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ASYMPTOTIC NORMALITY OF MULTIVARIATE LINEAR RANK STATISTICS IN THE NON-I.I.D. CASE

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Asymptotic normality is established for multivariate linear rank statistics of general type in the non-i.i.d. case covering null hypotheses as well as almost arbitrary alternatives. The functions generating the regression constants and the scores are allowed to have a finite number of discontinuities of the first kind, and to tend to infinity near 0 and 1. The proof is based on properties of empirical df's in the non-i.i.d. case and is patterned on the 1958 Chernoff–Savage method. As special cases e.g. rank statistics used for testing against regression and rank statistics for testing independence are included.

1. Introduction. There exists a variety of theorems on asymptotic normality of both univariate and multivariate rank statistics. Although these results are obviously related, separate proofs are given and in general different techniques are used. It is our purpose to give a unifying approach to these various results. We shall present two theorems establishing asymptotic normality for a general class of multivariate rank statistics and, apart from regularity conditions, almost arbitrary underlying continuous distribution functions (df's) which may correspond to the null hypothesis or to local or fixed alternatives. As such these theorems are more general than existing results. As special cases they contain or extend many of the results found in the literature and include, e.g., asymptotic normality for simple linear rank statistics as well as rank statistics for independence, under the null hypothesis and under alternatives. Specializing our theorems to particular cases it turns out that the present conditions are rather close to the best conditions that appear in the literature, although they are occasionally slightly stronger.

The technique is almost entirely based on the properties of empirical distribution functions in the non-i.i.d. case as developed in van Zuijlen (1976a), (1976b) and (1978) and might be called the Chernoff–Savage approach. It is generally applicable in problems of this kind (cf. Ruymgaart and van Zuijlen (1977) where virtually the same technique is applied to linear combinations of functions of order statistics). Recently, the Pyke–Shorack approach based on empirical processes has been employed in Rüschendorf (1976) to derive the asymptotic distribution for a general class of multivariate rank statistics under an assumption concerning the weak convergence of the reduced multivariate sequential empirical process.

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Let k be a fixed positive integer and for each $N = 1, 2, \cdots$ let $X_{nN} = (X_{1nN}, X_{2nN}, \cdots, X_{knN})$, $n = 1, 2, \cdots, N$, be N independent k-dimensional random vectors with joint continuous distribution function F_{nN} and marginal df's F_{1nN}, $F_{2nN}, \cdots, F_{knN}$. For each N, moreover, let $\mathbb{F}_N$ be the joint empirical df based on the N random vectors $X_{1N}, X_{2N}, \cdots, X_{NN}$ and, for $i = 1, 2, \cdots, k$, denote the marginal empirical df of the independent random variables $X_{i1N}, X_{i2N}, \cdots, X_{iNN}$ by $\mathbb{F}_{iN}$ and the ranks of these rv's by $R_{i1N}, R_{i2N}, \cdots, R_{iNN}$. We have the relations

$$R_{inN} = N\mathbb{F}_{iN}(X_{inN}) \qquad \text{for} \quad i = 1, 2, \cdots, k\,. \tag{1.1}$$

All random vectors are supposed to be defined on a single probability space $(\Omega, \mathscr{A}, P)$. We define $\bar{F}_N = N^{-1} \sum_{n=1}^N F_{nN}$ and $\bar{F}_{iN} = N^{-1} \sum_{n=1}^N F_{inN}$ for $i = 1, 2, \cdots, k$.

The rank statistics that we are interested in are called multivariate linear rank statistics; these are of the type

$$S_N = N^{-1} \sum_{n=1}^N c_{nN} a_N(R_{1nN}, R_{2nN}, \cdots, R_{knN})\,. \tag{1.2}$$

Here, for $n_i = 1, 2, \cdots, N$, $i = 1, 2, \cdots, k$, the $a_N(n_1, n_2, \cdots, n_k)$ are given real numbers, called scores, and the c_{nN}, for $n = 1, 2, \cdots, N$, are given real constants, called regression constants. For this terminology see Hájek and Šidák (1967). An important subclass of the statistics of the form (1.2) are those for which the scores have product structure, viz

$$T_N = N^{-1} \sum_{n=1}^N c_{nN} \prod_{i=1}^k a_{iN}(R_{inN})\,, \tag{1.3}$$

where, for $n = 1, 2, \cdots, N$ and $i = 1, 2, \cdots, k$, the $a_{iN}(n)$ are the scores. Statistics of the more general form

$$\sum_{j=1}^m \lambda_j T_{jN}\,, \tag{1.4}$$

with $\lambda_1, \lambda_2, \cdots, \lambda_m$ real constants and each T_{jN} of the type (1.3), occupies an intermediate position between (1.2) and (1.3).

To motivate the study of the statistics mentioned in (1.3) or (1.4), let us observe that most of the rank statistics considered in the literature are of this form. In Puri and Sen (1969) and (1971), functions of statistics of the type (1.3) are proposed as permutationally (conditionally) distribution-free tests for some specified problems; in Shirahata (1973) it is shown that in many natural multivariate models locally most powerful rank tests are based on such rank statistics. To get an insight into the situations that are covered in the present set-up, we shall consider some examples.

EXAMPLE 1.1 (simple linear rank statistics). Choosing $k = 1$, (1.3) reduces to

$$T_{1N} = N^{-1} \sum_{n=1}^N c_{nN} a_{1N}(R_{1nN})\,. \tag{1.5}$$

Statistics of this type are of particular importance for testing the null hypothesis of randomness against regression. These statistics and two-sample rank statistics which are a special case of (1.5) have been extensively studied in the literature, see e.g., Chernoff and Savage (1958); Govindarajulu, Le Cam and Raghavachari (1967); Hájek and Šidák (1967); Hájek (1968); Pyke and Shorack (1968); and Dupač and Hájek (1969).

EXAMPLE 1.2 (rank statistics for independence). Choosing $k = 2$ and $c_{nN} = 1$, for $n = 1, 2, \cdots, N$, (1.3) reduces to

$$T_{2N} = N^{-1} \sum_{n=1}^{N} a_{1N}(R_{1nN}) a_{2N}(R_{2nN}) . \tag{1.6}$$

Statistics of this type are particularly well suited for testing the null hypothesis of independence against alternatives with an underlying bivariate df exhibiting a positive (negative) stochastic dependence. The asymptotic distribution under fixed alternatives can be found in e.g., Bhuchongkul (1964); Ruymgaart, Shorack and van Zwet (1972); and Ruymgaart (1973) and (1974).

EXAMPLE 1.3 (generalization of a model of Hájek and Šidák). Let $k \geqq 2$. We consider a generalization to the k-dimensional "regression" case of the bivariate dependence model proposed in Hájek and Šidák (1967), page 75 (see also Bhuchongkul (1964)). This generalization is due to Shirahata (1973). Let $X_{nN} = (X_{1nN}, X_{2nN}, \cdots, X_{knN})$, $n = 1, 2, \cdots, N$, be random vectors defined by

$$X_{inN} = X_{inN}^{*} + c_{nN} \Delta Z_{nN} , \qquad i = 1, 2, \cdots, k ,$$

where $\{X_{inN}^{*}\}_{n=1}^{N}$, for $i = 1, 2, \cdots, k$, and $\{Z_{nN}\}_{n=1}^{N}$ are mutually independent and each sequence is an i.i.d. sequence of random variables, the c_{nN} are known constants and Δ is an unknown parameter. For $i = 1, 2, \cdots, k$, let f_{iN} denote the density function of X_{inN}^{*} and $f_{iN}^{(1)}$ its derivative. According to Shirahata (1973) under certain regularity conditions the locally most powerful rank test for testing $\Delta = 0$ (independence) against $\Delta > 0$ is based on the rank statistic

$$T_{3N} = \mathscr{E}(Z_{nN}) \sum_{n=1}^{N} c_{nN} \sum_{i=1}^{k} \mathscr{E}_0 \left[\frac{f_{iN}^{(1)}(X_{inN})}{f_{iN}(X_{inN})} \,\middle|\, R_{inN} \right] . \tag{1.7}$$

If either $\mathscr{E}(Z_{nN}) = 0$ or $c_{nN} = 1$ for $n = 1, 2, \cdots, N$, then (1.7) reduces to a constant. In this case the locally most powerful rank test is based on the rank statistic

$$\tilde{T}_{3N} = \sum_{n=1}^{N} c_{nN} \left(\sum_{i,j=1; i \neq j}^{k} \mathscr{E}_0 \left[\frac{f_{iN}^{(1)}(X_{inN})}{f_{iN}(X_{inN})} \,\middle|\, R_{inN} \right] \mathscr{E}_0 \left[\frac{f_{jN}^{(1)}(X_{jnN})}{f_{jN}(X_{jnN})} \,\middle|\, R_{jnN} \right] \right) .$$

Both T_{3N} and $\tilde{T}_{3N}$ are of the type (1.4).

EXAMPLE 1.4 (generalization of models of Farlie and Witting and Nölle). Let $k \geqq 2$. In a similar manner as in Example 1.3 the bivariate dependence models proposed in Farlie (1960) and in Witting and Nölle (1970), page 130, can be generalized to the k-dimensional "regression" case. Focussing on the latter model the sample elements $X_{nN} = (X_{1nN}, X_{2nN}, \cdots, X_{knN})$ have df $F_{nN\Delta}$, $n = 1, 2, \cdots, N$, where

$$F_{nN\Delta}(x_1, x_2, \cdots, x_k) = (1 - c_{nN}\Delta) \prod_{i=1}^{k} F_{iN}(x_i) + c_{nN}\Delta \prod_{i=1}^{k} F_{iN}^2(x_i) ,$$

for $0 \leqq \Delta < 1$. Choosing the regression constants all equal to 1 we find (see Shirahata (1973)) that the locally most powerful rank test for testing $\Delta = 0$ against $\Delta > 0$ is based on the rank statistic

$$T_{4N} = \sum_{n=1}^{N} \prod_{i=1}^{k} R_{inN}\,. \tag{1.8}$$

This statistic is of the type (1.3) and obviously is a generalization to the multivariate case of Spearman's rank statistic.

Let us now return to the statistic T_N. It is well known that locally optimal scores can be determined if one has in mind particular parametric alternatives. In many such cases (see also the examples given) these optimal scores are so-called exact scores derived from suitable functions J_i on (0, 1) according to

$$a_{iN}^*(n) = \mathscr{E}J_i(\xi_{n:N})\,, \qquad \text{for } i = 1, 2, \cdots, k,\ n = 1, 2, \cdots, N\,, \tag{1.9}$$

where $\xi_{n:N}$ is the nth order statistic of a sample of size N from the uniform distribution on (0, 1). These exact scores, however, are not only hard to compute, but also hard to manipulate in the asymptotic theory. For this reason one frequently uses the scores

$$a_{iN}(n) = J_i(\mathscr{E}(\xi_{n:N})) = J_i\left(\frac{n}{N+1}\right), \qquad i = 1, 2, \cdots, k,\ n = 1, 2, \cdots, N\,, \tag{1.10}$$

called the approximate scores derived from J_i. Under a suitable condition ((1.17) below) approximate scores are as good as exact scores in the sense of Pitman-efficiency. The regression constants c_{nN} can always be generated by some function J_{0N} according to

$$c_{nN} = J_{0N}\left(\frac{n}{N+1}\right), \qquad n = 1, 2, \cdots, N\,. \tag{1.11}$$

Note that in contrast to the scores, the regression constants are generated by a function which is allowed to depend on N. This has the advantage that we also contain in our theory rank statistics used for the regression problem and the k-sample problem. In fact this dependence is already needed to cover the two-sample situation.

For methodological reasons it will be convenient to introduce the regression constants with the aid of the additional set of mutually independent rv's X_{01N}, $X_{02N}, \cdots, X_{0NN}$, independent of all random vectors considered so far and also defined on the same probability space. Let $\mathscr{U}_{a,b}$ denote the uniform df on the interval (a, b) and let us assume that the df F_{0nN} of X_{0nN} satisfies

$$F_{0nN} = \mathscr{U}_{(n-1)/N,\,n/N}\,, \qquad \text{for } n = 1, 2, \cdots, N\,. \tag{1.12}$$

For the ranks of these rv's this entails that

$$R_{0nN} = n\,, \qquad \text{for } n = 1, 2, \cdots, N\,, \tag{1.13}$$

with probability 1. For $n = 1, 2, \cdots, N$ the joint df of the $(k + 1)$-dimensional random vector $(X_{0nN}, X_{1nN}, \cdots, X_{knN})$ will be written as G_{nN}, the corresponding $(k+1)$-dimensional empirical df by $\mathbb{G}_N$ and its first marginal empirical df (based on $X_{01N}, X_{02N}, \cdots, X_{0NN}$) by $\mathbb{F}_{0N}$. It should be observed that

$$G_{nN} = F_{0nN} \times F_{nN} = \mathscr{U}_{(n-1)/N, n/N} \times F_{nN}, \qquad \text{for } n = 1, 2, \cdots, N, \tag{1.14}$$

and that $N^{-1} \sum_{n=1}^{N} \mathscr{U}_{(n-1)/N, n/N} = \mathscr{U}_{0,1}$, the uniform df on (0, 1). Analogous to previous notation we shall write $\bar{G}_N = N^{-1} \sum_{n=1}^{N} G_{nN}$.

In order to give an alternative expression for T_N in the case of approximate scores we have to introduce the modified marginal empirical df's

$$\mathbb{F}_{iN}^* = [N/(N+1)]\mathbb{F}_{iN}, \qquad \text{for } i = 0, 1, \cdots, k. \tag{1.15}$$

Combining (1.3) with (1.1), (1.10), (1.11), (1.13) and (1.15), it follows that T_N equals

$$T_N = \int J_{0N}(\mathbb{F}_{0N}^*) \prod_{j=1}^{k} J_j(\mathbb{F}_{jN}^*)\, d\mathbb{G}_N, \tag{1.16}$$

with probability 1. Here the integration is extended over the $(k+1)$-dimensional number space. The extension of each of the original k-dimensional random vectors with a 1-dimensional dummy random coordinate, each having one of the uniform df's in (1.12), has the effect that the statistic T_N can be entirely expressed in terms of empirical df's.

Our main result—Theorem 2.1 in Section 2—is the asymptotic normality of a suitably standardized version of T_N for approximate scores, where the next three points should be kept in mind. In the first place we remark that the generating functions are allowed to tend to infinity near 0 and 1, and to have a finite number of discontinuities of the first kind. The price for allowing these discontinuities is a local differentiability condition on the underlying df's. In the second place there appears to be a natural balance between the respective orders of magnitude of the generating functions near 0 and 1. In the particular case (1.5) e.g., this leads to quite a spectrum of possible orders of magnitude of J_{0N} and J_1 near 0 and 1, whereas in Hájek (1968) and Dupač and Hájek (1969) only two possibilities are considered. In the third place the asymptotic normality is established for almost arbitrary triangular arrays of underlying df's. Hence asymptotic normality for a triangular array corresponding to a set of local alternatives is included as a special case. From the latter result we can immediately derive the asymptotic power of the corresponding tests, which is used for the computation of asymptotic relative efficiencies. It is worthwhile noting that in contrast to e.g., the theorems in Chernoff and Savage (1958) and Ruymgaart (1973), we do not need uniformity of the convergence on a subclass of arrays of underlying df's to achieve the computation of the limiting distribution under local alternatives.

The proof of the asymptotic normality of the statistic considered will be given by way of a decomposition in a sum of leading terms, which is asymptotically normally distributed, and a remainder term, which is asymptotically negligible. In Section 3 this decomposition for the standardized version of T_N for approximate scores is presented and the asymptotic normality of the leading terms is established.

The proof of the asymptotic negligibility of the corresponding remainder term will rely almost completely on properties of the empirical df's as is suggested by the representation of T_N in (1.16). We shall restrict ourselves to some remarks displaying the general idea of this proof which is long and uninteresting. For details we refer to van Zuijlen (1976b). Apart from a component due to the introduction of the dummy random variables $X_{01N}, X_{02N}, \cdots, X_{0NN}$, and apart from the dimension, the components of this remainder term are very similar to the higher order terms in Ruymgaart (1973) and (1974), the main difference being that in the present case we have N possibly different underlying df's, whereas in Ruymgaart (1973) and (1974) there is one single fixed underlying df. The proof of the asymptotic negligibility, however, can be given in essentially the same way, because it turns out that all the lemmas used in Ruymgaart (1973) and (1974) remain valid, properly modified if necessary, under the present circumstances with not necessarily identical underlying df's and with the averaged df in the role of the single fixed underlying df. These lemmas are based on the properties of the empirical df in the non-i.i.d. case, which are obtained in van Zuijlen (1976a), (1976b) and (1978).

Under the assumption that

$$(1.17) \qquad N^{-\frac{1}{2}} \textstyle\sum_{n=1}^{N} c_{nN}[\prod_{i=1}^{k} a_{iN}^{*}(R_{inN}) - \prod_{i=1}^{k} a_{iN}(R_{inN})] = o_P(1)\,, \qquad \text{as} \quad N \to \infty\,,$$

one immediately derives an asymptotic result for the statistic T_N in the case of exact scores from the corresponding Theorem 2.1 on approximate scores. Condition (1.17) is well known in the literature (see e.g., Bhuchongkul (1964), Chernoff and Savage (1958) and Ruymgaart (1973)). A verification of the condition is a problem in itself (see e.g. Ruymgaart (1973)). In general an additional condition on the generating functions is needed. The details are given in Section 4.

It is possible to prove asymptotic normality of a suitably standardized version of S_N (see (1.2)), in the case where the scores $a_N(n_1, n_2, \cdots, n_k)$ are generated by some continuous function J on $(0, 1)^k$ according to

$$(1.18) \qquad a_N(n_1, n_2, \cdots, n_k) = J\left(\frac{n_1}{N+1}, \frac{n_2}{N+1}, \cdots, \frac{n_k}{N+1}\right), \qquad n_i = 1, 2, \cdots, N,\ i = 1, 2, \cdots, k\,.$$

The continuity condition can even be weakened (cf. van Zuijlen (1976b)).

2. Statement of the main theorem. Before presenting the theorem let us introduce some more notation and conventions, to be used throughout the present and the subsequent sections. We shall use the left-continuous version of the inverse of a univariate df. The standard normal df will be denoted by

$$(2.1) \qquad \mathscr{N}(y) = (2\pi)^{-\frac{1}{2}} \textstyle\int_{(-\infty,y]} \exp(-z^2/2)\, dz \qquad \text{for} \quad y \in (-\infty, \infty)\,.$$

For convenience we shall only use q-functions and reproducing u-shaped functions (for a definition see the appendix in Shorack (1972)) of a special but common type, based on the function

$$r(t) = \{t(1-t)\}^{-1} \quad \text{for} \quad t \in (0, 1) . \tag{2.2}$$

For an arbitrary positive integer m the m-fold Cartesian product of a set S with itself will be denoted by S^m. For each m, moreover, let us define

$$\mathscr{F}_m = \{F : F \text{ is an } m\text{-variate df which is continuous on } \mathbb{R}^m\} . \tag{2.3}$$

In the theorem the df's F_{nN} will be restricted to $\mathscr{F}_k$.

With respect to the generating functions we shall assume that the J_{0N} ($N = 1, 2, \cdots$) and J_i ($i = 1, 2, \cdots, k$) have a finite number of discontinuities of the first kind only. Without loss of generality it can and will be assumed that these generating functions are right-continuous.

For any finite set S let $\# S$ denote the number of elements in S and for any function f the ith derivative is written as $f^{(i)}(f^{(0)} = f)$.

ASSUMPTION 2.1 (generating functions). (a) For $N = 1, 2, \cdots$ the function J_{0N} has discontinuities of the first kind only and a continuous derivative $J_{0N}^{(1)}$ on the set $(0, 1) - \mathscr{D}_{0N}$.

(b) For $i = 1, 2, \cdots, k$ the function J_i has discontinuities of the first kind only and a continuous derivative $J_i^{(1)}$ on the set $(0, 1) - \mathscr{D}_i$.

(c) There exist positive numbers $l_0, l_1, \cdots, l_k$ and τ such that for $N = 1, 2, \cdots$ and $i = 1, 2, \cdots, k$,

$$\mathscr{D}_{0N} \subset (\tau, 1-\tau) , \quad \# \mathscr{D}_{0N} \leqq l_0 \qquad \text{and} \qquad \mathscr{D}_i \subset (\tau, 1-\tau) , \quad \# \mathscr{D}_i \leqq l_i ,$$

(d) There exist positive numbers $a_0, a_1, \cdots, a_k$ and K_1, satisfying $a \equiv \sum_{j=0}^k a_j < \frac{1}{2}$, such that, with r defined in (2.2) we have for $\nu = 0, 1$, $N = 1, 2, \cdots$ and $i = 1, 2, \cdots, k$,

$$|J_{0N}^{(\nu)}| \leqq K_1 r^{a_0+\nu} \qquad \text{and} \qquad |J_i^{(\nu)}| \leqq K_1 r^{a_i+\nu} , \tag{2.4}$$

wherever these functions are defined on (0, 1).

The price for discontinuities in the scores generating functions is a kind of local differentiability condition on the transformations

$$\Phi_{nN} = F_{nN}(\bar{F}_{1N}^{-1}, \bar{F}_{2N}^{-1}, \cdots, \bar{F}_{kN}^{-1}) \tag{2.5}$$

of the F_{nN} to the k-dimensional unit cube $[0, 1]^k$ for $n = 1, 2, \cdots, N$. We shall say that Φ_{nN} possesses a density ϕ_{nN} (with respect to Lebesgue measure on $[0, 1]^k$) on the Borel set $B_0 \subset [0, 1]^k$ if, for each Borel set $B \subset B_0$, we have

$$\int_B d\Phi_{nN} = \int_B \phi_{nN}(t_1, t_2, \cdots, t_k)\, dt_1\, dt_2 \cdots dt_k . \tag{2.6}$$

To formulate the assumption on the underlying df's, let us define for $\eta > 0$,

$$\mathscr{Q}_{\eta,i} = \bigcup_{s \in \tilde{\mathscr{D}}_i} (s-\eta, s+\eta) , \qquad \text{for } i = 1, 2, \cdots, k , \tag{2.7}$$

where $\tilde{\mathscr{D}}_i$ is the set of discontinuity points of J_i. Note that $\tilde{\mathscr{D}}_i \subset \mathscr{D}_i$.

ASSUMPTION 2.2 (underlying df's). There exist positive numbers $\eta, b_1, b_2, \cdots, b_k$ and K_2 such that for $N = 1, 2, \cdots, n = 1, 2, \cdots, N$ and $i = 1, 2, \cdots, k$, Φ_{nN} (see (2.5)) has a continuous density ϕ_{nN} on $(0, 1)^{i-1} \times \mathscr{Q}_{\eta,i} \times (0, 1)^{k-i}$, satisfying

$$|\phi_{nN}(t_1, t_2, \cdots, t_k)| \leqq K_2 \prod_{j=1; j\neq i}^{k} \{r(t_j)\}^{b_j}, \tag{2.8}$$

for $(t_1, t_2, \cdots, t_k)$ in this set. Moreover, for every $(t_1, t_2, \cdots, t_{i-1}, t_{i+1}, \cdots, t_k) \in (0, 1)^{k-1}$, every $t_i \in \tilde{\mathscr{D}}_i$ (see (2.7)) and every $i = 1, 2, \cdots, k$,

$$\begin{aligned} \sup_{n,N} |\phi_{nN}(t_1, \cdots, t_{i-1}, t, t_{i+1}, \cdots, t_k) - \phi_{nN}(t_1, \cdots, t_k)| \\ \rightarrow 0 \quad \text{as} \quad t \rightarrow t_i . \end{aligned} \tag{2.9}$$

REMARK 2.1. If J_j is continuous, then $\tilde{\mathscr{D}}_j = \emptyset$ and $\mathscr{Q}_{\eta,j} = \emptyset$ so that Assumption 2.2 is vacuous for $i = j$.

To standardize the location of the statistics T_N we shall use the quantities

$$\mu_N = \mu_N(F_{1N}, F_{2N}, \cdots, F_{NN}) = \int J_{0N}(\bar{F}_{0N}) \prod_{j=1}^{k} J_j(\bar{F}_{jN})\, d\bar{G}_N . \tag{2.10}$$

The quantity μ_N arises in the fundamental decomposition of T_N in (3.10). The quantities used to standardize the scale of the T_N will be given in the implicit form

$$\sigma_N^2 = \sigma_N^2(F_{1N}, F_{2N}, \cdots, F_{NN}) = \operatorname{Var}(A_N + \textstyle\sum_{i=1}^{k} A_{iNc} + \sum_{i=1}^{k} A_{iNd}) , \tag{2.11}$$

where A_N and the A_{iNc} and A_{iNd} also arise in (3.10). Under the conditions of the theorem below these quantities are well defined.

THEOREM 2.1. *Let an arbitrary triangular array of underlying* df's $F_{nN} \in \mathscr{F}_k$, $n = 1, 2, \cdots, N$, $N = 1, 2, \cdots$ *be given, such that for the resulting triangular array of transformed* df's Φ_{nN} *Assumption* 2.2 *is fulfilled. Let the generating functions satisfy Assumption* 2.1 *and let the constants* a_j *(appearing in Assumption* 2.1) *and the constants* b_j *(appearing in Assumption* 2.2) *satisfy* $a_j + b_j < 1$ *for* $j = 1, 2, \cdots, k$. *Then the quantities* μ_N *and* σ_N^2, *defined in* (2.10) *and* (2.11) *are finite. If, moreover,* $\liminf_{N\rightarrow\infty} \sigma_N^2 > 0$ *we have*

$$\sup_{-\infty<z<\infty} |P(N^{\frac{1}{2}}(T_N - \mu_N)/\sigma_N \leqq z) - \mathscr{N}(z)| \rightarrow 0 , \qquad as \quad N \rightarrow \infty , \tag{2.12}$$

for T_N *as in* (1.16), i.e., *the case of approximate scores.*

3. Asymptotic normality of the leading terms.

Before writing down the leading terms of the standardized version of the statistic T_N for approximate scores let us make some introductory remarks.

We introduce for $N = 1, 2, \cdots$ a $(k + 1)$-dimensional random vector

$$(Y_{0N}, Y_{1N}, \cdots, Y_{kN}) \qquad \text{with joint df} \quad \bar{G}_N , \tag{3.1}$$

where $\bar{G}_N$ is defined below (1.14). Besides the transformed df's in (2.5) it will be convenient to have at our disposal the transformation

$$\bar{\Psi}_N \equiv \bar{G}_N(\bar{F}_{0N}^{-1}, \bar{F}_{1N}^{-1}, \cdots, \bar{F}_{kN}^{-1}) = N^{-1} \sum_{n=1}^{N} \mathscr{U}_{(n-1)/N, n/N} \times \Phi_{nN} . \tag{3.2}$$

The transformed random vector $(\bar{F}_{0N}(Y_{0N}), \bar{F}_{1N}(Y_{1N}), \cdots, \bar{F}_{kN}(Y_{kN}))$ has joint df

$\bar{\Psi}_N$ because of the continuity of the underlying df's and by definition all the univariate marginal df's of $\bar{\Psi}_N$ are $\mathscr{U}_{0,1}$. If Assumption 2.2 holds one can show that $\bar{\Psi}_N$ has, for $i = 1, 2, \cdots, k$, a density $\bar{\phi}_N$ (with respect to Lebesgue measure on $(0, 1)^{k+1}$) on the set $(0, 1)^i \times \mathscr{C}_{\eta,i} \times (0, 1)^{k-i}$, where $\mathscr{C}_{\eta,i}$ is defined in (2.7). We have for $n = 1, 2, \cdots, N$, $i = 1, 2, \cdots, k$,

$$\bar{\phi}_N(t_0, t_1, \cdots, t_k) = \phi_{nN}(t_1, t_2, \cdots, t_k) , \tag{3.3}$$

for

$$(t_0, t_1, \cdots, t_k) \in ((n-1)/N, n/N) \times (0, 1)^{i-1} \times \mathscr{C}_{\eta,i} \times (0, 1)^{k-i} .$$

Anticipating the finiteness of all expectations and integrals involved let us consider for $N = 1, 2, \cdots, i \in \{1, 2, \cdots, k\}$ and $t_i \in (0, 1)$ the conditional expectation

$$\mathscr{E}(J_{0N}(\bar{F}_{0N}(Y_{0N})) \prod_{j=1;j\neq i}^k J_j(\bar{F}_{jN}(Y_{jN})) \,|\, \bar{F}_{iN}(Y_{iN}) = t_i) . \tag{3.4}$$

Under Assumption 2.2, again, one of the possible determinations of (3.4) equals $h_{iN}(t_i)$, where

$$\begin{aligned} h_{iN}(t_i) = \textstyle\sum_{n=1}^N (\int_{((n-1)/N, n/N)} J_{0N}(t_0)\, dt_0) \int_{(0,1)^{k-1}} \prod_{j=1;j\neq i}^k J_j(t_j) \\ \times \phi_{nN}(t_1, t_2, \cdots, t_k)\, dt_1 \cdots dt_{i-1}\, dt_{i+1} \cdots dt_k , \end{aligned} \tag{3.5}$$

provided t_i is restricted to $\mathscr{C}_{\eta,i}$.

Throughout the sequel the symbol M will be employed as a generic constant, independent of N.

LEMMA 3.1. *Let the function h_{iN} be defined as in* (3.5). *Under the conditions of Theorem* 2.1 *we have for $N = 1, 2, \cdots$ and $i = 1, 2, \cdots, k$ that $|h_{iN}(t_i)| \leqq M_i$, for $t_i \in \mathscr{C}_{\eta,i}$, where M_i is a number independent of N. Moreover, for $N = 1, 2, \cdots$ and $i = 1, 2, \cdots, k$, h_{iN} is a continuous function of t_i for $t_i \in \mathscr{C}_{\eta,i}$, and for each i the set of functions $\{h_{iN}, N = 1, 2, \cdots\}$ is equicontinuous on $\tilde{\mathscr{D}}_i$* (cf. (2.7)).

PROOF. From the assumptions in Theorem 2.1 it is immediate that

$$\begin{aligned} |h_{iN}(t_i)| &\leqq M \textstyle\sum_{n=1}^N (\int_{((n-1)/N, n/N)} r^{a_0}(t_0)\, dt_0) \\ &\quad \times \int_{(0,1)^{k-1}} \prod_{j=1;j\neq i}^k r^{a_j+b_j}(t_j)\, dt_1 \cdots dt_{i-1}\, dt_{i+1} \cdots dt_k \\ &= M \int_0^1 r^{a_0}(t_0)\, dt_0 \prod_{j=1;j\neq i}^k \int_0^1 r^{a_j+b_j}(t_j)\, dt_j = M_i . \end{aligned}$$

For the second statement it suffices to show that for $n = 1, 2, \cdots, N$,

$$\int_{(0,1)^{k-1}} \prod_{j=1;j\neq i}^k J_j(t_j)\phi_{nN}(t_1, \cdots, t_k)\, dt_1 \cdots dt_{i-1}\, dt_{i+1} \cdots dt_k \tag{3.6}$$

is a continuous function of t_i, for $t_i \in \mathscr{C}_{\eta,i}$. Let $t_i, t_i + \xi \in \mathscr{C}_{\eta,i}$. Because of Assumption 2.2 in Theorem 2.1 we have that

$$\begin{aligned} \phi_{nN}(t_1, \cdots, t_{i-1}, t_i + \xi, t_{i+1}, \cdots, t_k) - \phi_{nN}(t_1, \cdots, t_k) \\ \to 0 \quad \text{as} \quad \xi \to 0 , \end{aligned} \tag{3.7}$$

for each $(t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_k) \in (0, 1)^{k-1}$. The continuity of h_{iN} follows from (2.4), (2.8), (3.7) and the dominated convergence theorem since $a_j + b_j < 1$ for

$j = 1, 2, \cdots, k$. Analogously, the equicontinuity can be established with the aid of (2.9). □

In view of Assumption 2.1 and the way in which we shall conduct the proof of Theorem 2.1 it is no loss of generality to assume that for $i = 1, 2, \cdots, k$ the generating functions J_i have only one discontinuity (say at s_i), so that

$$J_i(t) = J_{ic}(t) + \Lambda_i c(t - s_i) , \tag{3.8}$$

where J_{ic} is the continuous part of J_i and where

$$\begin{aligned} c(z) &= 1 \quad \text{for} \quad z \in [0, \infty) , \\ &= 0 \quad \text{elsewhere.} \end{aligned} \tag{3.9}$$

We are now in a position to give the basic decomposition, which holds with probability 1,

$$N^{\frac{1}{2}}(T_N - \mu_N) = A_N + \sum_{i=1}^{k} A_{iNc} + \sum_{i=1}^{k} A_{iNd} + E_N , \tag{3.10}$$

where

$$A_N = N^{\frac{1}{2}} \int J_{0N}(\bar{F}_{0N}) \prod_{j=1}^{k} J_j(\bar{F}_{jN}) \, d(\mathbb{G}_N - \bar{G}_N) , \tag{3.11}$$

$$A_{iNc} = N^{\frac{1}{2}} \int (\mathbb{F}_{iN} - \bar{F}_{iN}) J_{iN}^{(1)}(\bar{F}_{iN}) J_{0N}(\bar{F}_{0N}) \prod_{j=1; j \neq i}^{k} J_j(\bar{F}_{jN}) \, d\bar{G}_N , \tag{3.12}$$

$$A_{iNd} = N^{\frac{1}{2}} \Lambda_i h_{iN}(s_i)(\mathbb{F}_{iN}(\bar{F}_{iN}^{-1}(s_i)) - s_i) , \tag{3.13}$$

and E_N is a remainder term which is of second order. Remark that for $\Lambda_i \neq 0$ the conditional expectation $h_{iN}(s_i)$ is well defined; if $\Lambda_i = 0$ then A_{iNd} is defined to be zero. This section is devoted to establishing the asymptotic normality of the A-terms, i.e., under the conditions of Theorem 2.1 we shall show, with σ_N defined in (2.11), that

$$\begin{aligned} &\sup_{-\infty < z < \infty} |P((A_N + \textstyle\sum_{i=1}^{k} A_{iNc} + \sum_{i=1}^{k} A_{iNd})/\sigma_N \leqq z) - \mathcal{N}(z)| \\ &\quad \to 0 \quad \text{as} \quad N \to \infty . \end{aligned} \tag{3.14}$$

We begin by noting that with probability 1,

$$A_N + \sum_{i=1}^{k} A_{iNc} + \sum_{i=1}^{k} A_{iNd} = N^{-\frac{1}{2}} \sum_{n=1}^{N} Z_{nN} , \tag{3.15}$$

where

$$Z_{nN} = A_{nN} + \sum_{i=1}^{k} A_{inNc} + \sum_{i=1}^{k} A_{inNd} , \tag{3.16}$$

and

$$A_{nN} = J_{0N}(\bar{F}_{0N}(X_{0nN})) \prod_{j=1}^{k} J_j(\bar{F}_{jN}(X_{jnN})) - \mu_N , \tag{3.17}$$

$$\begin{aligned} A_{inNc} = \int &[c(\bar{F}_{iN} - \bar{F}_{iN}(X_{inN})) - \bar{F}_{iN}] \\ &\times J_{iN}^{(1)}(\bar{F}_{iN}) J_{0N}(\bar{F}_{0N}) \prod_{j=1; j \neq i}^{k} J_j(\bar{F}_{jN}) \, d\bar{G}_N , \end{aligned} \tag{3.18}$$

$$A_{inNd} = \Lambda_i h_{iN}(s_i)[c(s_i - \bar{F}_{iN}(X_{inN})) - s_i] . \tag{3.19}$$

It should be observed that the rv Z_{nN} depends on the random vector X_{nN} only. Consequently these rv's $Z_{1N}, Z_{2N}, \cdots, Z_{NN}$ are mutually independent.

Next we show that there exists a $\delta > 0$ such that

$$\limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} \mathscr{E}|Z_{nN}|^{2+\delta} < \infty . \tag{3.20}$$

This will be achieved by proving the stronger assertion that

$$\limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} \mathscr{E}|A_{nN}|^{2+\delta} < \infty , \tag{3.21}$$

and that for $i = 1, 2, \cdots, k$,

$$\limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} \mathscr{E}|A_{inNc}|^{2+\delta} < \infty , \tag{3.22}$$

$$\limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} \mathscr{E}|A_{inNd}|^{2+\delta} < \infty . \tag{3.23}$$

We note in passing that this result will ensure the finiteness of the expectations and integrals considered so far. The proof relies on Hölder's inequality in the form

$$\int |\prod_{i=0}^{k} f_i(\bar{F}_{iN})|\, d\bar{G}_N \leqq \prod_{i=0}^{k} [\int_0^1 |f_i(s_i)|^{\xi_i}\, ds_i]^{1/\xi_i} , \tag{3.24}$$

where $f_0, f_1, \cdots, f_k$ are measurable functions on (0, 1) such that the above integrals exist and where $\xi_0, \xi_1, \cdots, \xi_k > 1$ satisfy $\sum_{i=0}^{k} \xi_i^{-1} = 1$.

Application of (3.24) with $\xi_i = a/a_i$ (here $a = \sum_{i=0}^{k} a_i$) yields

$$\begin{aligned} N^{-1} \sum_{n=1}^{N} \mathscr{E}(|A_{nN}|^{2+\delta}) &\leqq M \int r^{a_0(2+\delta)}(\bar{F}_{0N}) \prod_{j=1}^{k} r^{a_j(2+\delta)}(\bar{F}_{jN})\, d\bar{G}_N \\ &\leqq M \prod_{i=0}^{k} [\int_0^1 r^{(2+\delta)a}(s)\, ds]^{a_i/a} < \infty , \end{aligned} \tag{3.25}$$

provided $\delta > 0$ is chosen sufficiently small to ensure that $(2 + \delta)a < 1$. Since $a < \frac{1}{2}$ by Assumption 2.1, this can always be achieved. Apparently the bound in (3.25) is independent of N so that (3.21) is proved.

To prove (3.22) for arbitrary $i \in \{1, 2, \cdots, k\}$ we note that for $\delta \in (0, \frac{1}{2}]$ and $u, v \in (0, 1)$, (see Ruymgaart (1973), page 27)

$$|c(u - v) - u| \leqq M[r(v)]^{\frac{1}{2}-\delta}[r(u)]^{-\frac{1}{2}+\delta} . \tag{3.26}$$

From (3.26) and Assumption 2.1 we find,

$$\begin{aligned} N^{-1} &\sum_{n=1}^{N} \mathscr{E}(|A_{inNc}|^{2+\delta}) \\ &\leqq N^{-1} \sum_{n=1}^{N} \mathscr{E}[M(r(\bar{F}_{iN}(X_{inN})))^{\frac{1}{2}-\delta} \\ &\qquad \times \int (r(\bar{F}_{iN}))^{-\frac{1}{2}+\delta}(r(\bar{F}_{iN}))^{a_i+1} \prod_{j=0;j\neq i}^{k} (r(\bar{F}_{jN}))^{a_j}\, d\bar{G}_N]^{2+\delta} \\ &\leqq M \int_0^1 (r(s))^{(\frac{1}{2}-\delta)(2+\delta)}\, ds[\int \prod_{j=0;j\neq i}^{k} (r(\bar{F}_{jN}))^{a_j}(r(\bar{F}_{iN}))^{a_i+\frac{1}{2}+\delta}\, d\bar{G}_N]^{2+\delta} . \end{aligned}$$

Since for every $\delta > 0$, $(\frac{1}{2} - \delta)(2 + \delta) < 1$ it suffices to consider the last factor in the last bound, which is bounded above by

$$\begin{aligned} \prod_{j=0;j\neq i}^{k} &\{\int_0^1 [r(s_i)]^{a_j/[a_j+(\frac{1}{2}-a-2\delta)/k]}\, ds_j\}^{a_j+(\frac{1}{2}-a-2\delta)/k} \\ &\times \{\int_0^1 [r(s_i)]^{(a_i+\frac{1}{2}+\delta)/(a_i+\frac{1}{2}+2\delta)}\, ds_i\}^{(a_i+\frac{1}{2}+2\delta)} < \infty . \end{aligned} \tag{3.27}$$

This follows from an application of (3.24) with $\xi_j^{-1} = a_j + (\frac{1}{2} - a - 2\delta)/k$ for $j \in \{0, 1, \cdots, k\}$ but $j \neq i$, and $\xi_i^{-1} = a_i + \frac{1}{2} + 2\delta$. Because $a < \frac{1}{2}$ we have for $0 < 2\delta < \frac{1}{2} - a$ that $\xi_i > 1$ for $j = 0, 1, \cdots, k$. The bound in (3.27) is independent of N, so that (3.22) is proved.

Finally let us note that because of Lemma 3.1 for $\Lambda_i \neq 0$,

$$N^{-1} \sum_{n=1}^{N} \mathscr{E}(|A_{inNd}|^{2+\delta}) \leqq M|\Lambda_i||h_{iN}(s_i)|^{2+\delta} \leqq M|\Lambda_i|M_i^{2+\delta} , \tag{3.28}$$

so that the contribution due to the purely discrete part of the generating functions is bounded by a finite constant independent of N. It is obvious that the minimum over the finite number of δ's considered so far is a δ for which (3.21), (3.22) and (3.23) are simultaneously satisfied and hence we have proved (3.20). Moreover, from the proof of (3.20) and Fubini's theorem it follows that

$$\mathscr{E} \sum_{n=1}^{N} Z_{nN} = 0 . \tag{3.29}$$

Asymptotic normality of the A-terms (3.14) follows by a version of the central limit theorem due to Esseen (see Theorem 1, page 43 in Esseen (1945)), using (3.20), (3.29) and the fact that the $\sigma_N{}^2$ are given to be bounded away from zero for N sufficiently large.

4. Exact scores. Theorem 2.1 is an asymptotic result on rank statistics in the case where approximate scores (cf. (1.10)) are used. Clearly, a result like Theorem 2.1 also holds in the case where exact scores (cf. (1.9)) are used, provided condition (1.17) is satisfied. Assumption 4.1 is a strengthening of Assumption 2.1 which ensures that condition (1.17) holds.

ASSUMPTION 4.1 (generating functions). The generating functions satisfy Assumption 2.1 with 2.1(b) replaced by the assumption that the function J_i is continuous throughout (0, 1) and has a second derivative $J_i^{(2)}$ on $(0, 1)$-$\mathscr{D}_i$ for $i = 1, 2, \cdots, k$. In addition to Assumption 2.1(d) the second condition in (2.4) also holds for $\nu = 2$.

LEMMA 4.1. *Let for $n = 1, 2, \cdots, N$, $N = 1, 2, \cdots, i = 1, 2, \cdots, k$, the exact scores $a_{iN}^*(n)$ and the approximate scores $a_{iN}(n)$ be defined as in* (1.9) *and* (1.10) *respectively. Suppose that Assumption* 4.1 *is satisfied. Then, with probability one,*

$$N^{-\frac{1}{2}} \sum_{n=1}^{N} c_{nN} \,|\prod_{i=1}^{k} a_{iN}^*(R_{inN}) - \prod_{i=1}^{k} a_{iN}(R_{inN})| \to 0 \qquad as \quad N \to \infty , \tag{4.1}$$

uniformly in the continuous underlying df's $F_{1N}, F_{2N}, \cdots, F_{NN}$, $N = 1, 2, \cdots$.

PROOF. First we remark that

$$\begin{aligned} &\prod_{i=1}^{k} a_{iN}^*(R_{inN}) - \prod_{i=1}^{k} a_{iN}(R_{inN}) \\ &\quad = \sum_{i=1}^{k} \{\prod_{j=1}^{i-1} a_{jN}(R_{jnN})[a_{iN}^*(R_{inN}) - a_{iN}(R_{inN})] \prod_{j=i+1}^{k} a_{jN}^*(R_{jnN})\} . \end{aligned} \tag{4.2}$$

With the aid of Assumption 4.1, (1.9), (1.10), (1.11) and the remarks in Ruymgaart (1973), page 87, we find for every $i \in \{1, 2, \cdots, k\}$ that with probability one

$$\begin{aligned} &N^{-\frac{1}{2}} \sum_{n=1}^{N} c_{nN}|\prod_{j=1}^{i-1} a_{jN}(R_{jnN})[a_{iN}^*(R_{inN}) - a_{iN}(R_{inN})] \prod_{j=i+1}^{k} a_{jN}^*(R_{jnN})| \\ &\quad \leqq MN^{-\frac{1}{2}} \sum_{n=1}^{N} \left|J_{0N}\left(\frac{R_{0nN}}{N+1}\right)\right| \prod_{j=1;j\neq i}^{k} r^{a_j}\left(\frac{R_{jnN}}{N+1}\right) \\ &\qquad \times |a_{iN}^*(R_{inN}) - a_{iN}(R_{inN})| \\ &\quad \leqq MN^{-\frac{1}{2}} \sum_{n=1}^{N} \prod_{j=0;j\neq i}^{k} r^{a_j}\left(\frac{R_{jnN}}{N+1}\right) |a_{iN}^*(R_{inN}) - a_{iN}(R_{inN})| \\ &\quad \leqq MN^{-\frac{1}{2}} \sum_{n=1}^{N} \prod_{j=0;j\neq i}^{k} r^{a_j}\left(\frac{Q_{jnN}}{N+1}\right) |a_{iN}^*(n) - a_{iN}(n)| , \end{aligned} \tag{4.3}$$

where, for $j = 0, 1, \cdots, k$, $j \neq i$, $(Q_{j1N}, Q_{j2N}, \cdots, Q_{jNN})$ is a random permutation of $(1, 2, \cdots, N)$. From the derivation of (7.14) and from (7.25) in Chernoff and Savage (1958) it is clear that

$$|a_{iN}^*(1) - a_{iN}(1)| \leqq MN^{a_i}, \tag{4.4}$$

and, for $1 < n \leqq N/2$, that

$$|a_{iN}^*(n) - a_{iN}(n)| \leqq MN^{a_i}\left|\mathscr{N}\left(\frac{-n^{\frac{1}{2}}}{M}\right) + \frac{1}{N} + \frac{1}{n^{1+a_i}}\right| + \left|J_i\left(\frac{n}{N}\right) - J_i\left(\frac{n}{N+1}\right)\right|, \tag{4.5}$$

where the function $\mathscr{N}$ is defined in (2.1). Hence,

$$MN^{-\frac{1}{2}} \sum_{n=1}^{[N/2]} \prod_{j=0;j\neq i}^{k} r^{a_j}\left(\frac{Q_{jnN}}{N+1}\right) |a_{iN}^*(n) - a_{iN}(n)|$$
$$\leqq MN^{-\frac{1}{2}} \prod_{j=0;j\neq i}^{k} r^{a_j}\left(\frac{N}{N+1}\right) N^{a_i}$$
$$+ MN^{-\frac{1}{2}} \sum_{n=2}^{[N/2]} \prod_{j=0;j\neq i}^{k} r^{a_j}\left(\frac{N}{N+1}\right) N^{a_i}\left|\mathscr{N}\left(\frac{-n^{\frac{1}{2}}}{M}\right) + N^{-1} + n^{-1-a_i}\right|$$
$$+ MN^{-\frac{1}{2}} \sum_{n=2}^{[N/2]} \prod_{j=0;j\neq i}^{k} r^{a_j}\left(\frac{Q_{jnN}}{N+1}\right)\left|J_i\left(\frac{n}{N}\right) - J_i\left(\frac{n}{N+1}\right)\right|.$$

It is obvious that the first two terms in this expression converge to zero as N tends to infinity. Application of the mean value theorem shows that the last term is bounded above by

$$MN^{-\frac{3}{2}} \sum_{n=2}^{[N/2]} \prod_{j=0;j\neq i}^{k} r^{a_j}\left(\frac{Q_{jnN}}{N+1}\right) r^{a_i+1}\left(\frac{n}{N+1}\right),$$

which, in view of Lemma 2.4.3 in van Zuijlen (1976b), is bounded above by

$$MN^{-\frac{3}{2}} \sum_{n=1}^{N} r^{a+1}\left(\frac{n}{N+1}\right) \to 0 \quad \text{as} \quad N \to \infty .$$

By a symmetric argument we can cover the range $N/2 < n \leqq N$, so that (4.3) converges to zero as N tends to infinity. Combination of this with (4.2) completes the proof of (4.1). □

THEOREM 4.1. *Let an arbitrary triangular array of underlying* df's $F_{nN} \in \mathscr{F}_k$, $n = 1, 2, \cdots, N$, $N = 1, 2, \cdots$ *be given and let the generating functions satisfy Assumption* 4.1. *Then the quantities* μ_N *and* σ_N^2, *defined in* (2.10) *and* (2.11) *are finite. If, moreover,* $\liminf_{N\to\infty} \sigma_N^2 > 0$, *we have*

$$\sup_{-\infty<z<\infty} |P(N^{\frac{1}{2}}(T_N - \mu_N)/\sigma_N \leqq z) - \mathscr{N}(z)| \to 0 \quad \textit{as} \quad N \to \infty, \tag{4.6}$$

for T_N *as in* (1.3) *with* a_{iN} *replaced by* a_{iN}^* *defined in* (1.9), i.e., *the case of exact scores.*

PROOF. This is immediate from Theorem 2.1, Lemma 4.1 and the equality

$$N^{\frac{1}{2}}\sigma_N^{-1}(N^{-1}\sum_{n=1}^N c_{nN}\prod_{i=1}^k a_{iN}^*(R_{inN}) - \mu_N)$$
$$= N^{\frac{1}{2}}\sigma_N^{-1}(N^{-1}\sum_{n=1}^N c_{nN}\prod_{i=1}^k a_{iN}(R_{inN}) - \mu_N)$$
$$+ N^{\frac{1}{2}}\sigma_N^{-1}\sum_{n=1}^N c_{nN}[\prod_{i=1}^k a_{iN}^*(R_{inN}) - \prod_{i=1}^k a_{iN}(R_{inN})] ,$$

for N sufficiently large. □

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