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A NONLINEAR RENEWAL THEORY WITH APPLICATIONS TO SEQUENTIAL ANALYSIS II

BY T. L. LAI[1] AND D. SIEGMUND[2]
Columbia University and Stanford University

This paper continues earlier work of the authors. An analogue of Blackwell's renewal theorem is obtained for processes $Z_n = S_n + \xi_n$, where S_n is the nth partial sum of a sequence $X_1, X_2, \cdots$ of independent identically distributed random variables with finite positive mean and ξ_n is independent of $X_{n+1}, X_{n+2}, \cdots$ and has sample paths which are slowly changing in a sense made precise below. As a consequence, asymptotic expansions up to terms tending to 0 are obtained for the expected value of certain first passage times. Applications to sequential analysis are given.

1. Introduction. Let $X_1, X_2, \cdots$ be independent identically distributed random variables with positive mean μ and finite variance σ^2. Let $S_n = X_1 + \cdots + X_n$ and $Z_n = S_n + \xi_n$, where for each n, ξ_n is independent of $X_{n+1}, X_{n+2}, \cdots$. This paper continues the program begun by Lai and Siegmund (1977) of developing a renewal theory for Z_n under conditions which guarantee that the sample paths of the ξ_n process are slowly changing in a suitable sense made precise below. In order to facilitate comparison of these conditions for different theorems and to provide a convenient reference, the main result of Lai and Siegmund (1977) is stated as Theorem 1. The interested reader may find the informal discussion contained in that paper helpful in motivating the decomposition of Z_n and the conditions imposed on ξ_n.

For $b \geqslant 0$ define

(1) $$T = T(b) = \inf\{n : Z_n > b\}$$

and

(2) $$\tau = \tau(b) = \inf\{n : S_n > b\}, \qquad \tau_+ = \tau(0).$$

THEOREM 1 (Lai and Siegmund (1977)). *Let $\frac{1}{2} < \alpha \leqslant 1$ and assume that*

(3) $$b^{-\alpha}(T - b\mu^{-1}) \to 0$$

in probability. Suppose that for each $\eta > 0$ there exist n' and $\rho > 0$ such that for all $n \geqslant n'$

(4) $$P\{\max_{n \leqslant j \leqslant n + \rho n^\alpha} |\xi_j - \xi_n| \geqslant \eta\} < \eta.$$

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If X_1 is nonlattice, then

$$\lim_{b\to\infty} P\{Z_T - b \leqslant x\} = (ES_{\tau_+})^{-1} \int_{(0,x]} P\{S_{\tau_+} > y\}dy. \tag{5}$$

The first results of this paper are an analogue of Blackwell's renewal theorem and a corollary.

THEOREM 2. *Suppose there exists $\frac{1}{2} < \alpha \leqslant 1$ such that the following three conditions hold:*

$$E|X_1|^{2/\alpha} < \infty, \tag{6}$$

$$\text{for each } \varepsilon > 0 \quad \Sigma_1^\infty P\{|\xi_n| > n^\alpha \varepsilon\} < \infty, \tag{7}$$

and for each $\eta > 0$ there exist n' and $\rho > 0$ such that

$$\Sigma_{n \leqslant j \leqslant n+\rho n^\alpha} P\{|\xi_j - \xi_n| \geqslant \eta\} < \eta \qquad n \geqslant n'. \tag{8}$$

If X_1 is nonlattice, then

$$\Sigma_1^\infty P\{b < Z_n \leqslant b + h\} \to h/\mu \qquad b \to \infty. \tag{9}$$

COROLLARY. *Suppose there exists $\frac{1}{2} < \alpha \leqslant 1$ such that* (6), (7) *and* (8) *hold. Let $p > 0$ and assume $E(X_1^+)^{p+1} < \infty$. If $\{A_b, b \geqslant b_0\}$ is a family of events and $\{((\xi_T - \xi_{T-1})^+)^p I_{A_b}, b \geqslant b_0\}$ is uniformly integrable, then so is $\{(Z_T - b)^p I_{A_b}, b \geqslant b_0\}$. In particular, if $E(\sup_n(\xi_n - \xi_{n-1})^+)^p < \infty$, then $\{(Z_T - b)^p, b \geqslant b_0\}$ is uniformly integrable.*

Theorem 2 and its corollary are proved in Section 2. Theorem 4 of Section 2 is a somewhat different renewal theorem required by some applications (cf. Section 3). Theorems 1 and 2 together imply the main result of this paper, Theorem 3, which contains an asymptotic expansion for ET(b) up to terms which vanish as $b \to \infty$.

In order to cover diverse applications the statement of Theorem 3 is quite complicated. In most cases only one of the assumed conditions requires careful checking. To understand the role of these conditions and the problems associated with verifying them it may be helpful to consider a special case. (A more general treatment of this example appears in Section 4.) Let $x_1, x_2, \cdots$ be independent and identically distributed with expectation $\tilde{\mu} \neq 0$ and variance $\tilde{\sigma}^2$, $s_n = x_1 + \cdots + x_n$, and $T = \inf\{n: |s_n| > [n(\log n + 2b)]^{\frac{1}{2}}\}$. This stopping rule may be rewritten in the form (1) with $S_n = \tilde{\mu}(s_n - \frac{1}{2}n\tilde{\mu})$ and $\xi_n = -\frac{1}{2}\log n + (s_n - n\tilde{\mu})^2/2n$. The random variables ξ_n may be further decomposed into a deterministic part $f(n) = -\frac{1}{2}\log n$ and a random part $V_n = (s_n - n\tilde{\mu})^2/2n$ (cf. equation (12)). Conditions (13), (14) and (17) of Theorem 3 require that $f(n)$ and V_n individually fulfill the conditions (7) and (8) imposed on ξ_n in Theorem 2. In our special case (13) and (15) are obviously satisfied, and (14), (16) and (17) may be replaced by a single moment condition on x_1 (see Proposition 1 below). In more general cases the decomposition of ξ_n given by (12) may not hold on the entire sample space, but is required to hold on events A_n which by (11) fill out sample space very rapidly. The remaining condition (10) is frequently more difficult to check and often requires a special argument.

Let $\mathfrak{F}_n = \mathfrak{B}((X_1, \xi_1), \cdots, (X_n, \xi_n))$, $n = 1, 2, \cdots$.

THEOREM 3. *Assume that for some* $\delta > 0$

$$P\{T \leqslant \delta b\} = o(b^{-1}) \qquad b \to \infty. \tag{10}$$

Assume that there exists a sequence of events $A_n \in \mathfrak{F}_n$ *such that*

$$\Sigma_n P\Big(\bigcup_{k=n}^{\infty} \tilde{A}_k\Big) < \infty \qquad (\tilde{A} = \text{complement of } A), \tag{11}$$

and on A_n

$$\xi_n = f(n) + V_n, \tag{12}$$

where for some $\frac{1}{2} < \alpha \leqslant 1$ *the following conditions hold*:

(13) $f:[0, \infty) \to \mathbb{R}$ *satisfies* $|x^{-\alpha}f(x)| + \sup_{x \leqslant y \leqslant x + x^{\alpha}}|f(y) - f(x)| \to 0$

as $x \to \infty$; V_n *is* $\mathfrak{F}_n$ – *measurable and satisfies*

$$\Sigma_n P\{\sup_{k \geqslant n} k^{-\alpha}|V_k| > \varepsilon\} < \infty \qquad \varepsilon > 0; \tag{14}$$

(15) V_n *converges in distribution to a random variable* V;

(16) *the sequence* $V_n^* = \max_{n \leqslant j \leqslant n + n^{\alpha}}|V_j|$ *is uniformly integrable*;

and for each $\eta > 0$ *there exist* n' *and* $\rho > 0$ *such that*

$$\Sigma_{n \leqslant j \leqslant n + \rho n^{\alpha}} P\{|V_j - V_n| \geqslant \eta\} < \eta \qquad n \geqslant n'. \tag{17}$$

Suppose X_1 *is nonlattice and* (6) *holds. Then*

$$\mu ET = b - f(\mu^{-1}b) - EV + ES_{\tau_+}^2/2ES_{\tau_+} + o(1) \tag{18}$$

as $b \to \infty$.

In many cases the behavior of V_n is governed by a term like the $(s_n - n\tilde{\mu})^2/2n$ of the example given above. To the extent that this is so, the conditions (14)–(17) on V_n may be replaced by a single moment condition on x_1. Proposition 1 is designed to facilitate checking these conditions in such cases.

PROPOSITION 1. *Let* $Y_1, Y_2, \cdots$ *be independent and identically distributed with mean* 0 *and finite variance* $\tilde{\sigma}^2$. *Let* u_n *and* w_n *be random variables such that for some positive constants* c, $c_n \to 0$, *and* β

$$|u_n - c| < n^{-\beta} \qquad \text{and} \qquad |w_n| \leqslant c_n. \tag{19}$$

Assume that $E|Y_1|^p < \infty$ *for some* $p > 2$. *Let* $V_n = u_n(\Sigma_1^n Y_k)^2/n + w_n$. *Then* $V_n \to_{\mathfrak{L}} V$, *where* V *has the distribution of* $c\tilde{\sigma}^2\chi_1^2$. *Also* V_n *satisfies* (14) *for any* $\alpha \geqslant 4/p - 1$ *and* (16) *for any* $0 < \alpha \leqslant 1$. *In addition, for*

$$0 < \alpha \leqslant \min(p\beta/2, p/(p+2)),$$

given any $\eta > 0$ *there exist* n' *and* $\rho > 0$ *such that*

$$\Sigma_{n \leqslant j \leqslant n + \rho n^{\alpha}} P\{\max_{n \leqslant i \leqslant j}|V_i - V_n| > \eta\} < \eta \qquad n \geqslant n'. \tag{20}$$

In particular (17) *holds.*

The proofs of Theorem 3 and Proposition 1 are given in Section 3, which also contains some information on $\mathrm{Var}(T)$ as $b \to \infty$. Some applications are discussed in Section 4, and Section 5 contains a comparison of the results of this paper with those of Woodroofe (1976a, 1977).

2. Nonlinear Blackwell's theorems.

PROOF OF THEOREM 2. The notation below is chosen to facilitate comparisons with the proof of Theorem 1 of Lai and Siegmund (1977), which contains similar basic ideas although their technical implementation is different.

Let α, η, and ρ be as in the statement of the theorem. Set

(21) $$n_0 = \mu^{-1}(b + h), \qquad n_1 = [n_0 - \rho n_0^{\alpha}/4], \qquad n_2 = [n_0 + \rho n_0^{\alpha}/4].$$

By Lemma 1 below for m sufficiently large and fixed, for all sufficiently large b

(22) $$\Sigma_{m \leqslant n \leqslant n_1} P\{b \leqslant Z_n \leqslant b + h\} < \eta,$$

and also

(23) $$\Sigma_{n \geqslant n_2} P\{b \leqslant Z_n \leqslant b + h\} \to 0 \quad \text{as} \quad b \to \infty.$$

Obviously,

(24) $$\Sigma_{n < m} P\{b \leqslant Z_n \leqslant b + h\} \to 0 \quad \text{as} \quad b \to \infty.$$

It remains to estimate the series of terms $P\{b \leqslant Z_n \leqslant b + h\}$ for $n_1 < n < n_2$. For each $n_1 < n < n_2$

(25) $$P\{b \leqslant Z_n \leqslant b + h\} \leqslant P\{|\xi_n - \xi_{n_1}| \geqslant \eta\} + P\{b - \eta \leqslant Z_{n_1} + (S_n - S_{n_1}) \leqslant b + h + \eta\}.$$

By (8) and (21) for all large b

(26) $$\Sigma_{n_1 < n < n_2} P\{|\xi_n - \xi_{n_1}| > \eta\} < \eta.$$

Furthermore,

(27) $$\Sigma_{n_1 < n < n_2} P\{b - \eta \leqslant Z_{n_1} + (S_n - S_{n_1}) \leqslant b + h + \eta\} = Eg(b - Z_{n_1}),$$

where

(28) $$g(t) = \Sigma_{j < n_2 - n_1} P\{t - \eta \leqslant S_j \leqslant (t - \eta) + h + 2\eta\}.$$

It will be shown in Lemma 2 below that as a consequence of Blackwell's theorem

(29) $$Eg(b - Z_{n_1}) \to (h + 2\eta)/\mu.$$

Then by (22), (23), (24), (25), (26) and (29)

$$\limsup_{b \to \infty} \Sigma_1^{\infty} P\{b \leqslant Z_n < b + h\} \leqslant 2\eta + (h + 2\eta)/\mu.$$

Letting $\eta \to 0$ gives one inequality. The inequality in the other direction follows by a similar but easier argument, which completes the proof.

LEMMA 1. *Under conditions* (6) *and* (7) *for m sufficiently large and fixed, for all large b* (22) *holds; also* (23) *holds.*

PROOF. Let $0 < \varepsilon < \rho\mu/9$. Note that for all large b and $n \geqslant n_2$, if $S_n - n\mu \geqslant -\varepsilon n^\alpha$ and $\xi_n \geqslant -\varepsilon n^\alpha$, then by (21)

$$S_n + \xi_n \geqslant n\mu - 2\varepsilon n^\alpha \geqslant n_2\mu - 2\varepsilon n_2^\alpha > n_0\mu = b + h.$$

From (6) it follows that $\Sigma P\{|S_n - n\mu| > n^\alpha \varepsilon\} < \infty$ (cf. Baum and Katz (1965), Theorem 3) and hence by (7) as $b \to \infty$

$$\Sigma_{n \geqslant n_2} P\{b \leqslant Z_n \leqslant b + h\} \leqslant \Sigma_{n \geqslant n_2}(P\{|S_n - n\mu| > \varepsilon n^\alpha\} + P\{|\xi_n| > \varepsilon n^\alpha\}) \to 0.$$

This proves (23), and (22) follows by a similar argument if m is chosen so large that

$$\Sigma_{n \geqslant m}(P\{|S_n - n\mu| > \varepsilon n^\alpha\} + P\{|\xi_n| > \varepsilon n^\alpha\}) < \eta.$$

LEMMA 2. *Under conditions* (7) *and* (8), *for g defined by* (28), *the limit* (29) *holds.*

PROOF. It suffices to show

$$g(b - Z_{n_1}) \to (h + 2\eta)/\mu \quad \text{a.s.} \qquad b \to \infty \tag{30}$$

and that g is bounded, for then (29) follows by dominated convergence. Let $\nu(b) = n_2 - n_1$. By (7) and the strong law of large numbers $Z_{n_1} = \mu n_1 + o(n_1^\alpha) = b - \rho\mu^{1-\alpha}b^\alpha/4 + o(b^\alpha)$. Hence by (28), to prove (30) it suffices to show that for arbitrary real numbers $z(b) = \rho\mu^{1-\alpha}b^\alpha/4 + o(b^\alpha)$

$$\Sigma_{j < \nu(b)} P\{z(b) \leqslant S_j \leqslant z(b) + h + 2\eta\} \to (h + 2\eta)/\mu. \tag{31}$$

But if $j \geqslant \nu(b)$ and $S_j \geqslant 2j\mu/3$, then by (21) $S_j \geqslant \rho\mu^{1-\alpha}b^\alpha/3 + o(1) > z(b) + h + 2\eta$ for all large b, and it follows that

$$\Sigma_{j \geqslant \nu(b)} P\{S_j \leqslant z(b) + h + 2\eta\} \leqslant \Sigma_{j \geqslant \nu(b)} P\{S_j < 2j\mu/3\} \to 0.$$

Thus (31), and with it (30), follow from Blackwell's theorem. That g is bounded is a consequence of

$$g(t) \leqslant 1 + \Sigma_{n=0}^\infty P\{-h - 2\eta \leqslant S_n \leqslant h + 2\eta\} < \infty. \tag{32}$$

The series in (32) converges because the random walk $\{S_n\}$ is transient (cf. Feller (1966), pages 199 ff.).

PROOF OF COROLLARY TO THEOREM 2. Assume $E(\sup_{n \geqslant 1}(\xi_n - \xi_{n-1})^+)^p < \infty$. For $x > 0$ and all large b

$$P\{Z_T - b \geqslant 2x\} \leqslant \Sigma_{n=0}^\infty P\{Z_n \leqslant b, Z_n + X_{n+1} \geqslant b + x\} + P\{\sup_n(\xi_n - \xi_{n-1})^+ \geqslant x\}. \tag{33}$$

Also

(34)

$$\begin{aligned}\Sigma_{n=0}^\infty & P\{Z_n \leqslant b, Z_n + X_{n+1} \geqslant b + x\} \\ &= \textstyle\int[(-\infty, b)] P\{X_1 \geqslant b + x - y\}\Sigma_{n=0}^\infty P\{Z_n \varepsilon dy\} \\ &\leqslant \Sigma_{k=-\infty}^{[b]+1} P\{X_1 \geqslant b + x - k\}\Sigma_{n=0}^\infty P\{k - 1 < Z_n \leqslant k\} \\ &\leqslant \text{const.}\left(\textstyle\int_x^\infty P\{X_1 \geqslant y\}\, dy + P\{X_1 \geqslant x - 1\}\right).\end{aligned}$$

To see the last inequality in (34), note that by Theorem 2 there exists a k_0 such that for all $k \geqslant k_0$, $\Sigma_{n=0}^{\infty} P\{k < Z_n \leqslant k + 1\} \leqslant 2/\mu$, while (6) and (7) imply $\Sigma P\{Z_n \leqslant k_0 + 1\} < \infty$ (cf. Baum and Katz (1965), Theorem 3). The uniform integrability of $(Z_T - b)^p$ follows from (33) and (34). If only $\{((\xi_T - \xi_{T-1})^+)^p I_{A_b}\}$ is assumed to be uniformly integrable, (33) may be replaced by

$$P(\{Z_T - b \geqslant 2x\} \cap A_b) \leqslant \Sigma_0^{\infty} P\{Z_n < b, Z_n + X_{n+1} \geqslant b + x\}$$
$$+ P(\{\xi_T - \xi_{T-1} \geqslant x\} \cap A_b),$$

and the rest of the proof follows as above.

The following theorem is equivalent to one of the main results of Woodroofe (1976a) in a number of special cases, although its abstract formulation and proof are different. (See Section 5 for a more systematic comparison of the results of this paper with those of Woodroofe (1976a).)

THEOREM 4. *Suppose there exists* $\frac{1}{2} < \alpha \leqslant 1$ *such that conditions* (6) *and* (7) *hold, and for each* $\eta > 0$ *there exist* n' *and* $\rho > 0$ *such that*

$$\Sigma_{n \leqslant j \leqslant n+\rho n^{\alpha}} P\{\max_{n \leqslant i \leqslant j} |\xi_i - \xi_n| \geqslant \eta\} < \eta \qquad n \geqslant n'. \tag{35}$$

If X_1 *is nonlattice, then for all* $y > 0$

$$\Sigma_{n=0}^{\infty} P\{T > n, Z_n > b - y\} \rightarrow \mu^{-1} \textstyle\int_{-y}^{0} P\{S_n > t \text{ for all } n \geqslant 0\} dt. \tag{36}$$

PROOF. Like Theorem 2 the proof of Theorem 4 consists of reducing the general case to the case $\xi_n \equiv 0$. This reduction is similar to the proof of Theorem 2 and is omitted. However, unlike Theorem 2, which reduces to Blackwell's theorem in the case $\xi_n \equiv 0$, the corresponding version of Theorem 3 does not seem to have appeared in the literature (although under stronger assumptions it is implied by Theorem 3.1 of Woodroofe (1976a)).

Assume then that $\xi_n \equiv 0$ so that $Z_n = S_n$ and $T = \tau$. Let $M_n = \max(0, S_1, \cdots, S_n)$ and $M^* = \min(0, S_1, S_2, \cdots)$. Then (36) becomes

$$\Sigma_{n=0}^{\infty} P\{M_n \leqslant b, S_n > b - y\} \rightarrow \mu^{-1} \textstyle\int_{(-y, 0]} P\{_m{}^* > t\} dt. \tag{37}$$

Let $\sigma(n)$ denote the nth (strict) ascending ladder time, i.e., $\sigma(0) = 0$ and for $n \geqslant 1$, $\sigma(n) = \inf\{n : n > \sigma(n-1), S_n > S_{\sigma(n-1)}\}$ ($\sigma(1) = \tau_+$). Let $\tau_- = \inf\{n : n \geqslant 1, S_n \leqslant 0\}$. By considering the (uniquely defined) smallest index $k \leqslant n$ for which $S_k = M_n$ one obtains for $0 \leqslant y < b$

$$P\{M_n \leqslant b, S_n > b - y\} = \Sigma_{k=0}^{n} \textstyle\int_{(b-y, b]} P\{S_i < S_k \forall i < k, S_k \in dx,$$
$$S_j \leqslant S_k \forall k \leqslant j \leqslant n, S_n - S_k > b - y - x\}.$$

Summing these terms for $n = 0, \cdots$, and reversing the order of summation yields

$$(38)\quad \Sigma_{n=0}^{\infty} P\{M_n \leqslant b, S_n > b - y\}$$
$$= \Sigma_{k=0}^{\infty}\Sigma_{n=k}^{\infty}\int_{(b-y,b]} P\{_{si} < S_k \forall i < k, S_k \in dx\} \times P\{\tau_+ > n - k, S_{n-k} > b - y - x\}$$
$$= \int_{[0,y)} \Sigma_{n=0}^{\infty} P\{\tau_+ > n, S_n > x - y\} \Sigma_{k=0}^{\infty} P\{S_i < S_k \forall i < k, S_k \in b - dx\}.$$

Now

$$(39)\quad \Sigma_{k=0}^{\infty} P\{S_i < S_k \forall_i < k, S_k \in b - dx\} = \Sigma_{k=0}^{\infty}\Sigma_{n=0}^{\infty} P\{\sigma(n) = k, S_k \in b - dx\} = \Sigma_{n=0}^{\infty} P\{S_{\sigma(n)} \in b - dx\}.$$

Since $P\{\tau_+ > n, S_n > x - y\} = P\{S_n - S_i \leqslant 0 \forall 0 \leqslant i \leqslant n, S_n > x - y\}$, it follows that

$$(40)\quad \Sigma_{n=0}^{\infty} P\{\tau_+ > n, S_n > x - y\} = \Sigma_{n=0}^{\infty} P\{S_i \geqslant S_n \forall 0 \leqslant i \leqslant n, S_n > x - y, S_j > S_n \forall j > n\}/P\{\tau_- = \infty\}$$
$$= E\tau_+ P\{M^* > x - y\} \qquad 0 \leqslant x < y.$$

The last equality uses the well-known fact that $E\tau_+ = 1/P\{\tau_- = \infty\}$ (cf. Feller (1966), page 379). Since $P\{M^* > t\}$ is decreasing in t, applying Blackwell's renewal theorem to the right-hand side of (39) and taking into account (38), (39) and (40) yields

$$\Sigma_{n=0}^{\infty} P\{M_n \leqslant b, S_n > b - y\} \to E\tau_+ \int_{[0,y)} P\{M^* > x - y\} dx / ES_{\tau_+}$$
$$= \mu^{-1} \int_{(-y,0]} P\{_m{}^* > t\} dt.$$

REMARK. The condition that $\sigma^2 = \text{Var} X_1 < \infty$ was not used in the preceding proof for the special case $\xi_n \equiv 0$.

3. Expansions of *ET(b)* and Var *T(b)*. Intuitively the random variables Z_n and $T(b)$ are the same variables in Theorem 3 as in Theorems 1 and 2. For technical reasons, in the proof that follows new random variables Z_n' and $T'(b)$ will be defined in terms of the original Z_n and $T(b)$, and Theorems 1 and 2 will be applied to these new variables.

PROOF OF THEOREM 3. Let $\{\varepsilon_n\}$ be a sequence of positive numbers tending to 0 to be further specified below. Since $f(x) = o(x^\alpha)$, for appropriately chosen $\{\varepsilon_n\}$ there exists an integer n_0 such that $|f(x)| < \varepsilon_n n^\alpha$ for all $x \geqslant n \geqslant n_0$. Let $L_1 = \sup\{n : \tilde{A}_n \text{ occurs}\}$, $L_2 = \sup\{n: |V_n| \geqslant \varepsilon_n n^\alpha\}$, $L_3 = \sup\{n: |S_n - n\mu| \geqslant \varepsilon_n n^\alpha\}$, and $L = 1 + \max(n_0, L_1, L_2, L_3)$. By (11) $EL_1 < \infty$. By (6) (cf. Baum and Katz (1965), Theorem 3) and (14) for appropriately chosen $\{\varepsilon_n\}$ $EL_3 < \infty$ and $EL_2 < \infty$. Hence, $EL < \infty$.

Let $\xi_n' = 0$ for $n \leqslant n_0$ and for $n > n_0$ set $\xi_n' = \xi_n I_{A_n \{|V_n| < \varepsilon_n n^\alpha\}}$. Let $Z_n' = S_n + \xi_n'$, $T'(b) = \inf\{n : Z_n' > b\}$, and $B = B_b = \{L < \min(T, T')\}$. Note that $|\xi_n| \leqslant 2\varepsilon_n n^\alpha$ for all n. For all $\varepsilon > 0$ and for sufficiently large b on B

$$(41) \quad T_b = T_b', \qquad Z_{T_b} = Z'_{T'_b}, \qquad \text{and} \quad b\mu^{-1} - \varepsilon b^\alpha < T' < b\mu^{-1} + \varepsilon b^\alpha.$$

The two equalities in (41) are obvious. The inequality in (41) follows from

$$b < Z'_{T'} \leqslant \mu T' + 3\varepsilon'_T(T')^\alpha$$

and the corresponding inequality for $T' - 1$, which hold on B. Hence by Wald's lemma

$$(42) \quad \mu ET - \int_{\tilde{B}} S_T dP = \int_B S_{T'} dP = bP(B) + \int_B (Z'_{T'} - b) dP - \int_B \xi'_{T'} dP.$$

It will be shown in Lemma 3 below that

$$(43) \qquad bP(\tilde{B}) \to 0$$

and in Lemma 4 that

$$(44) \qquad \int_{\tilde{B}} S_T dP \to 0$$

as $b \to \infty$. It is easy to see that condition (3) is satisfied for the random variables T' and hence with the help of (13), (15), (16), (41) and (43)

$$(45) \qquad \int_B \xi'_{T'} dP = f(b/\mu) + EV + o(1).$$

Finally, it will be shown in Lemma 5 as an application of Theorems 1 and 2 that

$$(46) \qquad \int_B (Z'_{T'} - b) dP \to ES^2_{\tau_+}/2ES_{\tau_+}.$$

Relations (42)–(46) yield the theorem.

Lemma 3. *Under the conditions* (6), (11) *and* (14) *the relation* (43) *holds.*

Proof. The conditions of the lemma imply that $EL < \infty$, where L is defined as in the proof of Theorem 3. To prove (43) it suffices to show that

$$(47) \qquad bP\{T \leqslant L\} \to 0$$

and

$$(48) \qquad bP\{T' \leqslant L\} \to 0.$$

Now for arbitrary $\delta > 0$

$$P\{T \leqslant L\} \leqslant P\{L \geqslant \delta b\} + P\{T \leqslant \delta b\}$$
$$\leqslant (\delta b)^{-1} \int_{\{L \geqslant \delta b\}} L\, dP + P\{T \leqslant \delta b\}.$$

Hence (47) follows from (10) and the finiteness of EL. Since by definition $|\xi_n'| < 2\varepsilon_n n^\alpha$, it may be shown that $bP\{T' \leqslant \delta b\} \to 0$, and then (48) follows from a similar argument.

Lemma 4. *Under the conditions* (6), (11) *and* (14) *the relation* (44) *holds.*

PROOF. By the Schwarz inequality and Wald's lemma for squared sums

$$\begin{aligned}|\int_{\tilde{B}} S_T dP| &= |\int_{\tilde{B}}(S_T - \mu T)dP + \mu\int_{\tilde{B}} T\, dP|\\ &\leqslant \{E(S'_T - \mu T)^2 P(\tilde{B})\}^{\frac{1}{2}} + \mu\int_{\tilde{B}} T\, dP\\ &\leqslant \{\sigma^2 ETP(\tilde{B})\}^{\frac{1}{2}} + \mu\int_{\{T \leqslant L\}} L\, dP + \mu\int_{\{T' \leqslant L < T\}} T\, dP.\end{aligned}$$

It is easy to see that $ET = 0(b)$ and hence by Lemma 3

$$E(T)P(\tilde{B}) \to 0 \qquad b \to \infty.$$

The conditions of the lemma imply $EL < \infty$ and hence

$$\int_{\{T \leqslant L\}} LdP \to 0.$$

To complete the proof it remains to show

$$\int_{\{T' \leqslant L < T\}} T\, dP \to 0.$$

If $T > L$, then $|S_{T-1} - \mu(T-1)| < \varepsilon_{T-1} T^\alpha$ and $|\xi_{T-1}| < 2\varepsilon_{T-1} T^\alpha$, so $b > S_{T-1} + \xi_{T-1} \geqslant \mu(T-1) - 3\varepsilon_{n_0} T^\alpha$. Thus for all large b $\{T > L\} \subset \{T < 2b/\mu\}$ and it follows that

$$\int_{\{T' \leqslant L < T\}} T\, dP \leqslant 2b/\mu P\{T' \leqslant L\} \to 0$$

by Lemma 3. This completes the proof.

LEMMA 5. *Under the conditions of Theorem* 3 *the relation* (46) *holds.*

PROOF. Lemma 3 and (41) show that condition (3) holds for the stopping times T'. Also the conditions of the corollary to Theorem 2 are satisfied with the events A of the corollary being the events B defined in the proof of Theorem 4 (recall especially (13), (16) and (41)). It follows from Theorem 1 that $Z'_{T'} - b$ converges in distribution and from the corollary to Theorem 2 that the $(Z'_{T'} - b)I_{B_b}$ are uniformly integrable. The lemma now follows by simple computation.

PROOF OF PROPOSITION 1. The convergence in law of V_n is immediate from the central limit theorem. That (14) is satisfied provided $p \geqslant 4/(1+\alpha)$ follows from Theorem 3 of Baum and Katz (1965).

Let $U_n = \Sigma_1^n Y_k$. The calculations given below prove (16) and (20). Several applications are made of Kolmogorov's inequality for submartingales (cf. Chow, Robbins and Siegmund (1971), page 24) and the inequality

$$E|U_n|p \leqslant Cn^{p/2} \tag{50}$$

(cf. Doob (1954), page 225). Here and in what follows C denotes constants which may differ from one appearance to the next. The proof of (16) is an immediate consequence of the inequalities

$$\begin{aligned}P\{\max_{n \leqslant j \leqslant n+n^\alpha}|U_j^2/j| > x\} &\leqslant P\{\max_{n \leqslant j \leqslant n+n^\alpha} U_j^2 > nx\}\\ &\leqslant (nx)^{-p/2} E|U_{[n+n^\alpha]}|^p \leqslant Cx^{-p/2}.\end{aligned}$$

To prove (20) note that for $i > n$

$$u_i U_i^2/i - u_n U_n^2/n = n^{-1}U_n^2(u_i - u_n)$$
$$+ u_i\{i^{-1}(U_i - U_n)^2 + 2i^{-1}U_n(U_i - U_n) - (ni)^{-1}(i-n)U_n^2\}.$$

Hence (20) is a consequence of the following inequalities.

$$\Sigma_{j=n}^{n+\rho n^\alpha} P\{\max_{n \leqslant i \leqslant j} i^{-1}(U_i - U_n)^2 > \eta\} \leqslant \Sigma_{j=n}^{n+\rho n^\alpha} P\{\max_{n \leqslant i \leqslant j}|U_i - U_n|^2 > \eta n\}$$
$$\leqslant (\eta n)^{-p/2}\Sigma_{j=n}^{n+\rho n^\alpha} E|U_j - U_n|^p$$
$$\leqslant C(\eta n)^{-p/2}\Sigma_{j=n}^{n+\rho n^\alpha}(j-n)^{p/2}$$
$$\leqslant C(\eta n)^{-p/2}(\rho n^\alpha)^{(p/2)+1}.$$

$$\Sigma_{j=n}^{n+\rho n^\alpha} P\{\max_{n \leqslant i \leqslant j} i^{-1}|U_n(U_i - U_n)| > \eta\}$$
$$\leqslant \Sigma_{j=n}^{n+\rho n^\alpha} E[P\{\max_{n \leqslant i \leqslant j}|U_i - U_n| > n\eta/|U_n| | U_n\}]$$
$$\leqslant (\eta n)^{-p}\Sigma_{j=n}^{n+\rho n^\alpha} E[|U_n|^p|U_j - U_n|^p] \leqslant C\eta^{-p}n^{-p/2}(\rho n^\alpha)^{(p/2)+1}.$$

$$\Sigma_{j=n}^{n+\rho n^\alpha} P\{\max_{n \leqslant i \leqslant j}(1 - n/j)U_n^2/n > \eta\} \leqslant \rho n^\alpha P\{n^{-1}U_n^2 \geqslant \eta n^{1-\alpha}/\rho\}$$
$$\leqslant C\rho(\rho/\eta)^{p/2}n^{-(p/2)+\alpha(1+p/2)}.$$

$$\Sigma_{j=n}^{n+\rho n^\alpha} P\{n^{-1}U_n^2 \max_{n \leqslant i \leqslant j}|u_i - u_n| > \eta\} \leqslant \rho n^\alpha P\{n^{-1}U_n^2(2n^{-\beta}) > \eta\}$$
$$\leqslant C\rho\eta^{-p/2}n^{\alpha - \beta p/2}.$$

The preceding results show that to a first order approximation the behavior of $\{Z_n\}$ and T is asymptotically the same as that of $\{S_n\}$ and τ, although differences appear with higher order asymptotic expansions. According to Chow, Robbins and Siegmund (1971, page 31) $\mathrm{Var}(\tau) \sim \sigma^2 b/\mu^3$ as $b \to \infty$; it should come as no surprise that under conditions similar to those of Theorem 3, one may show that $\mathrm{Var}\, T \sim \sigma^2 b/\mu^3$ also. Indeed, such a result has been proved by a different method and applied by Woodroofe (1976a, 1977) in several special cases. The details of such an analysis seem sufficiently similar to the proof of Theorem 3 that they have been omitted.

It would be more in the spirit of the present paper to obtain an expansion for $\mathrm{Var}\, T$ up to terms which vanish as $b \to \infty$. Unfortunately, the authors have been unable to produce such a result even in the simplest special (nonlinear) cases. For the *linear* case, in which $\xi_n \equiv 0$, $Z_n \equiv S_n$, and $\tau \equiv T$, it is possible to obtain an expansion of $\mathrm{Var}\, T = \mathrm{Var}\, \tau$ up to terms which vanish as $b \to \infty$ as an application of Theorem 4. This result seems to be new except under the further assumption that $X_1 \geqslant 0$—see Smith (1959).

THEOREM 5. *Assume that* $E(x_1^+)^3 < \infty$ *and* x_1 *is strongly nonlattice in the sense of Stone* (1965). *Then as* $b \to \infty$

$$\mathrm{Var}\, \tau = \mu^{-3}\sigma^2 b + \mu^{-2}K + o(1), \tag{51}$$

where K is given by

(52) $K = \sigma^2 ES^2_{\tau_+}/2\mu ES_{\tau_+} + \frac{3}{4}\{ES^2_{\tau_+}/ES_{\tau_+}\}^2 - \frac{2}{3}ES^3_{\tau_+}/ES_{\tau_+}$
$- (ES^2_{\tau_+}/ES_{\tau_+})E\{\min_{n\geq 0}S_n\} - 2\int_0^\infty E\{S_{\tau(x)} - x\}P\{\min_{n\geq 0}S_n \leq -x\}\,dx.$

PROOF. It is well known (and is the linear case of Theorem 3) that as $b \to \infty$

(53) $$\mu E\tau = b + ES^2_{\tau_+}/2ES_{\tau_+} + o(1).$$

Similarly, for $i = 1$ or 2

(54) $$E(S_\tau - b)^i \to (ES_{\tau_+})^{-1}\int_{(0,\infty)} x^i P\{S_{\tau_+} > x\}\,dx.$$

Also $E\tau^2 < \infty$, so by Wald's lemma for second moments (cf. Chow, Robbins and Siegmund (1971), page 23) and elementary algebra

$$\mu^2\mathrm{Var}\,\tau = E(\mu\tau - S_\tau + S_\tau - \mu E\tau)^2$$
$$= E(S_\tau - \mu\tau)^2 + E(S_\tau - \mu E\tau)^2 - 2E[(S_\tau - \mu\tau)(S_\tau - \mu E\tau)]$$

(55) $$= \sigma^2 E\tau + E(S_\tau - b + b - \mu E\tau)^2$$
$$- 2E[(S_\tau - b + b - \mu\tau)(S_\tau - b + b - \mu E\tau)]$$
$$= \sigma^2 E\tau - (\mu E\tau - b)^2 - E(S_\tau - b)^2 + 2E\{(\mu\tau - b)(S_\tau - b)\}.$$

Hence by (53) and (54)

(56) $$\mu^2\mathrm{Var}\,\tau = \sigma^2 b/\mu + \sigma^2 ES^2_{\tau_+}/2\mu ES_{\tau_+} + \tfrac{1}{4}\{ES^2_{\tau_+}/ES_{\tau_+}\}^2$$
$$- ES^3_{\tau_+}/3ES_{\tau_+} + o(1) + 2\mu E\{(\tau - E\tau)(S_\tau - b)\}.$$

It remains to evaluate the last term on the right-hand side of (56). By an easy renewal argument

(57) $$E\{(\tau - E\tau)(S_\tau - b)\} = \Sigma_{n=0}^\infty \int_{[0,\infty)}\{E(S_{\tau(x)} - x)$$
$$- E(S_{\tau(b)} - b)\}P\{\tau > n, S_n \in b - dx\}.$$

It follows from standard fluctuation identities (especially Feller (1966), page 570, equation (3.6)) that S_{τ_+} has a distribution which is strongly nonlattice in the sense of Stone (1965). Also $E(X_1^+)^3 < \infty$ implies $ES^3_{\tau_+} < \infty$. Hence by Theorem 3 of Stone (1965) applied to the renewal process determined by S_{τ_+}, equation (54) for $i = 1$ may be sharpened to read

(58) $$E(S_{\tau(b)} - b) = ES^2_{\tau_+}/2ES_{\tau_+} - H(b) + 0(b^{-2}\log b),$$

where

(59) $$H(b) = \int_b^\infty \int_x^\infty P\{S_{\tau_+} > y\}\,dy$$

is integrable at $+\infty$. Hence by (58)

$$|E(S_{\tau(x)} - x) - ES^2_{\tau_+}/2ES_{\tau_+}|$$

is a directly Riemann integrable function of x. It follows from (53), (57), (58) and Theorem 4 that

(60) $$E\{(\tau - E\tau)(S_\tau - b)\} \to \mu^{-1}\int_{[0,\infty)}\{E(S_{\tau(x)} - x) - ES_{\tau_+}^2/2ES_{\tau_+}\} P\{\min_{n\geqslant 0} S_n \geqslant -x\}\,dx.$$

Since $Z(x) = E(S_{\tau(x)} - x)$ satisfies the renewal equation $Z = z + F * Z$ with $F(y) = P\{S_{\tau_+} \leqslant y\}$, it may be shown by taking Laplace transforms and making a Taylor series expansion that

(61) $$\int_{[0,\infty)}\{E(S_{\tau(x)} - x) - ES_{\tau_+}^2/2ES_{\tau_+}\}\,dx = \tfrac{1}{4}\{ES_{\tau_+}^2/ES_{\tau_+}\}^2 - \tfrac{1}{6}ES_{\tau_+}^3/ES_{\tau_+},$$

and obviously

(62) $$\int_{[0,\infty)} P\{\min_{n\geqslant 0} S_n \leqslant -x\}\,dx = -E\{\min(0, S_1, S_2, \cdots)\}.$$

The theorem follows by substituting (60)–(62) into (56).

REMARK. Even in those cases where the moments of S_{τ_+} can be computed, the authors know of no general way to compute the integral appearing in (52). However, for numerical purposes the last two terms in (52) are "almost equal" and opposite in sign and can probably be neglected. To see that they are "almost equal" observe that $E\{S_{\tau(x)} - x\} \to ES_{\tau_+}^2/2ES_{\tau_+}$ as $x \to \infty$, and if this were actually an equality rather than just a limit relation, the two terms would be equal.

4. Applications to sequential analysis. Theorem 4 may be applied to yield asymptotic expansions for the expected sample size of a variety of sequential tests, including the classical sequential χ^2, t, and F tests. Many of these applications are conceptually similar, and for brevity only two have been included here. The first example was studied by Pollak and Siegmund (1975), who ignored the problem of the excess over the boundary in their analysis but otherwise provided a concrete model from which Theorem 3 has been abstracted.

For θ in some open interval J containing 0 assume that $\exp(\theta x - \psi(\theta))$ is a probability density function with respect to a probability distribution H and that $\psi(0) = \psi'(0) = 0$ and $\psi''(\theta) > 0$ for all θ. Let F be a probability on J and define

$$f(x, t) = \int_J \exp(yx - t\psi(y))\,dF(y).$$

Assume that $x_1, x_2, \cdots$ are independent, identically distributed random variables such that for some $\theta \in J - \{0\}$

$$Ex_1 = \psi'(\theta),$$

and that F' exists in some neighborhood of θ where it is continuous and positive. Let $s_n = \sum_1^n x_k$ and $Z_n = \log f(s_n, n)$. Take $0 < \gamma_1 < \gamma_2 < \frac{1}{2}$ and $A_n = \{|s_n - n\psi'(\theta)| < n^{\frac{1}{2}+\gamma_1}\}$. A straightforward modification of the proof of Theorem 1 of Pollak and

Siegmund (1975) shows that on A_n

(63) $$Z_n = \theta s_n - n\psi(\theta) + \log\int_J \exp[(y-\theta)s_n - n(\psi(y)-\psi(\theta))]dF(y)$$
$$= S_n - \tfrac{1}{2}\log n + \tfrac{1}{2}\log[2\pi(F'(\theta))^2/\psi''(\theta)]$$
$$+ u_n(s_n - n\psi'(\theta))^2/[2\psi''(\theta)n] + w_n.$$

Here $S_n = \theta s_n - n\psi(\theta)$ and u_n and w_n are random variables for which $|u_n - 1| \leqslant n^{-\frac{1}{2}+\gamma_2}$ and $|w_n| \leqslant c_n$ with c_n nonrandom and converging to 0. Now assume that for some $p \geqslant 4$

(64) $$E|x_1|^p < \infty.$$

Choose $\gamma_1 + 2\gamma_2 \leqslant \frac{1}{2}$ and set $\alpha = \frac{1}{2} + \gamma_1$ and $\beta = \frac{1}{2} - \gamma_1$, so $\alpha \leqslant \min(2\beta, \frac{2}{3})$. Let $V_n = u_n(s_n - n\psi'(\theta))^2/[2\psi''(\theta)n] + w_n$ on A_n and 0 elsewhere. It follows from (64) and Theorem 3 of Baum and Katz (1965) that (11) holds, and by Proposition 1 that (14), (16) and (17) are satisfied. It is easy to see that (10) need not hold without further assumptions, but it does hold if either $\exp(\theta x - \psi(\theta))$ is the true density function of x_1, i.e.,

(65) $$P\{x_1 \in dx\} = \exp(\theta x - \psi(\theta))dH(x),$$

or (64) holds for some $p > 4$ and for some $\sigma_*^2 > 0$

(66) $$\psi''(\theta) \geqslant \sigma_*^2 \qquad \text{for all} \quad \theta \in J.$$

That (65) implies (10) follows from Lemma 3 of Pollak and Siegmund (1975). A simple application of the Hájek-Rényi-Chow inequality to modify the proof of their Lemma 7 shows that (64) with $p > 4$ and (66) imply (10). Hence by Theorem 3 as $b \to \infty$

(67)
$$I(\theta)E(T) = b + \frac{1}{2}\left[\log(b/I(\theta)) - \log\{2\pi[F'(\theta)]^2/\psi''(\theta)\} - \tilde{\sigma}^2/\psi''(\theta)\right]$$
$$+ ES_{\tau_+}^2/2ES_{\tau_+} + o(1),$$

where $I(\theta) = \theta\psi'(\theta) - \psi(\theta)$ and $\tilde{\sigma}^2 = \operatorname{Var} x_1$.

The approximation (67) without the term involving S_{τ_+} was given by Pollak and Siegmund (1975). Classical random walk theory leads to an evaluation of $ES_{\tau_+}^2/2ES_{\tau_+}$ in terms of

(68) $$\Sigma_1^\infty n^{-1}\int_{(S_n \leqslant 0)} S_n\,dP,$$

which in general is very difficult to compute. For the special case in which the x_i are $N(\theta, 1)$,

(69) $$ES_{\tau_+}^2/2ES_{\tau_+} = 2 + \theta^2/2 - 2\theta B(\theta/2),$$

where

$$B(\theta) = \Sigma_1^\infty\{n^{-\frac{1}{2}}\phi(\theta n^{\frac{1}{2}}) - \theta\Phi(-\theta n^{\frac{1}{2}})\}.$$

A brief table of values for B is given by Siegmund (1975). A simple useful approximation is given by

$$ES_{\tau_+}^2/2ES_{\tau_+} = \theta(.583 + \theta/8) + o(\theta^2) \qquad \theta \to 0.$$

For the numerical values considered in Tables 1 and 2 of Pollak and Siegmund (1975), use of the term $ES_{\tau_+}^2/2ES_{\tau_+}$ reduces an error of about 10% by a factor of roughly one-half. Except for special cases, computation of this term is quite difficult and perhaps not worth the necessary effort. This is in marked contrast to the approximation of error probabilities, where analyzing the excess over the boundary can lead to dramatic improvement in the accuracy of the approximation (cf. Siegmund (1975), or part I of this paper).

The second example involves a stopping rule suggested by Siegmund (1977) for testing whether a normal mean is 0 when the variance is unknown. Let $x_1, x_2, \cdots$ be independent and normally distributed with mean $\tilde{\mu}$ and variance $\tilde{\sigma}^2$. Put $s_n = x_1 + \cdots + x_n$, $\bar{x}_n = n^{-1}s_n$, $s_n^* = (s_n - n\tilde{\mu})/\tilde{\sigma}$, $v_n^2 = n^{-1}\Sigma_1^n(x_k - \bar{x}_n)^2$, $t_n^* = \Sigma_1^n(x_k - \tilde{\mu})^2/\tilde{\sigma}^2 - n$, and $\theta = \tilde{\mu}/\tilde{\sigma}$. Let $Z_n = n/2 \log\{1 + \bar{x}_n^2/v_n^2\}$, and for $m \geq 3$ define $T = T(b) = \inf\{n : n \geq m, Z_n > b\}$. Obviously $Z_n = n/2g(\bar{x}_n, n^{-1}\Sigma_1^n x_k^2)$, where $g(x, y) = -\log(1 - x^2/y)$. Expanding g in a Taylor series about $(\tilde{\mu}, \tilde{\sigma}^2 + \tilde{\mu}^2)$ and collecting terms yields

$$\begin{aligned} Z_n &= \frac{n}{2}\log(1 + \theta^2) + \theta(1 + \theta^2)^{-1}s_n^* - \frac{1}{2}\theta^2(1 + \theta^2)^{-1}t_n^* \\ &\quad + (1 + \theta^2)^{-2}(1 + \theta^4)s_n^{*2}/n + \frac{1}{2}(1 + \theta^2)^{-2}\theta^2(\theta^2 + 2)t_n^{*2}/n \\ &\quad -2(1 + \theta^2)^{-2}\theta s_n^* t_n^*/n + w_n, \end{aligned}$$

where $|w_n| \leq W(n^{-2}|s_n^*|^3 + n^{-2}|t_n^*|^3)$ for some function W which equals 0 at 0 and is continuous there.

Let $A_n = \{n^{-2}|s_n^*|^3 < \varepsilon_n, n^{-2}|t_n^*|^3 < \varepsilon_n\}$, where $\varepsilon_n \to 0$ sufficiently slowly that (11) holds. Let $S_n = n/2 \log(1 + \theta^2) + (1 + \theta^2)^{-1}\theta s_n^* - \frac{1}{2}(1 + \theta^2)^{-1}\theta^2 t_n^*$, $V_n^{(1)} = s_n^{*2}/n$, $V_n^{(2)} = t_n^{*2}/n$, and $V_n^{(3)} = s_n^* t_n^*/n$. Also let

$$V_n = \frac{1}{2}(1 + \theta^2)^{-2}\{(1 + \theta^4)V_n^{(1)} + \theta^2(\theta^2 + 2)V_n^{(2)} - 2\theta V_n^{(3)}\} + w_n$$

on A_n and 0 elsewhere. With the aid of the identity $2V_n^{(3)} = (s_n^* + t_n^*)^2/n - V_n^{(1)} - V_n^{(2)}$ and Proposition 1 it is easy to check that $V_n^{(i)}$ for $i = 1, 2,$ and 3 and hence V_n satisfy (14)–(17).

As in the preceding example (10) requires a special argument. Let P_0 denote the probability under which the x's have expectation $\tilde{\mu} = 0$. From the trivial inequality $P_0\{T \leq n\} \leq \Sigma_{k=m}^n P_0\{Z_k > b\}$ and an analysis of the tail of the t-distribution, it may be shown for $x > 0$ that

$$(70) \qquad P_0\{T \leq bx\} = 0\left(b^{\frac{1}{2}}\exp\left[-b\left(1 - \frac{2}{m}\right)\right]\right)$$

as $b \to \infty$. By Lemma 3 of Pollak and Siegmund (1975), for arbitrary $\theta \neq 0$ and

$y > 0$

(71) $$P\{T \leqslant n\} \leqslant 1 - \Phi(y) + \exp\left(\theta^2 n/2 + y\theta n^{\frac{1}{2}}\right) P_0\{T \leqslant n\}.$$

Putting $y = b^{\frac{1}{4}}$ in (71) and appealing to (70) yields (10) for any $\delta < 2\theta^{-2}(1 - 2/m)$. Hence by Theorem 3, for $\theta \neq 0$ and $m \geqslant 3$

(72) $$\left[\log(1 + \theta^2)\right] ET = 2(b - 1) - 2\theta^4/(1 + \theta^2)^2 + ES_{\tau_+}^2/ES_{\tau_+} + o(1),$$

as $b \to \infty$.

5. Comparison with Woodroofe's results. The purpose of this section is to discuss briefly the relation of the results of this paper to similar results obtained recently by Woodroofe (1976a, 1977) by completely different methods. In general terms the methods of this paper and its companion develop renewal theory for nonlinear functions of a random walk S_n by expanding the function and applying classical renewal theory to the dominant linear term. In contrast Woodroofe considers the first passage of a random walk S_n to a nonlinear boundary which he analyzes by expanding the boundary around an appropriate point. One consequence of this difference in formulation is that in this paper and its companion a fairly small number of theorems provide a unified theory, whereas Woodroofe is required to reapply his method with its fairly elaborate computations to deal with different stopping rules. A technical difference is that Woodroofe requires a blanket smoothness condition on the distribution of his random variables, which has no counterpart in the present development. Other technical differences are described below.

Let $x_1, x_2, \cdots$ be independent identically distributed random variables with positive expectation $\tilde{\mu}$ and finite variance $\tilde{\sigma}^2$. Let $s_n = x_1 + \cdots + x_n$. Woodroofe (1976a) studies the behavior of the stopping rule

(73) $$T_1 = \inf\{n : s_n > cn^\gamma\} \qquad c > 0, 0 \leqslant \gamma < 1$$

as $c \to \infty$. (Actually for some results a slightly more general class of stopping rules is considered, but since Woodroofe gives no application for these more general rules, and since their introduction would complicate this discussion, they have been omitted.) Statistical applications of the stopping rule (73) have been described by Woodroofe (1976b) and Siegmund (1977). Under the additional restriction

(74) $$P\{x_1 \leqslant 0\} = 0.$$

Woodroofe (1977) studies the behavior of

(75) $$T_2 = \inf\{n : n \geqslant m, s_n < cn^\gamma L(n)\} \qquad c > 0, \gamma > 1, m = 1, 2, \cdots$$

as $c \to 0$, where $L(n) = 1 + \text{const.}/n + o(n^{-1})$ as $n \to \infty$. Both (73) and (75) may be written in the form

(76) $$T = \inf\{n : n \geqslant m, (n + \delta + \delta_n) g(n^{-1}s_n) > b\},$$

where $g(x) = (x^+)^{1/(1-\gamma)}$, $b = g(c)$, and $\delta_n \to 0$. Suppose more generally that

$g : (-\infty, \infty) \to [0, \infty)$ is three times continuously differentiable in a neighborhood of $\tilde{\mu}$ and that $g'(\tilde{\mu}) > 0$. Let $\varepsilon_n \to 0$ and $A_n = \{n^{-2}|s_n - n\tilde{\mu}|^3 < \varepsilon_n\}$. By Taylor's theorem, on A_n

$$ng(n^{-1}s_n) = ng(\tilde{\mu}) + (s_n - n\tilde{\mu})g'(\tilde{\mu}) + \frac{(s_n - n\tilde{\mu})^2}{2n}g''(\tilde{\mu}) + w_n,$$

where $|w_n| \leqslant W(n^{-2}|s_n - n\tilde{\mu}|^3)$ for some function W which vanishes at 0 and is continuous there. Let $S_n = ng(\tilde{\mu}) + (s_n - n\tilde{\mu})g'(\tilde{\mu})$ and $V_n = (s_n - n\tilde{\mu})^2/2n + (\delta + \delta_n)g(n^{-1}s_n)$ on A_n and 0 otherwise. It may be shown as in the second example of Section 4 with the aid of Proposition 1 that for suitable ε_n (11) and (14)–(17) are satisfied with $\alpha = \frac{2}{3}$, provided $E|x_1|^4 < \infty$. Thus Theorem 3 applies to give an asymptotic expansion for $E(T)$ provided that (10) holds, and as always a special argument is required here.

For the stopping rules (73) and (75) proofs of (10) under appropriate conditions follow from Woodroofe (1976a, Lemma 7.1 and 1977, Lemma 2.3). Use of the Hájek-Rényi-Chow inequality (cf. Chow, Robbins and Siegmund (1971), page 25) together with (50) would simplify these arguments.

Hence Woodroofe's expansions of ET_1 and ET_2 follow from Theorem 3. In Woodroofe's work a central role is played by results resembling Theorem 4, which form the basis for subsequent calculations. One example is Theorem 3.1 of Woodroofe (1976a), which says that if $E|x_1|^3 < \infty$ and Woodroofe's blanket smoothness condition is satisfied, then for T_1 defined by (71)

$$(77) \quad \Sigma_1^\infty P\{T_1 > n, s_n > cn^\gamma - y\} \to [(1-\gamma)\tilde{\mu}]^{-1} \times \int_{-y}^{0} P\{s_n \geqslant n\gamma\tilde{\mu} - x \text{ for all } n \geqslant 1\}dx.$$

Deriving (77) from Theorem 4 requires the slightly stronger moment condition $E|x_1|^p < \infty$ for some $p \geqslant 1 + 5^{\frac{1}{2}}$. With (73) rewritten in the form of (76) (with $\delta = \delta_n = 0$) the inequality $s_n > cn^\gamma - y$ becomes $Z_n = ng(n^{-1}s_n) > (b^{1-\gamma} - yn^{-\gamma})^{1/(1-\gamma)}$. As in the proof of Theorem 2, it is easily shown that only the terms with $n_1 \leqslant n \leqslant n_2$, where n_1 and n_2 are defined in (21), are nonnegligible in evaluating the limit of the left-hand side of (77). For these values of n, which are $\sim b/\mu$, a simple expansion gives $(b^{1-\gamma} - yn^{-\gamma})^{1/(1-\gamma)} = b - \mu^\gamma y/(1-\gamma) + o(1)$. Now Theorem 4 and a simple change of variable yield (77). A similar argument applies to the stopping rule T_2 of (75).

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DEPARTMENT OF MATHEMATICAL STATISTICS
COLUMBIA UNIVERSITY
MATHEMATICS BUILDING
NEW YORK, N. Y. 10027

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305