$\sup_{0<u<1}|n^{\frac{1}{2}}(F_n(u) - u) - B_n(u)| = O\Big(n^{-\frac{1}{2}}\log n\Big) \quad \text{a.s.}$$

Therefore

$$a_n \sup_{\varepsilon_n < u < \delta_n}|V_n(u) - B_n(u)(u(1-u))^{-\frac{1}{2}}| = O\Big((2\log_2 n)^{\frac{1}{2}}(\log n)(n\mu_n)^{-\frac{1}{2}}\Big) = o(1) \quad \text{a.s. } \square$$

In (16) we shall apply

LEMMA 3. *Let* $\{\nu(u)\}_{u \geqslant 0}$ *a Poisson process* $(\lambda = 1)$ *with paths constant except for upward jumps of height* 1, $c > 0$, $\alpha \in \mathbb{R}$. *Then for* $\gamma(u) = u^{-\frac{1}{2}}(\nu(u) - u)$

(13) $$\alpha a_n \sup_{0<u<(\log n)^c}|\gamma(u)| - b_n \to_{\mathbb{P}} -\infty.$$

PROOF. We now follow the line of proof of Lemma (6.1) in [7]. Of course we only need to show (13) for $\alpha > 0$. Then (13) is equivalent to

$$\lim_{n\to\infty} \mathbb{P}\Big(\sup_{0<u<(\log n)^c}|\gamma(u)| > \alpha^{-1}(2\log_2 n)^{\frac{1}{2}} + \alpha^{-1}\big(t + 2^{-1}\log_3 n\big)(2\log_2 n)^{-\frac{1}{2}}\Big) = 0$$

for all $t \in \mathbb{R}$. So it suffices to show

(14) $$\lim_{n\to\infty} \mathbb{P}\Big(\sup_{1<u\leqslant n^c}|\gamma(u)| > \alpha^{-1}(2\log n)^{\frac{1}{2}}\Big) = 0.$$

For the proof of (14) we choose some $d > 1$, define $n_k = d^k$ and $c_0 = c/\log d$. Then

$$\mathbb{P}\Big(\sup_{1<u\leqslant n^c}|\gamma(u)| > \alpha^{-1}(2\log n)^{\frac{1}{2}}\Big) \leqslant \sum_{k=0}^{[c_0\log n]}\mathbb{P}\Big(\sup_{n_k<u\leqslant n_{k+1}}|\nu(u) - u|n_k^{-\frac{1}{2}} \geqslant \alpha^{-1}(2\log n)^{\frac{1}{2}}\Big) = P_n.$$

Since

$$\mathbb{P}(\sup_{a<u\leqslant b}|\nu(u) - u| \geqslant z) \leqslant \tfrac{4}{3}\mathbb{P}\big(|\nu(b) - b| \geqslant z - 2(b - a)^{\frac{1}{2}}\big)$$

(cf. (4.2) in O'Reilly (1974)) we have for large enough n

$$\mathbb{P}\big(\sup_{n_k<u\leqslant n_{k+1}}|\nu(u) - u| \geqslant \alpha^{-1}(2n_k \log n)^{\frac{1}{2}}\big)$$
$$\leqslant \tfrac{4}{3}\mathbb{P}\big(|\nu(n_{k+1}) - n_{k+1}| \geqslant \alpha^{-1}d^{k/2}(\log n)^{\frac{1}{2}}\big) = p(n, k).$$

The upper bound ($c_1, c_2 > 0$ suitable constants)

$$p(|\nu(x) - x| > z) \leqslant 2\exp[-c_1z^2x^{-1}] + 2\exp[-c_2z]$$

(cf. (8) in Čibisov (1966)) yields constants $c_3, c_4 > 0$ so that

$$p(n, k) \leqslant \tfrac{8}{3}\{\exp[-c_3 \log n] + \exp[-c_4d^{k/2}(\log n)^{\frac{1}{2}}]\}$$
$$= \tfrac{8}{3}\{p_1(n, k) + p_2(n, k)\}.$$

In $p_1(n, k)$, k no longer occurs, so that

$$\Sigma_{k=0}^{[c_0 \log n]}p_1(n, k) \leqslant n^{-c_3}(1 + c_0 \log n) \to 0.$$

With $c(n) = c_4(\log n)^{\frac{1}{2}}$, since $c(n) \to \infty$ and $\int_1^\infty x^{-1}e^{-x}\, dx < \infty$, we obtain

$$\Sigma_{k=0}^{[c_0 \log n]}p_2(n, k) \leqslant \int_0^{[c_0 \log n]+1} \exp[-c(n)d^{(x-1)/2}]\, dx$$
$$\leqslant 2(\log d)^{-1}\int_{d^{-\frac{1}{2}}c(n)}^{\infty} x^{-1}e^{-x}\, dx \to 0,$$

completing the proof of (14). □

LEMMA 4. *Let* $\mu_n = (\log n)^3/n$. *Then*

(15) $$a_n[W_n(0, \mu_n) \vee W_n(1 - \mu_n, 1)] - b_n \to_{\mathbb{P}} -\infty.$$

PROOF. Following the ideas of the proof of Lemma 7.2 in [7], let $\xi_1, \xi_2 \cdots$ be strictly positive irv's distributed according to the density function $\mathbf{1}_{(0,\infty)}(u)e^{-u}$ and $\eta_i = \Sigma_{j=1}^{i}\xi_j$. If $U_1^n \leqslant \cdots \leqslant U_n^n$ are the order statistics of independent $U(0, 1)$ – rv's $U_1, \cdots, U_n$, we then have $\mathcal{L}(U_1^n, \cdots, U_n^n) = \mathcal{L}(\eta_1/\eta_{n+1}, \cdots, \eta_n/\eta_{n+1})$ (cf. e.g. Breiman (1968), page 285). We therefore may assume F_n to be the empirical df of the η_i/η_{n+1}. Then $\nu(u) = \Sigma_{i\geqslant 1}\mathbf{1}_{\{\eta_i\leqslant u\}}$, $u > 0$ is a Poisson process according to Lemma 3 with $F_n(u) = n^{-1}\nu(u\eta_{n+1})$ $(0 < u < 1)$. So we get

$$W_{1n} = \sup_{0<u<\mu_n} n^{\frac{1}{2}}u^{-\frac{1}{2}}|F_n(u) - u|$$
$$= \sup_{0<u<\mu_n}(nu)^{-\frac{1}{2}}|\nu(u\eta_{n+1}) - nu| = \sup_{0<u<\eta_{n+1}\mu_n}(\eta_{n+1}/n)^{\frac{1}{2}}|\gamma(u) + \Delta_n(u)|,$$

where $\gamma(u) = u^{-\frac{1}{2}}(\nu(u) - u)$ and $\Delta_n(u) = u^{\frac{1}{2}}(1 - n/\eta_{n+1})$. For $\alpha_n = ((\log n)/n)^{\frac{1}{2}}$ we define $\Omega_n = \{|1 - \eta_{n+1}/n| \leqslant \alpha_n\}$. From $n^{\frac{1}{2}}(1 - \eta_{n+1}/n) \to_d N(0, 1)$, $\mathbb{P}(|1 -$

$(\eta_{n+1}/n)^{\frac{1}{2}}| \leqslant \alpha_n) \geqslant \mathbb{P}(\Omega_n) \to 1$ follows. On Ω_n we have

(16)
$$\begin{aligned} a_n W_n(0, \mu_n) - b_n &\leqslant a_n(1-\mu_n)^{-\frac{1}{2}} W_{1n} - b_n \\ &\leqslant (1+o(1))a_n\Big[\sup\nolimits_{0<u<n\mu_n(1+\alpha_n)}|\gamma(u)| \\ &\quad + (n\mu_n(1+\alpha_n))^{\frac{1}{2}}|1 - n/\eta_{n+1}|\Big] - b_n. \end{aligned}$$

Now $a_n\mu_n^{\frac{1}{2}} \to 0$, implying $a_n(n\mu_n(1+\alpha_n))^{\frac{1}{2}}|1 - n/\eta_{n+1}| \to_{\mathbb{P}} 0$, so that by Lemma 3, $a_n W_n(0, \mu_n) - b_n \to_{\mathbb{P}} -\infty$. The proof is completed by symmetry arguments for the region $(1-\mu_n, 1)$. □

PROOF OF (1). Of course we only need to consider $c > 0$, for then the case $c = 0$ is a trivial consequence. According to Remark 2 it here suffices to show (5). Now except for at most finitely many n we have $\mu_n < \delta_n$, $\varepsilon_n < 1 - \mu_n$, so that for large enough n

(17)
$$\begin{aligned} V_n(\varepsilon_n, \delta_n) = V_n(\varepsilon_n, \mu_n \vee \varepsilon_n) \vee V_n(\mu_n \vee \varepsilon_n, \delta_n \wedge (1-\mu_n)) \\ \vee V_n(\delta_n \wedge (1-\mu_n), \delta_n), \end{aligned}$$

and, due to (15), we need only consider the middle expression on the right side of (17), for which we get the limit distribution from (10) and (12). □

Similarly, one obtains (2). For the proofs of (3) and (4), we need two more lemmas.

LEMMA 5. *For* ε_n, $1 - \delta_n \geqslant \lambda_n = n^{-1}\log n$ *we have*

$$\hat{V}_n(\varepsilon_n, \delta_n) - V_n(\varepsilon_n, \delta_n) = o_{\mathbb{P}}(1/a_n)\big[= o_{\mathbb{P}}(1/a(\tau_n))\big].$$

PROOF. Due to Theorem 3.1 of Csáki (1977), we have

$$\sup\nolimits_{\lambda_n<u<1}|1 - u^{-1}F_n(u)| = 0\big((\log_2 n/(n\lambda_n))^{\frac{1}{2}}\big) \text{ a.s.}$$

and

$$\sup\nolimits_{0<u<1-\lambda_n}|1 - (1-F_n(u))/(1-u)| = 0\big((\log_2 n/(n\lambda_n))^{\frac{1}{2}}\big) \text{ a.s.}$$

Now $a_n\hat{V}_n(\varepsilon_n, \delta_n) = a_n V_n(\varepsilon_n, \delta_n)(1 + 0((\log_2 n/\log n)^{\frac{1}{2}}))$ a.s. By (9),

$$a_n V_n(\varepsilon_n, \delta_n)0\big((\log_2 n/\log n)^{\frac{1}{2}}\big) \to_{\mathbb{P}} 0,$$

from which the assertion follows. □

LEMMA 6. *With* $\lambda_n = n^{-1}\log n$, *we have*

$$a_n\big[\hat{W}_n(0, \lambda_n) \vee \hat{W}_n(1-\lambda_n, 1)\big] - b_n \to_{\mathbb{P}} -\infty.$$

PROOF. Here we follow the line of the proof of (2.15) in Eicker (1978). Let $0 \leqslant U_1^n \leqslant \cdots \leqslant U_n^n \leqslant 1$ be the order statistics of a random sample from $U(0, 1)$.

First, since

$$n(\log n)^{-1} U^n_{[\log^2 n]} \to \infty \quad \text{a.s.}$$

(cf. Theorem 6 in Kiefer (1970)), we have on some Ω_{1n} with $\mathbb{P}(\Omega_{1n}) \to 1$

$$\sup_{U_1^n \leqslant u \leqslant \lambda_n} |\hat{V}_n(u)| \leqslant \sup_{U_1^n \leqslant u \leqslant U^n_{[\log^2 n]}} |\hat{V}_n(u)|.$$

Defining

$$w_{kn} = (k(1 - k/n)/n)^{\frac{1}{2}},$$

$$B_{kn} = \left\{ k/n - t_{\log n} w_{kn} n^{-\frac{1}{2}} \leqslant U_k^n \leqslant (k-1)/n + t_{\log n} w_{k-1,n} n^{-\frac{1}{2}} \right\}$$

and

$$B_n = \left\{ \sup_{U_1^n \leqslant u \leqslant U^n_{[\log^2 n]}} |\hat{V}_n(u)| < t_{\log n} \right\},$$

we have $B_n = \cap_{k=2}^{[\log^2 n]} B_{kn}$. With the notations of the proof of our Lemma 4, we obtain

$$B_{kn} \supset \Omega_n\Big((1 + \alpha_n)\big(k - t_{\log n} w_{kn} n^{\frac{1}{2}}\big) \leqslant \eta_k \leqslant (1 - \alpha_n)\big(k - 1 + t_{\log n} w_{k-1,n} n^{\frac{1}{2}}\big)\Big)$$

$$\supset \Omega_n\big(k^{-\frac{1}{2}} |\eta_k - k| < u_{kn}\big),$$

where

$$u_{kn} = (2 \log_2 n)^{\frac{1}{2}} (1 + ((\log_3 n) + 2t)/(4 \log_2 n))(1 - \alpha_n)(1 - k^{-1})^{\frac{1}{2}} (1 - k/n)^{\frac{1}{2}}$$
$$- c_n (k/n)^{\frac{1}{2}} - (1 + o(1)) k^{-\frac{1}{2}}.$$

In the region $2 \leqslant k \leqslant \log^2 n$ we have

$$u_{kn} > (2 \log_2 n)^{\frac{1}{2}} \left\{ 2^{-1} (1 + (\log_3 n + 2t + o(t))/(4 \log_2 n)) - \left(\tfrac{1}{2} + o(1)\right)(\log_2 n)^{-\frac{1}{2}} \right\}$$
$$= (\log_2 n)^{\frac{1}{2}} (1 + o(1)) > t_{\log(\log^2 n)}(1/o(1))$$

for some $o(1) \downarrow 0$ with sufficient slowness. Theorem 2 of [5] yields for $[\log^2 n]$ partial sums

$$\liminf_{n\to\infty} \mathbb{P}\big(\cap_{2 \leqslant k \leqslant \log^2 n} B_{kn}\big) \geqslant \lim_{n \to \infty} \mathbb{P}\Big(\max_{k=1,\dots [\log^2 n]} k^{-\frac{1}{2}} |\eta_k - k| < t_{\log[\log^2 n]}(1/o(1))\Big) = 1. \qquad \square$$

Now (3) follows from (1) and the last two lemmas. Similarly, (4) follows.

Acknowledgment. The author is indebted to Professor Pál Révész for several helpful conversations, to the referee for some valuable suggestions and to Professor Friedhelm Eicker for his guidance.

REFERENCES

[1] ANDERSON, T. W. and DARLING, D. A. (1952). Asymptotic theory of certain "goodness of fit" criteria based on stochastic processes. *Ann. Math. Statist.* **23** 193–212.

[2] BREIMAN, L. (1968). *Probability*. Addison-Wesley, Reading.

[3] ČIBISOV, D. M. (1966). Some theorems on the limiting behavior of the empirical distribution function. *Selected Translations Math. Statist. Prob.* **6** 147–156.

[4] CSÁKI, E. (1977). The law of the iterated logarithm for normalized empirical distribution function. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **38** 147–167.

[5] DARLING, D. A. and ERDÖS, P. (1956). A limit theorem for the maximum of normalized sums of independent random variables. *Duke Math. J.* **23** 143–155.

[6] EICKER, F. (1978). The asymptotic distribution of suprema of standardized empirical processes. *Ann. Statist.* **7** 116–138.

[7] JAESCHKE, D. (1975). Über die Grenzverteilung des Maximums der normierten empirischen Verteilungsfunktion. Dissertation, Dortmund Univ.

[8] KIEFER, J. (1970). Iterated logarithm analogues for sample quantiles when $p_n \downarrow 0$. *Proc. Sixth Berkeley Symp. Math. Statist. Probability* **I** 227–244.

[9] KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1975). An approximation of partial sums of independent r.v.'-s, and the sample df.I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **32** 111–131.

[10] NOÉ, M. (1972). The calculation of distributions of two-sided Kolmogorov-Smirnov type statistics. *Ann. Math. Statist.* **43** 58–64.

[11] O'REILLY, N. E. (1974). On the weak convergence of empirical processes in sup-norm metrics. *Ann. Probability* **2** 642–651.

[12] STECK, G. P. (1971). Rectangle probabilities for uniform order statistics and the probability that the empirical distribution function lies between two distribution functions. *Ann. Math. Statist.* **42** 1–11.

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