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A NOTE ON A FINITE POPULATION PLAN[1]

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Wynn gives an example of a balanced incomplete block design for 24 varieties in 56 blocks of three plots each which involves only 24 distinct blocks. It is shown that this design can be obtained from a hierarchic group divisible association scheme, and is one of a set of eight possible designs.

1. Introduction. Wynn (1977) discusses the selection of a sample of k distinct elements from a set of v elements (varieties). He is led in particular to consider balanced incomplete block designs in which some of the blocks are repeated. Let D be a BIBD with parameters v, b, r, k, λ; the support of D is the set of distinct blocks, and the number, b^*, of blocks is the support size. Wynn considers an example of a design for $v = 24$, $k = 3$ with $b = 56$ and $b^* = 24$. The purpose of this note is to show the connection between Wynn's design and partially balanced designs with the hierarchic group divisible scheme.

2. Hierarchic GD designs. We denote the varieties by the integers 1 through 8. We divide them into two groups of four and then subdivide the groups into two subgroups of two varieties each as follows. The first group has subgroups 1, 3 and 2, 4; the second has subgroups 5, 7 and 6, 8. Following Raghavarao (1960) we define two varieties to be first associates if they are in the same subgroup, second associates if they are in different subgroups of the same group and third associates if they are in different groups.

The 56 triples of the eight varieties are arranged in an array in Table I. Each row corresponds to a set of eight blocks and is a subdesign.

Table I

D_{11}	125	148	167	278	236	347	358	456
D_{12}	126	145	178	258	237	348	356	467
D_{13}	127	146	158	256	238	367	345	478
D_{14}	128	147	156	267	235	346	378	458
D_{21}	136	138	157	357	245	247	268	468
D_{22}	135	137	168	368	246	248	257	457
D_3	123	124	134	234	567	568	578	678

The first four rows are subdesigns with $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$. The next two rows have $\lambda_1 = 2$, $\lambda_2 = 0$, $\lambda_3 = 1$ and the last row has $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 0$. We may obtain a BIBD with the desired parameters by taking any one of the first

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four subdesigns four times, together with either D_{21} or D_{22} twice and D_3 once. Wynn's design uses for its support D_{11}, D_{21} and D_3.

3. Group divisible designs. Two designs with $b^* = 32$ may be obtained in a similar way by using the ordinary group divisible association schemes.

If we divide the varieties into two groups of four each: 1, 3, 5, 7 and 2, 4, 6, 8, the subdesign

$$D_4\colon 135, 137, 157, 357, 246, 248, 268, 468$$

has $\lambda_1 = 2$, $\lambda_2 = 0$. Clatworthy (1973) has a design, R58, with $b = 24$, $\lambda_1 = 2$, $\lambda_2 = 3$. Repeating R58 twice and adding D_4 once gives a BIBD with $b^* = 32$.

Alternatively we may divide the varieties into four groups of two each and use two more of the designs listed by Clatworthy. His R54 has $b = 8$, $\lambda_1 = 0$, $\lambda_2 = 1$; his R56 has $b = 24$, $\lambda_1 = 6$, $\lambda_2 = 2$. If we take R54 four times and R56 once, we again have a BIBD with $b^* = 32$.

4. Conclusion. The interest in these designs lies mainly in the fact that they are constructed by combining and repeating partially balanced designs. It is clear that other examples can be found where this technique will produce balanced designs with $b^* < b$: and it is tempting to ask whether all balanced incomplete blocks may be obtained by using partially balanced designs in this way. However, Foody and Hedayat (1977) have approached the problem of BIB designs with repeated blocks from a more general point of view and have developed some theoretical techniques for generating such designs which, for small v and k, can be implemented by a computer program. In particular they have produced designs with $v = 8$, $k = 3$, and $b = 56$ for all values of b^* in the interval $22 \leqq b^* \leqq 50$ and for $b^* = 52$. Even when b^* is a multiple of eight they find designs in which the support is not equireplicate.

In their example with $b^* = 48$ they omit a set of eight triples and duplicate a corresponding set of eight blocks. In each set varieties 1 and 7 appear four times and 2 and 5 appear twice. Neither set can be expressed as a combination of one or more partially balanced designs.

We are led therefore to the following theorem which encompasses the special cases that we have considered in detail. The proof of the theorem is similar to that presented by Foody and Hedayat and will not be included.

Let D be a balanced incomplete block design with parameters v, b, r, k, λ with support d^* consisting of b^* blocks. Let D_1 and D_2 be two designs for the v varieties each consisting of b' distinct blocks of size k with incidence matrices $\mathbf{N}_1$ and $\mathbf{N}_2$; D_1 and D_2 need not involve all the varieties. Let D_j^* denote the set of blocks of D_j that are also in d^*, and let $\bar{D}_j$ denote the remaining blocks of D_j.

THEOREM. *If* (i) $\bar{D}_1$ *and* $\bar{D}_2$ *are identical sets and* (ii) $\mathbf{N}_1\mathbf{N}_1' = \mathbf{N}_2\mathbf{N}_2'$, *then we may omit* D_1^* *from* D *and replace it by* D_2^*, *and thereby obtain another balanced design* D' *with the same parameters as* D. *The support size of* D' *is* $b^* - b^{**}$, *where* b^{**} *is the number of blocks of* D_1 *that are not contained in* D_2.

The case in which D_1 and D_2 contain repeated blocks is easily handled.

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