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SEQUENTIAL BAHADUR EFFICIENCY

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The notion of Bahadur efficiency for test statistics is extended to the sequential case and illustrated in the specific context of testing one-sided hypotheses about a normal mean. An analog of Bahadur's theorem on the asymptotic optimality of the likelihood ratio statistic is seen to hold in the normal case. Some possible definitions of attained level for a sequential experiment are considered.

1. Introduction. Let $X_1, X_2, \cdots$ be a data sequence of i.i.d. (abstract) random variables whose distributions are indexed by a parameter θ, ranging in a parameter space Θ. We are interested in sequential tests of $H_0: \theta \in \Theta_0$ vs. $H_1: \theta \in \Theta_1 = \Theta - \Theta_0$, where Θ_0 is a nonempty proper subset of Θ. We shall be particularly concerned with the one-sided testing problem for a normal mean, by which is meant that under θ, the data are i.i.d. $N(\theta, 1)$ random variables, $\Theta = (-\infty, \infty)$ and $\Theta_0 = (-\infty, 0]$. Below, we suggest an extension to the sequential case of Bahadur's theory of the stochastic comparison of tests (more precisely, of test statistics) and of Bahadur efficiency. To that end, a brief synopsis of the non-sequential theory is perhaps in order (cf. Bahadur (1971)).

Let P_θ denote the (joint) distribution of the data sequence under θ and let $P_\theta{}^n$ denote the marginal distribution of $(X_1, \cdots, X_n)$. If, for $\theta, \omega \in \Theta$, we have $P_\theta{}^1 \ll P_\omega{}^1$, we let $l_1(\theta:\omega) = l(X_1; \theta:\omega) = \log(dP_\theta{}^1/dP_\omega{}^1)$ denote the indicated log-likelihood ratio statistic for X_1. Similarly, $l_n(\theta:\omega) = \sum_{j=1}^n l(X_j; \theta:\omega) = \log(dP_\theta{}^n/dP_\omega{}^n)$. We define the Kullback–Leibler information number

$$K(\theta;\omega) = \int l_1(\theta:\omega)\, dP_\theta \tag{1.1}$$

if $P_\theta{}^1 \ll P_\omega{}^1$; otherwise $K(\theta;\omega) = \infty$. We also let

$$K(\theta;\Theta_0) = \inf\{K(\theta;\omega) : \omega \in \Theta_0\}\,. \tag{1.2}$$

The stochastic comparison of tests applies to a sequence $\{T_n\}$ of real-valued test statistics (for testing H_0 vs. H_1), where T_n is $\mathscr{B}(X_1, \cdots, X_n)$ measurable and large values of T_n are significant. The attained level of T_n is then defined to be $L_n = G_n(T_n)$, where $G_n(x) = \sup\{P_\omega(T_n' \geqq x) : \omega \in \Theta_0\}$. We note that $G_n(\cdot)$ is always nonincreasing, hence L_n is measurable. The limit (in probability, if it exists) $r(\theta) = P_\theta - \lim_n [-\log L_n]/n$ will be called the Bahadur index, or simply the index, of the sequence $\{T_n\}$ at θ. (Apart from a factor of 2, this is Bahadur's

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exact slope.) The index provides a measure of efficiency in that if another sequence $\{T_n'\}$ has index $r'(\theta)$, to obtain a similar behavior of the attained level with n' observations (i.e., $L'_{n'} \sim L_n$), one must have $n'r'(\theta) \sim nr(\theta)$. Thus $r(\theta)/r'(\theta)$ is, in this sense, the asymptotic relative efficiency of $\{T_n\}$ to $\{T_n'\}$; in short, the Bahadur efficiency. Clearly, large indices are desirable. It follows from a theorem of Raghavachari (1970) (see also Bahadur (1971), Theorem 7.5) that for any such sequence $\{T_n\}$, one has

$$P_\theta - \limsup_n [-\log L_n]/n \leqq K(\theta; \Theta_0) . \tag{1.3}$$

(For a sequence of statistics $\{Y_n\}$, we write $P_\theta - \limsup_n Y_n \leqq Y$ to mean $\lim_n P_\theta(Y_n > Y + \delta) = 0$ for all $\delta > 0$. Clearly $P_\theta - \lim Y_n = Y$ iff $P_\theta - \limsup_n Y_n \leqq Y$ and $P_\theta - \liminf_n Y_n \geqq Y$.) A sequence $\{T_n\}$ for which equality holds in (1.3), for all $\theta \in \Theta_1$, is called optimal. Under general conditions, Bahadur (1967a, 1971) has shown that the likelihood ratio statistic (sequence)

$$\Lambda_n = \sup_{\theta \in \Theta_1} \inf_{\omega \in \Theta_0} l_n(\theta : \omega)$$

is optimal. For the one-sided normal testing problem, $K(\theta; \Theta_0) = \frac{1}{2}\theta^2$ and Λ_n is a strictly monotone function of $S_n/n^{\frac{1}{2}}$, $S_n = X_1 + \cdots + X_n$, which is optimal in this case.

In a sequential context, we retain a statistic sequence $\{T_n\}$ and adjoin to this a (randomized) stopping time N for the data sequence. Unless otherwise noted, we assume $P_\theta(N < \infty) = 1$ for all $\theta \in \Theta$. T_N then denotes the stopped value of the sequence $\{T_n\}$. Assuming still that large values are significant, we define the attained level of T_N to be

$$L_N = H(T_N) , \qquad \text{where} \quad H(x) = \sup \{P_\omega(T_N \geqq x) : \omega \in \Theta_0\} . \tag{1.4}$$

To allow for asymptotic considerations, we introduce a family of test sequences and stopping times, $\{\{T(n, a)\}, N_a\}$, indexed by the real parameter a. Here, for each a, $T(n, a)$, $n = 1, 2, \cdots$ is a sequence of test statistics and N_a is a stopping time. We require that for all $\theta \in \Theta$, $P_\theta - \lim_a N_a = \infty$. (Throughout, limits on a are taken as $a \to \infty$.) In many examples, a is a parameter of the stopping boundary defining N_a and the boundary moves out as a increases. Also, in many examples, $T(n, a)$ does not depend on a. We write $T(a) = T(N_a, a)$ and denote the corresponding attained level by $L(a)$. The Bahadur index of $T(a)$ at θ is then defined to be $P_\theta - \lim_a [-\log L(a)]/N_a$, provided this limit exists.

In the sequel, we show that (1.3) remains true in the sequential case (Theorems 2.1 and 2.2). Moreover, for the normal testing problem, we exhibit (Section 3) a sequence $\{T_n\}$ which is optimal for *all* families of stopping times. (The statistic is a modification of the likelihood ratio statistic $S_n/n^{\frac{1}{2}}$.) We also indicate connections between our results and some putative definitions of attained level for a sequential experiment in Section 4. Connections with power curves are discussed in Section 5.

2. Bounds for the Bahadur index. In this section, we give two extensions of

Raghavachari's theorem to the sequential case. Theorem 2.1 requires only that $P_\theta - \lim_a N_a = \infty$ and provides a bound for limits in probability of $[-\log L(a)]/N_a$. With a condition somewhat stronger than $P_\theta(N_a \to \infty) = 1$, Theorem 2.2 provides an analogous pointwise result. It is shown by example that the hypothesis of Theorem 2.2 cannot be weakened to $P_\theta(N_a \to \infty) = 1$.

THEOREM 2.1. *Let $L(a)$ be the attained level of $T(a)$, the stopped value of $\{T(n, a)\}$ for the stopping time N_a. Then, provided $P_\theta - \lim_a N_a = \infty$,*

$$P_\theta - \limsup_a [-\log L(a)]/N_a \leqq K(\theta; \Theta_0) . \tag{2.1}$$

PROOF. We suppose first that Θ_0 is simple, say $\Theta_0 = \{\omega\}$, so that P_ω denotes the null distribution of the data sequence. We fix $\theta \in \Theta$ and write $K = K(\theta; \omega)$. We may suppose $K < \infty$, since otherwise (2.1) is trivially true. We then write $l_1(\theta) = l_1(\theta : \omega)$, etc. We also drop the affix a, writing $N = N_a$, etc. Let $H(x) = P_\omega(T \geqq x) = \sum_n P_\omega(N = n, T_n \geqq x)$. Thus $L = H(T)$. Let $B = (l_N(\theta) < N(K + \delta))$. By the SLLN, $P_\theta(\lim_n l_n/n = K) = 1$ and hence $P_\theta - \lim_a l_N/N = K$. Thus $\lim_a P_\theta B = 1$ for $\delta > 0$. Letting B^c denote the complement of B, for a fixed positive integer k and arbitrary $\delta > 0$, we have

$$P_\theta(L < e^{-N(K+2\delta)}) \leqq P_\theta(N < k) + P_\theta B^c + P_\theta D_k , \tag{2.2}$$

where $D_k = (N \geqq k, L < e^{-N(K+2\delta)}, B)$. We note that for $n \geqq k$, $E_n = (D_k, N = n) = (N = n, H(T_n) < e^{-n(K+2\delta)}, l_n < n(K + \delta))$. We obtain a bound for $P_\theta E_n$.

$$\begin{aligned} P_\theta E_n &= \textstyle\int_{E_n} \exp\{l_n(\theta : \omega)\}\, dP_\omega \\ &\leqq \exp\{n(K + \delta)\}P_\omega E_n \leqq \exp\{n(K + \delta)\}P_\omega(L < e^{-n(K+2\delta)}) \leqq e^{-n\delta} ; \end{aligned} \tag{2.3}$$

the last inequality follows from a result of Bahadur (1971, Theorem 7.4) which says that under any null distribution, an attained level statistic is superuniform on $[0, 1]$. On summing (2.3) over $n \geqq k$, we obtain

$$P_\theta D_k = \textstyle\sum_{n\geqq k} P_\theta E_n \leqq e^{-k\delta}/(1 - e^{-\delta}) . \tag{2.4}$$

We then see from (2.2) that

$$P_\theta(L < e^{-N(K+2\delta)}) \leqq P_\theta(N < k) + P_\theta B^c + e^{-k\delta}/(1 - e^{-\delta}) . \tag{2.5}$$

Letting $a \to \infty$ and then $k \to \infty$ and recalling that $P_\theta - \lim_a N = \infty$ and $\lim_a P_\theta B^c = 0$, it follows that $\lim_a P_\theta([-\log L]/N > K + 2\delta) = 0$ for any $\delta > 0$, which entails (2.1). The case of composite Θ_0 follows immediately since $L \geqq L(a, \omega)$ for $\omega \in \Theta_0$, where $L(a, \omega)$ is the attained level of T for the simple null hypothesis $\{\omega\}$. Thus we have

$$P_\theta - \limsup_a [-\log L]/N \leqq P_\theta - \limsup_a [-\log L(a, \omega)]/N \leqq K(\theta, \omega)$$

for all $\omega \in \Theta_0$, which entails (2.1). □

Raghavachari's theorem is actually a pointwise result for $[-\log L_n]/n$. A corresponding sequential version is given in the next theorem. The framework is the same as for Theorem 1, except that $\{N_a\}$ is replaced by a *sequence* of stopping times.

THEOREM 2.2. *Let N_k, $k = 1, 2, \cdots$ be a sequence of stopping times for which the following hold: there is a sequence of positive integers $n_k = n_k(\theta)$ such that*

$$P_\theta \bigcup_{k \geqq m} (N_k < n_k) \to 0 \quad \text{as} \quad m \to \infty \qquad \text{and} \tag{2.6}$$

$$\sum_k e^{-\delta n_k} < \infty \qquad \text{for all} \quad \delta > 0 . \tag{2.7}$$

Then, letting $L(k)$ denote the attained level for $T(k) = T(N_k, k)$,

$$P_\theta(\limsup_k [-\log L(k)]/N_k \leqq K(\theta; \Theta_0)) = 1 . \tag{2.8}$$

REMARK. (2.7) entails that $n_k \to \infty$, hence (2.6) is somewhat stronger than the requirement that $P_\theta(N_k \to \infty) = 1$. In fact, (2.6) $\Rightarrow P_\theta(\liminf_k N_k/n_k \geqq 1) = 1 \Rightarrow P_\theta(N_k < n_k(1 - \varepsilon)$ i.o.$) = 0$ for any $\varepsilon > 0$. Hence (2.6) is equivalent to the existence of $\{m_k\}$ for which $P_\theta(\liminf_k N_k/m_k \geqq 1) = 1$. (In many examples, one has the stronger condition that $P_\theta(N_k/n_k \to 1) = 1$.) One sees that (2.7) holds if for $i \neq k$, $n_i \neq n_k$ and in particular, if $n_k < n_{k+1}$ for all k. The hypothesis thus holds for the nonsequential case treated by Raghavachari: $N_k = k$. The hypothesis also holds if $N_k < N_{k+1}[P_\theta]$ for all k.

PROOF OF THEOREM. As in the proof of Theorem 2.1, we again assume first that Θ_0 is simple and that $K = K(\theta; \Theta_0) < \infty$. Let $B_m = \bigcap_{k \geqq m} (l_{N_k} < (K + \delta)N_k)$. Since $P_\theta(N_k \to \infty) = 1$, $\lim_m P_\theta B_m{}^c = 0$ by the SLLN. Then

$$\begin{aligned} P_\theta \bigcup_{k \geqq m} (L(k) < e^{-(K+2\delta)N_k}) \\ \leqq P_\theta \bigcup_{k \geqq m} (N_k < n_k) + P_\theta B_m{}^c + \sum_{k \geqq m} P_\theta D_k , \end{aligned} \tag{2.9}$$

where $D_k = (N_k \geqq n_k, L(k) < e^{-(K+2\delta)N_k}, B_m)$. As in Theorem 2.1 (cf. (2.4)), it follows that $P_\theta D_k \leqq e^{-\delta n_k}/(1 - e^{-\delta})$, hence

$$\begin{aligned} P_\theta \bigcup_{k \geqq m} (L(k) < e^{-(K+2\delta)N_k}) \\ \leqq P_\theta \bigcup_{k \geqq m} (N_k < n_k) + P_\theta B_m{}^c + (1 - e^{-\delta})^{-1} \sum_{k \geqq m} e^{-\delta n_k} . \end{aligned} \tag{2.10}$$

On letting $m \to \infty$, it follows from the hypothesis that the RHS of (2.10) tends to zero. Thus for any $\delta > 0$, $P_\theta([-\log L(k)]/N_k > K + 2\delta$ i.o.$) = 0$, which entails (2.8) for Θ_0 simple. As in Theorem 1, the result for Θ_0 composite then follows directly. □

An examination of the proof of Theorem 2.2 shows that it utilizes in an essential way the discreteness of the indexing parameter k. It is not clear if there is a satisfactory analog of Theorem 2.2 for a generally indexed collection $\{N_a\}$. This suggests that it is more appropriate to define indices via probability limits rather than pointwise limits; cf. Bahadur (1960a). This contention is reinforced by the following example, which shows that the conclusion of Theorem 2.2 can fail in a spectacular way if conditions (2.6) and (2.7) do not hold. We assume here the framework of the one-sided normal testing problem.

EXAMPLE. For $n \geqq 1$, let $T_{2n} = S_{2n}/(2n)^{\frac{1}{2}}$ and $T_{2n+1} = \infty$. We define a sequence of stopping times, $\{N_k\}$: for $n = 1, 2, \cdots$, let $b_n = e^{n^2}$ and let $\{C_k: b_n < k < b_{n+1}\}$ be a partition of $(-\infty, \infty)$, such that $P_0(S_{2n} \in C_k) \leqq e^{-n^2}$. This can be done since $b_{n+1} - b_n > e^{n^2}$. Then for $b_n < k < b_{n+1}$ and $n \geqq 1$, let

$$\begin{aligned} N = N_k &= 2n && \text{if} \quad S_{2n} \notin C_k \\ &= 2n + 1 && \text{if} \quad S_{2n} \in C_k . \end{aligned}$$

Clearly $P_\theta(N_k \to \infty) = 1$ for all $\theta \in \Theta$ and

$$\begin{aligned} T_N = T_{N_k} &= S_{2n}/(2n)^{\frac{1}{2}} && \text{if} \quad S_{2n} \notin C_k \\ &= \infty && \text{if} \quad S_{2n} \in C_k . \end{aligned}$$

To compute $L(k)$, the attained level of T_N, we note that for $b_n < k < b_{n+1}$ $\sup\{P_\omega(T_N \geqq x): \omega \leqq 0\} \leqq \sup\{P_\omega(S_{2n}/(2n)^{\frac{1}{2}} \geqq x): \omega \leqq 0\} + \sup\{P_\omega(S_{2n} \in C_k): \omega \leqq 0\} \leqq e^{-\frac{1}{2}x^2} + e^{-n^2} \leqq 2e^{-(\frac{1}{2}x^2 \wedge n^2)}$. Thus for $b_n < k < b_{n+1}$, $[-\log L(k)]/N \geqq \frac{1}{2}[(T_N{}^2/N) \wedge n] - n^{-1}$. However, $T_N = \infty$ for exactly one k in the range $b_n < k < b_{n+1}$. Hence $\sup\{[-\log L(k)]/N: b_n < k < b_{n+1}\} \geqq \frac{1}{2}n - n^{-1}$ (w.p. 1, under any $\theta \in \Theta$), so that

$$P_\theta(\limsup_k [-\log L(k)]/N = \infty) = 1 , \qquad \text{all} \quad \theta \in \Theta , \tag{2.11}$$

contrary to (2.8).

Note that, (2.11) notwithstanding,

$$P_\theta - \lim_k [-\log L(k)]/N = P_\theta - \lim_k \tfrac{1}{2}T_N{}^2/N = P_\theta - \lim_k \tfrac{1}{2}S_N{}^2/N^2 = \tfrac{1}{2}\theta^2 ,$$

so that $\{T_N\}$ has (maximal) index $\frac{1}{2}\theta^2$ at θ. It is easily seen from the above that the corresponding lim inf in (2.11) is $\frac{1}{2}\theta^2$ w.p. 1. This example thus displays a kind of superefficiency. The example rests heavily on the fact that N_k is essentially constant over long intervals of k-values, a possibility excluded by (2.6) and (2.7). The use of improper random variables is not essential here. One may, instead, map $(-\infty, \infty)$ into $(0, 1)$ in an order-preserving way and replace ∞ by 1 in the above example.

3. An optimal statistic. As cited above, Bahadur (1967a) showed under general conditions that the likelihood ratio statistic is optimal. It might thus be supposed that this result carries over to the sequential case. We show below (Theorem 3.1) for the one-sided normal testing problem that in a sense, this is so. More specifically, we exhibit a sequence $\{T_n\}$ which has the property that for *every* collection $\{N_a\}$ of stopping times for which $P_\theta - \lim_a N = \infty$, T_N has maximal index $\frac{1}{2}\theta^2$ under P_θ, $\theta > 0$. We also show by examples that none of the three likely candidates S_n, the likelihood ratio statistic $S_n/n^{\frac{1}{2}}$ and S_n/n are universally optimal in this sense. Counterexamples for S_n and $S_n/n^{\frac{1}{2}}$ are discussed after Theorem 3.1 and one for S_n/n is given in Section 4.

The following result applies to the one-sided normal testing problem.

THEOREM 3.1. *Let* $T_n = S_n/n^{\frac{1}{2}} - c_n$, *where, for some* $\delta > 0$, $(2(1 + \delta)\log n)^{\frac{1}{2}} \leqq c_n = o(n^{\frac{1}{2}})$. *Then for any collection* $\{N_a\}$ *of stopping times for which* $P_\theta - \lim_a N_a = \infty$, T_{N_a} *has exact slope* $\frac{1}{2}\theta^2$ *under* P_θ, $\theta > 0$.

PROOF. We drop the affix a, writing $N = N_a$, etc. For $x > 0$,

$$\begin{aligned}\sup\{P_\omega(T_N \geqq x) : \omega \leqq 0\} &= \sup\{P_\omega(S_N/N^{\frac{1}{2}} - c_N \geqq x) : \omega \leqq 0\} \\ &\leqq \textstyle\sum_{n=1}^\infty \sup\{P_\omega(S_n/n^{\frac{1}{2}} - c_n \geqq x) : \omega \leqq 0\} \\ &= \textstyle\sum_{n=1}^\infty P_0(S_n/n^{\frac{1}{2}} \geqq x + c_n) \leqq \sum_{n=1}^\infty \exp\{-\frac{1}{2}(x^2 + c_n^2)\} \\ &\leqq e^{-\frac{1}{2}x^2} \textstyle\sum_n n^{-(1+\delta)} = C(\delta)e^{-\frac{1}{2}x^2} .\end{aligned}$$

Letting L denote the attained level of T_N, it follows that $-\log L \geqq \frac{1}{2}(S_N/N^{\frac{1}{2}} - c_N)^2 - \log C(\delta)$ on $(S_N > N^{\frac{1}{2}}c_N)$, so that $P_\theta - \liminf_a [-\log L]/N \geqq P_\theta - \lim_a \{\frac{1}{2}(S_N/N - c_N/N^{\frac{1}{2}})^2 - N^{-1} \log C(\delta)\} = \frac{1}{2}\theta^2$ since $P_\theta - \lim_a c_N/N^{\frac{1}{2}} = 0$ and $P_\theta(S_N > N^{\frac{1}{2}}c_N) \to 1$. By Theorem 2.1, $P_\theta - \limsup_a [-\log L]/N \leqq \frac{1}{2}\theta^2$. Thus T_N has (maximal) exact slope $\frac{1}{2}\theta^2$ under P_θ. □

REMARK. The estimate used to establish the theorem is very crude and it is conjectured that $\{T_n\}$ remains optimal for all $c_n = o(n^{\frac{1}{2}})$ for which $n^{\frac{1}{2}}c_n$ is in the upper class of LIL (for S_n). By a similar argument, it can also be shown that another modification of $S_n/n^{\frac{1}{2}}$ that renders it optimal for all stopping times is $(S_n/n^{\frac{1}{2}})I(S_n > n^{\frac{1}{2}}c_n)$, where c_n satisfies the conditions given in Theorem 3.1.

We show next by example that $S_n/n^{\frac{1}{2}}$ need not be optimal. Heuristically, this may be seen by considering the stopping time $N = \inf\{n : |S_n| \geqq (an)^{\frac{1}{2}}\}$. Neglecting overshoot, $S_N/N^{\frac{1}{2}}$ assumes only the values $-a^{\frac{1}{2}}$ and $a^{\frac{1}{2}}$, having corresponding attained levels 1 and $\frac{1}{2}$. Since $P_\theta - \lim_a N = \infty$, the exact slope of $S_N/N^{\frac{1}{2}}$ is zero. (The argument becomes rigorous if S_n is replaced by a continuous-time Wiener process.) A rigorization of the above heuristics for discrete time appears to be very delicate. Instead, we consider another stopping boundary for which the analysis is simpler. We introduce the following notation. For $x > 0$, let $e_0(x) = l_0(x) = x$, $l_{k+1}(x) = \max\{\log l_k(x), 1\}$ and $e_{k+1}(x) = \exp\{e_k(x)\}$, $k = 0, 1, \cdots$. Thus l_k and e_k are inverse to each other on $[e_k(1), \infty)$. Let

$$N = \inf\{n : S_n \geqq (anl_3(n))^{\frac{1}{2}}\} . \tag{3.1}$$

By the SLLN and LIL, $P_\theta(N < \infty) = 1$, all $\theta \geqq 0$. (It would perhaps appear more natural, in the testing context, to define N by a two-sided boundary, with $|S_n|$ replacing S_n in (3.1). This requires no essential change in the following argument.) We note that a reasonable sequential test can be based on the data sequence stopped by N. One may, for example, reject H_0 for large values of $S_N/N^{\frac{1}{2}} - \log N$ (cf. Theorem 3.1).

PROPOSITION 3.2. *With N as in* (3.1), *$S_N/N^{\frac{1}{2}}$ has exact slope zero under all $\theta > 0$.*

PROOF. We use the following crude estimate:

$$P_0(N \leqq n) \leqq \textstyle\sum_{k=1}^n P_0(S_k/k^{\frac{1}{2}} \geqq (al_3(k))^{\frac{1}{2}}) \leqq ne^{-\frac{1}{2}a} . \tag{3.2}$$

We next estimate the tail of the statistic $S_N/(aN)^{\frac{1}{2}}$. Since $S_N/(aN)^{\frac{1}{2}} \geqq [l_3(N)]^{\frac{1}{2}}$, we have, for $x > 1$,

$$\begin{aligned}\sup\{P_\omega(S_N/(aN)^{\frac{1}{2}} \geqq x) : \omega \leqq 0\} &\geqq P_0(S_N/(aN)^{\frac{1}{2}} \geqq x) \geqq P_0(l_3(N) \geqq x^2) \\ &= P_0(N \geqq e_3(x^2)) \geqq 1 - e^{-\frac{1}{2}a}e_3(x^2)\end{aligned}$$

by (3.2). It will follow that L, the attained level of $S_N/N^{\frac{1}{2}}$ satisfies

$$P_\theta(L \to 1) = 1 , \qquad \theta > 0 , \tag{3.3}$$

if we show that

$$P_\theta(e^{-\frac{1}{2}a} e_3(S_N^2/aN) \to 0) = 1 , \qquad \theta > 0 . \tag{3.4}$$

To establish (3.4), we note first that

$$0 \leqq S_N - [aNl_3(N)]^{\frac{1}{2}} \leqq X_N . \tag{3.5}$$

It is an easy consequence of the Borel–Cantelli lemmas that for normal variables with unit variance,

$$P_\theta(\limsup_n X_n/(2 \log n)^{\frac{1}{2}} = 1) = 1 , \qquad \text{all} \quad \theta , \tag{3.6}$$

hence

$$X_N = O([\log N]^{\frac{1}{2}}) . \tag{3.7}$$

(For random variables Y_a and Z_a, we write $Z_a = O(Y_a)$ to mean $\limsup_a |Z_a/Y_a| < \infty$ w.p. 1. A corresponding meaning is given to the expression $Z_a = o(Y_a)$.) On dividing across by N in (3.5) and noting that $P_\theta(S_N/N \to \theta) = 1$, we see that $P_\theta(N/al_3(N) \to 1/\theta^2) = 1$, hence that

$$P_\theta(N/al_3(a) \to 1/\theta^2) = 1 , \qquad \theta \geqq 0 . \tag{3.8}$$

It follows from (3.5)—(3.8) that

$$S_N^2/(aN)^{\frac{1}{2}} = l_3(N) + O([l_1(a)]^{\frac{1}{2}}/a) . \tag{3.9}$$

On taking exponentials, we see that on $N > e_3(1)$,

$$\begin{aligned} e_1(S_N^2/aN) &= l_2(N)[1 + O([l_1(a)]^{\frac{1}{2}}/a)] \\ &= l_2(N) + O(l_2(a)[l_1(a)]^{\frac{1}{2}}/a) , \end{aligned}$$

hence that

$$\begin{aligned} e_2(S_N^2/aN) &= l_1(N) + O(l_2(a)[l_1(a)]^{\frac{3}{2}}/a) \qquad \text{and} \\ e_3(S_N^2/aN) &= N[1 + o(1)] . \end{aligned} \tag{3.10}$$

It follows from (3.10) and (3.8) that (3.4) and hence (3.3) hold. A fortiori, $S_N/N^{\frac{1}{2}}$ has exact slope zero under all $\theta > 0$. □

The next example shows that S_N need not be optimal. The no-overshoot heuristics are similar to those for $S_N/N^{\frac{1}{2}}$ above. Let

$$N = N_a = \inf\{n : |S_n| \geqq a\} . \tag{3.11}$$

The following result implies that S_N has index zero. Let $L(a)$ denote the attained level for S_N.

PROPOSITION 3.3. *As* $a \to \infty$,

$$\log L(a) = o_p(N_a) . \tag{3.12}$$

REMARK. Note that $N_a \to \infty$. In fact, from Theorem 2.1 of Berk (1973), we have for $\theta > 0$ that

$$\lim_a N_a/a = 1/\theta \quad [P_\theta]. \tag{3.13}$$

We establish (3.12) with the aid of the following lemmas.

LEMMA 3.4. *Let $N = N_a$ be as in* (3.11). *Let Φ denote the* $N(0, 1)$ df. *For* $x > 0$,

$$P_0(S_N \geqq a + x) \geqq (1 - \Phi[x + o(\log a)])e^{0(1)}. \tag{3.14}$$

PROOF. Fix a positive $k < a$. Then

$$\begin{aligned} P_0(S_N \geqq a + x) &\geqq P_0(S_{N-1} \geqq a - k, X_N \geqq x + k) \\ &= \textstyle\sum_n P_0(N \geqq n, S_{n-1} \geqq a - k, X_n \geqq x + k) \\ &= \textstyle\sum_n [1 - \Phi(x + k)]P_0(N \geqq n, S_{n-1} \geqq a - k) \\ &\geqq [1 - \Phi(x + k)] \textstyle\sum_n P_0(N = n, S_{n-1} \geqq a - k) \\ &= [1 - \Phi(x + k)]P_0(S_{N-1} \geqq a - k) \\ &\geqq [1 - \Phi(x + k)]P_0(X_N \leqq k, S_N \geqq a) \\ &= [1 - \Phi(x + k)][P_0(S_N \geqq a) - P_0(X_N > k)]. \end{aligned}$$

We note next that $P_0(X_N > k) \leqq E_0 \sum_1^N I(X_j > k) = E_0 N P_0(X_1 > k) \leqq e^{-\frac{1}{2}k^2} E_0 N$. We have from Theorem 2.4 of Berk (1973) that $E_0 N = O(a^2)$, hence, taking $k = (6 \log a)^{\frac{1}{2}}$, $P_0(X_N > k) = O(a^{-1})$. The desired conclusion then follows from the fact that $P_0(S_N \geqq a) = \frac{1}{2}$. □

LEMMA 3.5. *Let N be as in* (3.11). *Then for all θ, $E_\theta(S_N - a \mid S_N \geqq a) = O(1)$, hence $(S_N - a)I(S_N \geqq a) = O_p(1)$ for all $\theta \geqq 0$.*

PROOF. A familiar argument, due originally to Wald (1947) gives $E_\theta(S_N - a \mid S_N \geqq a) = E_\theta(X_N - (a - S_{N-1}) \mid X_N \geqq a - S_{N-1}) \leqq \sup\{E_\theta(X_1 - u \mid X_1 \geqq u): 0 \leqq u \leqq a + b\} = E_\theta(X_1 \mid X_1 \geqq 0) = E_0(X_1 + \theta \mid X_1 \geqq -\theta) = w(\theta)$, say. Clearly $w(\theta) < \infty$ and does not depend on a, which entails the desired conclusion. □

REMARK. For the stopping time $\tau = \tau_a = \inf\{n: S_n \geqq a\}$, it follows from known results in renewal theory (see, e.g., Feller (1971), page 371) that $S_\tau - a$ has a proper limit law as $a \to \infty$ and is a fortiori $O_p(1)$. This entails the second part of the lemma since on $(S_N \geqq a)$, $S_N = S_\tau$. (In fact, for $\theta > 0$, $S_N - a$ and $S_\tau - a$ have the same limit law for then $\lim_a P_\theta(S_N \geqq a) = 1$.)

PROOF OF PROPOSITION 3.3. We obtain (3.12) by noting that $\sup\{P_\omega(S_N \geqq a + x): \omega \leqq 0\} \geqq P_0(S_N \geqq a + x)$, so that Lemma 3.4 entails $-\log L(a) \leqq \frac{1}{2}[S_N - a + o(\log a)^2] + O(1)$. The desired conclusion then follows from Lemma 3.5 and (3.13). □

4. Bahadur efficiency and sequential experiments.

We consider here, in the context of the one-sided testing problem for a normal mean, possible definitions of the attained level of a sequential experiment. For us, the term "sequential experiment" is synonymous with "stopping time." In speaking of the attained level of an experiment, one has tacitly in mind, of course, an appropriate null hypothesis. We suppose that the stopping time is defined by a well-behaved (say, convex) continuation region in the (n, S_n) plane. A point in this plane is denoted by (n, s). In the nonsequential case, when the stopping time is $N \equiv n$, the stopping boundary is a vertical line at n and the second coordinate orders the boundary, providing a measure of significance or attained level for the possible outcomes (with $s = \infty$ the most significant value and $s = -\infty$ the least significant). It seems natural to try to similarly order other stopping boundaries and analogously define attained levels for the outcomes of a sequential experiment. For example, consider the triangular (Anderson) boundary corresponding to $N = \inf\{n: |S_n| \geqq a - n\}$, the upper boundary being the segment of the line $s = a - n$ between $(1, a - 1)$ and $(a, 0)$. Intuitively (neglecting overshoot), the portion of the boundary near $(1, a - 1)$ contains the most significant outcomes (outcomes that speak most against the null hypothesis $\theta \leqq 0$) while the boundary near $(1, 1 - a)$ contains the least significant outcomes. It seems natural to order the stopping boundary with the outcome $(1, a - 1)$ having the smallest attained level, proceeding clockwise around to the outcome $(1, 1 - a)$, which then has attained level one. The actual attained level of a point $(n, a - n)$ on the upper boundary, say, is the (maximum) null probability of stopping on the portion of the boundary between $(1, a - 1)$ and $(n, a - n)$. In this case, this is just the null probability (neglecting overshoot) that $S_N/N \geqq (a - n)/n$, so that this notion of attained level for the sequential experiment is just the attained level of S_N/N. This reasoning implicitly uses the idea that under θ, (n, S_n) tends to move along the ray $s = n\theta$, so that one expects large values of S_N/N under the alternative hypothesis.

These considerations apply as well to other familiar stopping rules defined by an upper and a lower stopping boundary, the SPRT for example. (For open boundaries such as the SPRT or the square-root boundary mentioned in Section 3, the ordering still proceeds clockwise starting from $n = 1$ on the upper boundary. However, one must make an imaginary transition at $n = \infty$ to reach the lower boundary.) For these stopping times (still neglecting overshoot), the attained level of the point (n, s) on the upper boundary is also the null probability of exiting at the upper boundary with $N \leqq n$. Thus another possible definition of attained level for the stopping time is the attained level of the statistic N, small values being significant (and subject to the proviso that one exits at the upper boundary). If overshoot is not neglected, the two definitions do not quite coincide, except asymptotically. We indicate below (Theorem 4.1) a class of boundaries for which this notion of attained level is optimal, in that the corresponding exact slope is maximal. Our considerations apply to S_N/N, though there are similar results for N. We also show by example that S_N/N is not optimal in every case, even for a convex continuation region whose boundary permits a clockwise ordering as discussed above.

The following gives conditions under which S_N/N is optimal. We consider continuation regions whose upper halves have asymptotic shapes: on homothetically contracting by a factor a, the contracted regions tend to a limit.

THEOREM 4.1. *Let* $N = \inf\{n: S_n \notin (-af_a(n/a), ag_a(n/a))\}$, *where the nonnegative boundary curves* f_a *and* g_a *satisfy the following*:

(i) *For* $n = 1, 2, \cdots$, $\lim_a af_a(n/a) = \infty$ *and* $\lim_{x\to\infty} f_a(x)/x = 0$.

(ii) $g_a^2(x)/x$ *is nonincreasing in* x.

(iii) $R_a(x) = g_a(x)/x$ *decreases with* a *to a continuous strictly decreasing limit* $R(x)$, *where* $R(0+) > 0$.

Then L, *the attained level of* S_N/N *satisfies*

$$P_\theta - \lim_a [-\log L]/N = \tfrac{1}{2}\theta^2\,, \qquad \theta > 0\,.$$

REMARK. Here is an example of a continuation region satisfying the hypothesis of the theorem. For $b > 0$ (which will depend on a in a manner specified below), let $N = \inf\{n: |S_n| \geqq (bn\log n)^{\frac{1}{2}}\}$. Setting $n/a = x$ and $ag_a(x) = (bn\log n)^{\frac{1}{2}}$, $g_a(x) = (bxa^{-1}[\log a + \log x])^{\frac{1}{2}}$. If we now set $b = a/\log a$, we see that $g_a(x) \to x^{\frac{1}{2}}$. Thus the asymptotic shape of a $(n\log n)^{\frac{1}{2}}$ boundary is a "root-n" boundary.

PROOF. If follows from the hypothesis that $R_a(\cdot)$ decreases continuously to zero. Let $\eta_a \doteq R_a^{-1}$. Then η_a is increasing and decreases with a to $\eta = R^{-1}$. For $\theta > 0$, we have

$$\lim_a P_\theta(S_N \geqq ag_a(N/a)) = 1\,, \tag{4.1}$$

which, together with the relation $S_{N-1} < ag_a(N/a - 1/a)$ entails (on dividing across by N) $P_\theta(\lim_a R_a(N/a) = \theta) = 1$. The hypotheses insure that R_a converges uniformly to R. Thus $P_\theta(\lim_a R(N/a) = \theta) = 1$ or

$$P_\theta(N/a \to \eta(\theta)) = 1\,. \tag{4.2}$$

Next we note that for $x > 0$, letting $g(x) = xR(x)$,

$$\begin{aligned}
&\sup\{P_\omega(S_N \geqq Nx): \omega \leqq 0\}\\
&\quad \leqq \textstyle\sum_n P_0(S_n \geqq nx \vee ag(n/a))\\
&\quad \leqq \textstyle\sum_{n\leqq a\eta(x)} P_0(S_n \geqq ag(n/a)) + \sum_{n>a\eta(x)} P_0(S_n \geqq nx)\\
&\quad \leqq \textstyle\sum_{n\leqq a\eta(x)} \exp\{-\frac{1}{2}a^2g^2(n/a)/n\} + \sum_{n>a\eta(x)} \exp\{-\frac{1}{2}nx^2\}\\
&\quad = \exp\{-\tfrac{1}{2}ag^2[\eta(x)]/\eta(x)[1+o(1)]\} + \exp\{-\tfrac{1}{2}a\eta(x)x^2[1+o(1)]\}\\
&\quad = \exp\{-\tfrac{1}{2}a\eta(x)x^2[1+o(1)]\}\,,
\end{aligned} \tag{4.3}$$

since $g[\eta(x)]/\eta(x) = x$. It follows that under $\theta > 0$, L, the attained level of S_N/N satisfies

$$-\log L \geqq \tfrac{1}{2}a(S_N/N)^2\eta(S_N/N)[1+o(1)]\,. \tag{4.4}$$

Since $P_\theta(S_N/N \to \theta) = 1$, it follows from (4.2) that S_N/N has index $\frac{1}{2}\theta^2$ under P_θ. □

The following example shows that the above intuitive notion of attained level for a sequential experiment is not always optimal. (Alternatively, S_N/N is not always an optimal statistic.) Let $N = \inf\{n: S_n \geqq a^{\frac{1}{3}} n^{\frac{2}{3}} \text{ or } n \geqq a^2\}$. The truncation assures that $P_\theta(N < \infty) = 1$ for all θ. It follows as in Theorem 4.1 that

$$N/a \to 1/\theta^3 = \eta(\theta)\,, \qquad \theta > 0\,. \tag{4.5}$$

(Asymptotically, the truncation has no effect for $\theta > 0$.) Moreover, for $x > 0$,

$$\sup\{P_\omega(S_N \geqq Nx) : \omega \leqq 0\} \geqq P_0(S_N \geqq Nx) \geqq P_0(X_1 \geqq a^{\frac{1}{3}}) = e^{-\frac{1}{2}a^{\frac{2}{3}}}[1 + o(1)]\,.$$

Thus L, the attained level of S_N/N satisfies $-\log L = O(a^{\frac{2}{3}})$, which, in view of (4.5) implies that S_N/N has exact slope zero under all P_θ. In the above example, we get the same result for any limiting boundary curve $g(x)$ for which $g^2(x)/x$ increases to ∞ as $x \uparrow \infty$. It follows from Theorem 3.1 that there are still optimal statistics in such cases: $S_N/N^{\frac{1}{2}} - \log N$, for example. In fact, it may be verified that $S_N/N^{\frac{1}{2}}$ is optimal here.

Theorem 4.1 suggests that Bahadur efficiency is not helpful for distinguishing among stopping times of interest. If the above intuitive notion of attained level of a sequential experiment is adopted, all of the stopping times subsumed by Theorem 4.1 seem equally efficient. If, instead of S_n/n, one uses $T_n = S_n/n^{\frac{1}{2}} - \log n$ to define the attained level of a sequential experiment, by Theorem 3.1 *all* stopping times have (absolute and relative) efficiency one. One reaches the same conclusion if the index of a sequential experiment is taken to be the supremum of all indices attained by statistics defined on the stopped data sequence. The relevance of this for sequential testing is not entirely clear. At any rate, the notions of Hodges–Lehmann and Chernoff efficiency do serve to distinguish among sequential tests; see Berk (1976).

5. Some connections with significance levels. The Bahadur index we have been considering is more properly called the stochastic Bahadur index. An allied notion, a nonstochastic Bahadur index, provides a measure of efficiency for critical regions; cf. Bahadur (1960a, b). In the nonsequential case, for critical regions defined by a sequence of test statistics $\{T_n\}$, this latter notion is the following: one selects critical values $c_n = c_n(\theta)$ so that $\beta < P_\theta(T_n \geqq c_n) < 1 - \beta$ for some $\beta \in (0, \frac{1}{2})$ and all n. Then, letting $\alpha_n = \alpha_n(\theta) = \sup\{P_\omega(T_n \geqq c_n) : \omega \in \Theta_0\}$ be the associated size, the nonstochastic index for (the critical regions defined by) T_n is

$$\rho(\theta) = \lim_n -\frac{1}{n}\log\alpha_n\,, \tag{5.1}$$

provided this limit exists.

Judging from the loose way in which the term "Bahadur efficiency" is used in the literature, the distinction between stochastic and nonstochastic indices is not always uppermost in people's minds. Happily, the two notions often coincide. Indeed, Bahadur (1967b, Proposition 11) has established a result of this nature in the nonsequential case: if $\{T_n\}$ has stochastic index $r(\theta)$ which is constant w.p. 1, then $\rho(\theta) = r(\theta)$. Some authors (Klotz (1965), Stone (1968), Tsutakawa (1968)) have computed only nonstochastic indices. In such cases, it would be of interest to know that this is also the stochastic index. One can, of course, verify the existence of a constant stochastic index and appeal to Bahadur's result. This seems tantamount to computing the stochastic index. An alternative is provided by Theorem 5.2 below, which avoids treating stochastic limits.

An appropriate sequential analog of (5.1) is not entirely obvious. Should one replace the divisor n by N, or, to retain the nonstochastic nature of the index, use $E_\theta N$ or some other deterministic quantity? Fortunately, in many cases one does not have to confront this difficulty, since there exist constants $\nu_a = \nu_a(\theta)$ for which

$$P_\theta - \lim_a N_a/\nu_a = 1 \,. \tag{5.2}$$

Usually, one has $\lim_a E_\theta N_a/\nu_a = 1$ as well. Assuming the choice of divisor to be thus resolved, Theorems 5.1 and 5.2 below give conditions under which the stochastic and nonstochastic indices coincide. Theorem 5.1 is similar to Bahadur's Proposition 11 (op. cit.), in that one essentially assumes the existence of a stochastic index. Theorem 5.2 is something of a converse proposition, in which one infers the existence of a stochastic index.

THEOREM 5.1. *Let $T(a)$ be the stopped statistic for N_a and let $L(a)$ denote the attained level of $T(a)$. Suppose that for some $\beta \in (0, \frac{1}{2})$, there are critical values $c_a = c_a(\theta)$ for which*

$$\beta < P_\theta(T(a) \geqq c_a) < 1 - \beta \,, \qquad \textit{all} \quad a \,. \tag{5.3}$$

Let $\alpha_a = \alpha_a(\theta) = \sup \{P_\omega(T(a) \geqq c_a) : \omega \in \Theta_0\}$. Then, if there are constants $\nu_a = \nu_a(\theta)$ for which

$$P_\theta - \lim_a [-\log L(a)]/\nu_a \doteq k \,, \qquad k = 1 \textit{ or } 0 \tag{5.4}$$

then also

$$\lim_a [-\log \alpha_a]/\nu_a = k \,. \tag{5.5}$$

If $k = 1$, we can take $\nu_a = -\log \alpha_a$. The possibility $k = 0$ allows for an index of zero. Verifying (5.4) is akin to verifying the existence of a stochastic index. If $P_\theta - \lim N_a/\nu_a$ exists and is not zero (w.p. 1), then (5.4) is equivalent to the existence of a stochastic index. The next result allows us, in a sense, to reverse the logical implication (5.4) $\Rightarrow$ (5.5) and deduce the existence of a stochastic index from corresponding behavior of $\log \alpha_a$.

THEOREM 5.2. *Let $T(a)$ and $L(a)$ be as in Theorem 5.1. Suppose there are subsets B_0 and B_1 of $[0, 1]$ and, for all $\beta \in B_0 \cup B_1$, critical values $c_a(\beta) = c_a(\beta, \theta)$ which satisfy*:

$$\inf \{\beta : \beta \in B_0\} = 0 \,, \qquad \sup \{\beta : \beta \in B_1\} = 1 \,. \tag{5.6}$$

$$\begin{aligned} &\text{(a)} \quad \forall\, \beta \in B_1 \,, \quad \liminf_a P_\theta(T(a) \geqq c_a(\beta)) \geqq \beta \\ &\text{(b)} \quad \forall\, \beta \in B_0 \,, \quad \limsup_a P_\theta(T(a) > c_a(\beta)) \leqq \beta \,. \end{aligned} \tag{5.7}$$

Let $\alpha_a(\beta) = \alpha_a(\beta, \theta) = \sup\{P_\omega(T(a) \geqq c_a(\beta)) : \omega \in \Theta_0\}$. *Then, if there are constants* $\nu_a = \nu_a(\theta)$ *for which*

$$\lim_a [-\log \alpha_a(\beta)]/\nu_a = k\,, \qquad \text{all} \quad \beta \in B_0 \cup B_1\,, \qquad k = 1 \text{ or } 0\,, \tag{5.8}$$

then also

$$P_\theta - \lim_a [-\log L(a)]/\nu_a = k\,. \tag{5.9}$$

REMARKS. It appears that in many cases, (5.8) is easier to verify than (5.4). Requirements (5.6) and (5.7) are actually equivalent to the following: there are critical values $c_a(0)$ and $c_a(1)$ so that

$$\begin{aligned} &\lim_a P_\theta(T(a) \geqq c_a(1)) = 1 = \lim_a P_\theta(T(a) \leqq c_a(0)) \qquad \text{and} \\ &\lim_a [\log \alpha_a(0)]/\log \alpha_a(1) = 1\,. \end{aligned} \tag{5.10}$$

With a little care, it may be possible to check (5.10) directly, thus shortcutting the verifications necessary for Theorem 5.2. This theorem can be used to justify that suitable limits involving $\log \alpha_a$ are, in fact, stochastic indices. We note also that Theorems 5.1 and 5.2 are not explicitly asymptotic: there is no (explicit) requirement that $N_a \to \infty$.

PROOF OF THEOREMS 5.1 AND 5.2. Since $\sup\{P_\omega(T(a) \geqq x) : \omega \in \Theta_0\}$ is nonincreasing in x, we have that

$$T(a) \geqq c_a \Rightarrow L(a) \leqq \alpha_a \qquad \text{and} \qquad T(a) \leqq c_a \Rightarrow L(a) \geqq \alpha_a\,. \tag{5.11}$$

From (5.3) it follows that $P_\theta(L(a) \leqq \alpha_a) > \beta > 0$ or that $\beta < P_\theta([-\log L(a)]/\nu_a \geqq [-\log \alpha_a]/\nu_a)$. Then, it follows from (5.4) that $\limsup_a [-\log \alpha_a]/\nu_a \leqq k$. Similarly, (5.3) and (5.11) entail $\beta < P_\theta(T(a) < c_a) \leqq P_\theta(L(a) \geqq \alpha_a)$, from which it similarly follows that $\liminf_a [-\log \alpha_a]/\nu_a \geqq k$. Thus (5.5) holds.

Theorem 5.2 is proved similarly: from (5.7a) and (5.11) we obtain $P_\theta([-\log L(a)]/\nu_a \geqq [-\log \alpha_a(\beta)]/\nu_a) \geqq \beta$, which, together with (5.8) entails $\liminf_a P_\theta([-\log L(a)]/\nu_a > k - \varepsilon) \geqq \beta$, all $\varepsilon > 0$, all $\beta \in B_1$. Since $\sup\{\beta : \beta \in B_1\} = 1$ (cf. (5.6)), it follows that $P_\theta - \liminf_a [-\log L(a)]/\nu_a \geqq k$. From (5.7b), one similarly concludes that $P_\theta - \limsup_a [-\log L(a)]/\nu_a \leqq k$, hence that (5.9) holds. □

We note that it is not always possible to obtain a stochastic index from significance levels. This is manifest in the nonsequential case if the stochastic index is random (not constant w.p. 1). Bahadur and Raghavachari (1972) discuss some nonsequential testing problems in which random stochastic indices occur.

In closing, we touch on the interpretation of a random stochastic index as a measure of efficiency. Suppose $\{T_i(n, a), N_{ia}\}$, $i = 1, 2$ are two collections of test statistics and stopping times for the same data sequence. Let $T_i(a)$, $i = 1, 2$ denote the corresponding stopped statistics. If, for $i = 1, 2$, $L_i(a)$, the attained level of $T_i(a)$, satisfies

$$P - \lim_a [-\log L_i(a)]/N_{ia} = r_i\,, \tag{5.12}$$

then the possibly random ratio r_1/r_2 does, in a sense, measure the asymptotic performance of $T_1(a)$ to $T_2(a)$: if

$$P - \lim_a N_{2a}/N_{1a} = r_1/r_2 , \tag{5.13}$$

then the log-attained levels are equalized asymptotically. That is,

$$P - \lim_a [\log L_1(a)]/[\log L_2(a)] = 1 . \tag{5.14}$$

That r_1/r_2 may not be constant then seems to mean that the asymptotic relative performance of T_1 to T_2 is different on different parts of the sample space. (If the convergence in (5.12) is w.p. 1, one could say that the relative performance is different for different data sequences.) In the sequential case, satisfying a condition like (5.13) with r_1/r_2 random seems to present no conceptual difficulties. However, an interpretation for the nonsequential case seems a bit elusive. If the convergence in (5.12) is w.p. 1, then choosing nonrandom sample sizes n_1 and n_2 so that $n_2/n_1 \to \lambda$ means that the behaviors of T_1 and T_2 are matched (asymptotically) on the set of sample sequences for which $r_1/r_2 = \lambda$. Of course, even when the convergence in (5.12) is only in probability, one has convergence w.p. 1 for a suitable subsequence.

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