The Annals of Statistics
1978, Vol. 6, No. 1, 191–199

ON THE TIME AND THE EXCESS OF LINEAR BOUNDARY CROSSINGS OF SAMPLE SUMS

BY CHUNG-SIUNG KAO

Brookhaven National Laboratory

For an infinite sequence of independent and identically distributed random variables $X, X_1, X_2, \cdots; X_n, \cdots$ for which $EX = 0$ and $\operatorname{Var} X = 1$, the behavior of crossings that $S_n = \sum_{1\leqq k\leqq n} X_k \geqq n\varepsilon$ ($\varepsilon > 0$) for some $n \geqq 1$ has recently been under intensive investigation, of which the subject matters are the largest excess $Z = \sup_{n\geqq 1}(S_n - n\varepsilon)^+$, the last time $M = \sup\{n; S_n \gneqq n\varepsilon, 1 \leqq n < \infty\}$, or $= 0$ if no such sup exists, and the number of crossings $N = \sum_{1\leqq n<\infty} I\{S_n \geqq n\varepsilon\}$ (I means the indicator function). This paper describes a striking distributional similarity between $\varepsilon^2 N$ and $\varepsilon^2(M - N)$ in the limiting sense as $\varepsilon \to 0$. Moreover, a new and systematic treatment for the moments problem unifies the previously published results as well as giving some new results. Existence of the limiting moments as $\varepsilon \to 0$ and of the moment generating function is also considered in detail. Most of the results for the one-sided crossings (i.e., $S_n \geqq n\varepsilon$) are then extended to cover their analogues in two-sided crossings (i.e., $|S_n| \geqq n\varepsilon$).

1. Introduction. Boundary crossings of the two linear types (i) $S_n \geqq n\varepsilon$ for some $n \geqq 1$ (one-sided), and (ii) $|S_n| \geqq n\varepsilon$ for some $n \geqq 1$ (two-sided), where ε is positive and $S_n = X_1 + X_2 + \cdots + X_n$ is the sum of a sequence of independent and identically distributed random variables $X, X_1, X_2, \cdots, X_n, \cdots$ having zero mean and unit variance, have recently received considerable attention. If we conveniently set $S_0 = 0$, then the subject matters have been (1) the largest excess of crossings ($Z = \sup_{n\geqq 1}(S_n - n\varepsilon)^+$ for the one-sided and $\tilde{Z} = \sup_{n\geqq 1}(|S_n| - n\varepsilon)^+$ for the two-sided), (2) the last time of crossings ($M = \sup\{n; n \geqq 0$ such that $S_n \geqq n\varepsilon\}$ for the one-sided and $\tilde{M} = \sup\{n; n \geqq 0$ such that $|S_n| \geqq n\varepsilon\}$ for the two-sided) and (3) the number of crossings ($N = \sum_{1\leqq n<\infty} I_{\{S_n \geqq n\varepsilon\}}$ for the one-sided and $\tilde{N} = \sum_{1\leqq n<\infty} I_{\{|S_n|\geqq n\varepsilon\}}$ for the two-sided, where I stands for the indicator function).

Robbins, Siegmund and Wendel (1968) as well as Müller (1968) started the first inquiry about M. They obtained the limiting distribution of $\varepsilon^2 M$ as $\varepsilon \to 0$ and conditions for the existence of limiting moments. Müller (1972) gave the limiting distribution of $\varepsilon^2\tilde{N}$ and $\varepsilon^2 N$ as $\varepsilon \to 0$. Slivka and Severo (1970) gave necessary and sufficient conditions for the existence of moments of $\tilde{N}$ in terms of the moments of X, and later Stratton (1972) extended their results to that of $|S_n| \geqq n^\alpha\varepsilon$ for some n with $\alpha \geqq 0$, for which case we would use $\tilde{Z}_\alpha$, $\tilde{M}_\alpha$ and $\tilde{N}_\alpha$ to denote corresponding largest excess, last time and number of crossings. In

Received July 1974; revised September 1976.

[1] Work performed under the auspices of the ERDA.

AMS 1970 *subject classifications*. Primary 60F10; Secondary 60F15, 60G50.

Key words and phrases. Sample sums, linear boundary crossings, largest excess, last time, total number, limiting distribution, limiting moments, moment generating function.

this connection, some results were obtained earlier by Baum and Katz (1965). Chow and Lai (1975) obtained bounds, in terms of moments of X, for the moments of Z_α and M_α as defined using the general boundary of Stratton, and they also determined the limiting distribution and limiting moments of Z_α and M_α as $\varepsilon \to 0$ in terms of boundary crossing in a Wiener process. It seems that these results are pieces of a unified structure in which Z, M and N are somehow connected, and are insufficient to cast a clear light on the whole structure. The theorems in Section 3 are either new or carry new proofs in such a way that they reflect that structure. The relations amongst moments of Z, M and N and of $\tilde{Z}$, $\tilde{M}$ and $\tilde{N}$ may then be seen more clearly from Theorem 3.1 and the remark in Section 4. Equally significant in this presentation is the interesting identity between the limiting distributions of $\varepsilon^2 N$ and $\varepsilon^2(M - N)$ as $\varepsilon \to 0$. This identity and other results in Section 2 are obtained with an approach based on the findings in Baxter (1956). Such approach, which is different from the approach by Müller in a Markovian context, enables us to visualize, to some extent, the relation between M and N in the limiting sense as $\varepsilon \to 0$.

In the context of the paper, there are basic relations frequently brought into use. It is easily seen that $M \geqq N$ and $\sup_{n\geqq 1}(S_n - n\varepsilon/2)^+ \geqq S_M - M\varepsilon/2 \geqq M\varepsilon - M\varepsilon/2 = M\varepsilon/2$. Therefore, we have the basic relations

$$2\varepsilon^{-1}\sup_{n\geqq 1}(S_n - n\varepsilon/2)^+ \geqq M \geqq N. \tag{1.1}$$

2. Some limiting distributions. The finding of limiting distributions in the following context is largely based on the results in Baxter's paper [2]. In addition, there is a common procedure in proving theorems in this section before final application of Baxter's formulas in the proof. The common procedure mainly includes a truncation process in the beginning and the application of Donsker's "invariance principle" next. For this reason, we only prove the theorem about the limiting distribution of $\varepsilon^2 N$ as $\varepsilon \to 0$ to demonstrate the procedure. It should be noted that Lemma 2 of Robbins-Siegmund-Wendel's paper [10] is fundamental to the truncation process. This lemma was used by the authors to prove that $\lim_{\varepsilon\to 0} P\{\varepsilon^2 M \leqq x\} = 2\Phi(x^{\frac{1}{2}}) - 1$ for $x \geqq 0$, which was then generalized to obtain a corollary that asserts $\lim_{\varepsilon\to 0} P\{\varepsilon \sup_{n\geqq 1}(S_n - n\varepsilon)^+ < x\} = 1 - e^{-2x}$ for $x \geqq 0$. The function Φ here stands as usual for the cumulative distribution function of the standard normal. Therefore, in Mann-Wald symbols, the largest excess is $O_p(\varepsilon^{-1})$ while the last time is $O_p(\varepsilon^{-2})$ as $\varepsilon \to 0$. As to the order of N, it will be shown to be $O_p(\varepsilon^{-2})$ next.

For our convenience in what follows, we would let $[x]$ denote the smallest integer greater than or equal to x, and let

$$F(y) = 1 - \int_y^\infty \left[\int_t^\infty (2\pi u^3)^{-\frac{1}{2}} e^{-u/2}\, du\right] dt \qquad \text{for } y > 0. \tag{2.1}$$

THEOREM 2.1. *For each* $x \geqq 0$, $\lim_{\varepsilon\to 0} P\{\varepsilon^2 N \leqq x\} = F(x)$.

PROOF. Let W_t denote the Wiener process, and let μ denote Lebesgue measure on the real line. Then, for $T > 0$ and any $m > 0$, define $W_t^{(m)}$ by

$$W_t^{(m)} = S_{[mt]}/m^{\frac{1}{2}} \quad \text{for} \quad t \quad \text{in} \quad (0, T) . \tag{2.2}$$

It is obvious that

$$P\{\varepsilon^2 N > x\} = P\{\#\{n \text{ in } N^+ : S_n \geqq n\varepsilon\} > x/\varepsilon^2\} , \tag{2.3}$$

where $\#$ refers to the cardinality of A or the number of elements in A, and N^+ means the set of all positive integers. Consider the truncation at mT and put

$$A_1 = \{\#\{n \text{ in } N^+ : S_n \geqq n\varepsilon, n \leqq mT\} > x/\varepsilon^2\} , \tag{2.4}$$

and

$$A_2 = \{\#\{n \text{ in } N^+ : S_n \geqq n\varepsilon, n > mT\} > 0\} .$$

For $m = 1/\varepsilon^2$ we have

$$\begin{aligned} P(A_1) &= P\{\mu\{t \text{ in } (0, \infty) : S_{[t]} \geqq [t]/m^{\frac{1}{2}}, [t] \leqq mT\} > mx\} \\ &\leqq P\{\mu\{t \text{ in } (0, mT) : S_{[t]} \geqq [t]/m^{\frac{1}{2}}\} > mx\} . \end{aligned}$$

Since $\mu\{t \text{ in } (0, mT); S_{[t]} \geqq [t]/m^{\frac{1}{2}}\}/m = \mu\{t \text{ in } (0, T); W_t^{(m)} \geqq [mt]/m\}$, it then holds that $P(A_1) = P\{\mu\{t \text{ in } (0, T); W_t^{(m)} \geqq [mt]/m\} > x\}$. Then as ε tends to 0, m tends to ∞, and by Donsker's "invariance principle," we have

$$\lim_{\varepsilon \to 0} P(A_1) = P\{\mu\{t \text{ in } (0, T); W_t \geqq t\} > x\} . \tag{2.5}$$

But $P(A_2) \leqq P\{M > mT\}$, so by Lemma 2 in [11] it holds that $P(A_2) \leqq 8/T$. Hence, by letting $T \to \infty$ in $P(A_1) \leqq P\{\varepsilon^2 N > x\} \leqq P(A_1) + P(A_2)$, (2.5) gives

$$\lim_{\varepsilon \to 0} P\{\varepsilon^2 N > x\} = P\{\mu\{t \text{ in } (0, \infty); W_t \geqq t\} > x\} .$$

Then from Baxter [2] the theorem is immediate.

To start looking into the distributional relation between M and N, we define M_w and N_w for Wiener process W_t by

$$\begin{aligned} M_w &= \sup\nolimits_{t \geqq 0} (t; W_t \geqq t) \quad \text{and} \\ N_w &= \mu\{t; t \geqq 0 \text{ such that } W_t \geqq t\} , \end{aligned} \tag{2.6}$$

and also define f on R^2, the Euclidean plane, by

$$f(x, y) = (2\pi y^3 e^y)^{-\frac{1}{2}} \quad \text{for} \quad y > x \geqq 0 \quad \text{and} \quad = 0 \quad \text{otherwise.} \tag{2.7}$$

Then, using Baxter's formula in [2], the probability density function (pdf) of (N_w, M_w) as well as of $(M_w - N_w, M_w)$ may be shown to be f. In accordance with the remarks made in the first paragraph of this section, we shall state the next two theorems without proofs.

THEOREM 2.2. *For any rectangle* $B = \{(x, y); x_1 \leqq x \leqq x_2, y_1 \leqq y \leqq y_2\}$ *in* R^2,

$$\begin{aligned} \lim_{\varepsilon \to 0} P\{(\varepsilon^2 N, \varepsilon^2 M) \text{ in } B\} &= \lim_{\varepsilon \to 0} P\{(\varepsilon^2 (M - N), \varepsilon^2 M) \text{ in } B\} \\ &= \int_B f(x, y)\, d(x, y) . \end{aligned}$$

THEOREM 2.3. *For each* $x \geqq 0$, $\lim_{\varepsilon \to 0} P\{\varepsilon^2 (M - N) \leqq x\} = F(x)$, *where* F *was* *defined by* (2.1) *and is obtained by integration on* f *to eliminate the second* (i.e., y) *argument*.

There is thus an exact similarity in the limiting sense as $\varepsilon \to 0$ between N and $M - N$, as clearly exposed in the foregoing theorems.

3. Finiteness of moments, moment generating functions and their limits. In this section, we shall discuss a number of propositions and references which lead to the following unified conclusion:

THEOREM 3.1. *For each* $\lambda \geqq 0$ *and* $\varepsilon > 0$ *the following conditions are equivalent*:

(a) $E(X^+)^{2+\lambda} < \infty$,
(b) $EZ^{1+\lambda} < \infty$,
(c) $EM^{1+\lambda} < \infty$,
(d) $EN^{1+\lambda} < \infty$,
(e) $\sum_{n=1}^{\infty} n^\lambda P(S_n \geqq n\varepsilon) < \infty$.

(Observe that the random variables Z, M and N depend on ε through their definitions.)

Of these, Chow, Robbins and Siegmund [4] (page 92) assert that (a) $\Leftrightarrow$ (b); Robbins, Siegmund and Wendel [11] assert without proof that (a) $\Leftrightarrow$ (c) when $\lambda > 0$; inequality (1.1) immediately leads to (b) $\Rightarrow$ (c) (but for different ε's); and inequality (1.1) immediately leads to (c) $\Rightarrow$ (d) (for the same ε's). It follows that (a) implies each of (b), (c) and (d). (The issue of different ε's is not a serious problem.) In Proposition 3.1, we shall show that (d) $\Rightarrow$ (e) (but for different ε's), and, in Proposition 3.2, show that (e) $\Rightarrow$ (a). Since condition (a) does not depend on ε, Theorem 3.1 must follow.

LEMMA 3.1. *If* $\{A_n\}$ *is any sequence of events and* $N = \sum_{1 \leqq k < \infty} I_{A_k}$ (*not necessarily finite*), *then*

$$EN^{1+\lambda} \geqq (1 + \lambda) \sum_{n=1}^{\infty} n^\lambda P(A_n) \qquad \text{for} \quad -1 < \lambda \leqq 0\,,$$

and

$$EN^{1+\lambda} \leqq (1 + \lambda) \sum_{n=1}^{\infty} n^\lambda P(A_n) \qquad \text{for} \quad 0 \leqq \lambda < \infty\,.$$

PROOF. Suppose $-1 < \lambda \leqq 0$. If $P(N = \infty) > 0$, then $EN^{1+\lambda} = \infty$ and there is nothing to prove. Therefore, assume $P(N = \infty) = 0$. Set $N_n = \sum_{1 \leqq k \leqq n} I_{A_k}$ and observe that

$$\sum_{n=k}^{\infty} I_{A_n[N_n = k]} = I_{[N \geqq k]}\,.$$

Thus $P(N \geqq k) = \sum_{k \leqq n < \infty} P(A_n[N_n = k])$, and, hence,

$$\begin{aligned}(1 + \lambda)^{-1} EN^{1+\lambda} &= \sum_{n=1}^{\infty} \int_0^n x^\lambda \, dx \, P(N = n) \\ &\geqq \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} k^\lambda\right) P(N = n) \\ &= \sum_{k=1}^{\infty} k^\lambda P(N \geqq k) \\ &= \sum_{k=1}^{\infty} k^\lambda \sum_{n=k}^{\infty} P(A_n[N_n = k]) \\ &\geqq \sum_{n=1}^{\infty} n^\lambda \sum_{k=1}^{n} P(A_n[N_n = k]) \\ &= \sum_{n=1}^{\infty} n^\lambda P(A_n)\,,\end{aligned}$$

which verifies the first inequality. If, instead, $0 \leqq \lambda < \infty$ and $P(N = \infty) = 0$, the previous argument goes through but with the inequalities reversed. The unwanted condition $P(N = \infty) = 0$ can be removed by invoking a simple truncation argument. Specifically, $P(N_m = \infty) = 0$ and, hence, for $0 \leqq \lambda < \infty$,

$$\begin{aligned} EN_m^{1+\lambda} &\leqq (1+\lambda)\{\textstyle\sum_{n=1}^{m} n^\lambda P(A_n) + \sum_{n=m+1}^{\infty} n^\lambda P(\emptyset)\} \\ &\leqq (1+\lambda) \textstyle\sum_{n=1}^{\infty} n^\lambda P(A_n) . \end{aligned}$$

Then, we let $m \to \infty$ and apply the monotone convergence theorem.

We shall only have cause to use the first part of this lemma, but the second part is of intrinsic interest and, in particular, shows immediately that condition (e) of Theorem 3.1 implies condition (d). It is the reverse direction which interests us at this point:

PROPOSITION 3.1. *If $EN^{1+\lambda} < \infty$ for some $\lambda \geqq 0$ and $\varepsilon > 0$, then $\sum_{1 \leqq n < \infty} n^\lambda P(S_n \geqq n\varepsilon') < \infty$ for some $\varepsilon' > 0$.*

PROOF. Let $N_n = \sum_{m=n+1}^{2n} I_{\{S_m - S_n \geqq -2n\varepsilon\}}$, which is independent of S_n for $n \geqq 1$. Moreover, $N \geqq N_n$ when $S_n \geqq 4n\varepsilon$. Thus

$$\begin{aligned} EN^{1+\lambda} &= E \textstyle\sum_{n=1}^{\infty} I_{\{S_n \geqq n\varepsilon\}} N^\lambda \geqq E \sum_{n=1}^{\infty} I_{\{S_n \geqq 4n\varepsilon\}} N_n^\lambda \\ &= \textstyle\sum_{n=1}^{\infty} P(S_n \geqq 4n\varepsilon) EN_n^\lambda . \end{aligned} \tag{3.1}$$

Let n_0 be the smallest integer greater than $\frac{1}{2}\varepsilon^{-2}$. Then, by Chebyshev's inequality, we have $P(S_m - S_n \geqq -2n\varepsilon) \geqq \frac{1}{2}$ for $n+1 \leqq m \leqq 2n$ and $n \geqq n_0$.

If $\lambda \geqq 1$, then, by Jensen's inequality, $EN_n^\lambda \geqq (EN_n)^\lambda$ and it follows from (3.1), that

$$EN^{1+\lambda} \geqq \textstyle\sum_{n=n_0}^{\infty} P(S_n \geqq 4n\varepsilon)(EN_n)^\lambda \geqq 2^{-\lambda} \sum_{n=n_0} P(S_n \geqq 4n\varepsilon) n^\lambda . \tag{3.2}$$

If $0 < \lambda < 1$, then we apply the first part of Lemma 3.1 by setting $A_m = \emptyset$ (the empty set) for $1 \leqq m \leqq n$ or $m > 2n$, and $= \{S_m - S_n \geqq -2n\varepsilon\}$ for $n+1 \leqq m \leqq 2n$. Note that here n is fixed. Then

$$EN_n^\lambda \geqq \lambda \textstyle\sum_{m=n+1}^{2n} m^{\lambda-1} P(S_m - S_n \geqq -2n\varepsilon) \geqq \frac{1}{2}\lambda \int_n^{2n} m^{\lambda-1}\, dm \qquad \text{for} \quad n \geqq n_0 ,$$

and it follows, from (3.1), that

$$EN^{1+\lambda} \geqq C \textstyle\sum_{n=n_0}^{\infty} P(S_n \geqq 4n\varepsilon) n^\lambda , \tag{3.3}$$

where $C = \frac{1}{2}\lambda(2^\lambda - 1) > 0$.

Thus, the proposition is immediate from (3.2) and (3.3).

It should be mentioned that the proof to be given for the next proposition is motivated by Erdös [6].

PROPOSITION 3.2. *For $\lambda \geqq 0$, $E(X^+)^{2+\lambda} < \infty$ if $\sum_{n=1}^{\infty} n^\lambda P\{S_n \geqq n\varepsilon\} < \infty$.*

PROOF. For $\lambda \geqq 0$, it may be shown that $E(X^+)^{2+\lambda} < \infty$ is equivalent to $\sum_{n=1}^{\infty} n^{1+\lambda} P\{X \geqq 2n\varepsilon\} < \infty$ for some $\varepsilon > 0$. For the fixed $\varepsilon > 0$ such that $\sum_{n=1}^{\infty} n^\lambda P\{S_n \geqq n\varepsilon\} < \infty$, we have $P\{S_n \geqq n\varepsilon\} \to 0$ as $n \to \infty$. It follows, by the assumption of i.i.d., that there is a constant $\rho > 0$ independent of k and n such that

$$P\{|\sum_{j=1, j\neq k}^{n} X_j| < n\varepsilon\} \geqq \rho \qquad \text{for} \quad n \geqq n_0 .$$

Let $R_k = \{X_k \geqq 2n\varepsilon\}$, and let $T_k = \{|\sum_{j=1, j\neq k}^{n} X_j| < n\varepsilon\}$. Then we have $\bigcup_{k=1}^{n} (R_k T_k) \subset \{S_n \geqq n\varepsilon\}$. Therefore, denoting by R' the complement of R we have

$$\begin{aligned} P\{S_n \geqq n\varepsilon\} &\geqq P(\bigcup_{k=1}^{n} (R_k T_k)) \\ &\geqq P(\bigcup_{k=1}^{n} (R_1 T_1)' \cdots (R_{k-1} T_{k-1})' R_k T_k) \\ &\geqq \sum_{k=1}^{n} P(R_1' \cdots R'_{k-1} R_k T_k) \\ &\geqq \sum_{k=1}^{n} [P(R_k T_k) - P(R_k(R_1 \cup \cdots \cup R_{k-1}))] \\ &\geqq \sum_{k=1}^{n} P(R_k)[P(T_k) - (k-1)P(R_1)] \\ &\geqq \sum_{k=1}^{n} P(R_k)[\rho - nP(R_1)] . \end{aligned}$$

Since $EX^2 < \infty$, we have $nP(R_1) = nP\{X \geqq 2n\varepsilon\} = o(1)$ as $n \to \infty$. Then there exists $n_1 > 0$ such that $nP(R_1) \leqq \rho/2$ for all $n \geqq n_1$. So $P\{S_n \geqq n\varepsilon\} \geqq \rho \sum_{k=1}^{n} P(R_k)/2 = \rho n P\{X \geqq 2n\varepsilon\}/2$ for $n \geqq m$, where $m = \max(n_0, n_1)$. Therefore $\sum_{n=1}^{\infty} n^{1+\lambda} P\{X \geqq 2n\varepsilon\} \leqq \sum_{n=1}^{m-1} n^{1+\lambda} + 2 \sum_{n=m}^{\infty} n^{\lambda} P\{S_n \geqq n\varepsilon\}/\rho < \infty$. This shows that $E(X^+)^{2+\lambda} < \infty$.

About the limiting behavior of moments, Robbins, Siegmund and Wendel stated without proof that, for $\lambda > 0$, $\lim_{\varepsilon\to 0} E(\varepsilon^2 M)^{1+\lambda} = \int_0^\infty x^{1+\lambda}\, dG(x)$ if $E(X^+)^{2+\lambda} < \infty$, where $G(x) = 2\Phi(x^{\frac{1}{2}}) - 1$ for $x \geqq 0$. In fact, it may be proved that $\lim_{\varepsilon\to 0} E(\varepsilon \sup_{n\geqq 1} (S_n - n\varepsilon)^+)^{1+\lambda} = \int_0^\infty x^{1+\lambda}\, dH(x)$ for $\lambda \geqq 0$, $H(x) = 1 - e^{-2x}$ $(x \geqq 0)$. Then the uniform integrability of $(\varepsilon \sup_{n\geqq 1} (S_n - n\varepsilon)^+)^{1+\lambda}$ as $\varepsilon \to 0$ and the relation of (1.1) together constitute the uniform integrability of $(\varepsilon^2 M)^{1+\lambda}$, $(\varepsilon^2 N)^{1+\lambda}$ and $(\varepsilon^2(M - N))^{1+\lambda}$ as $\varepsilon \to 0$. Thus, the statement by Robbins, Siegmund and Wendel on $\lim_{\varepsilon\to 0} E(\varepsilon^2 M)^{1+\lambda}$ is proved, and in light of the limiting distributions in Section 2, it is immediate that $\lim_{\varepsilon\to 0} E(\varepsilon^2 N)^{1+\lambda} = \lim_{\varepsilon\to 0} E(\varepsilon^2(M - N))^{1+\lambda} = \int_0^\infty x^{1+\lambda}\, dF(x)$. Chow and Lai (1975) obtained, based on bounds, a proof (Theorem 7) for the uniform integrability of $(\varepsilon \sup_{n\geqq 1} (S_n - n\varepsilon)^+)^{1+\lambda}$, while earlier the original proof by this author was by a lengthy combinatorial treatment. An interesting instance of the obtained limiting moments is $\lim_{\varepsilon\to 0} E(\varepsilon^2 N) = \lim_{\varepsilon\to 0} E(\varepsilon^2(M - N)) = \frac{1}{2} \lim_{\varepsilon\to 0} E(\varepsilon^2 M) = \frac{1}{2}$, which happens also to be the value of $\lim_{\varepsilon\to 0} E(\varepsilon \sup_{n\geqq 1} (S_n - n\varepsilon)^+)$.

From this point to the end of this section, we will assume that the moment generating function $Ee^{\theta X}$ exists in a positive neighborhood of 0, and will denote it by $\varphi(\theta)$ $(\theta \geqq 0)$. Then for sufficiently small $\varepsilon > 0$, there uniquely exists $\theta_0 > 0$ that satisfies $\varepsilon\theta_0 = \log \varphi(\theta_0)$, and we may define $\theta_1 \geqq 0$ by $\varepsilon\theta_1 - \log \varphi(\theta_1) = \sup_{0\leqq\theta\leqq\theta_0} (\varepsilon\theta - \log \varphi(\theta))$. It will be shown that $Ee^{\lambda Z} < \infty$ for some $\lambda > 0$ (Theorem 3.2), and that $\lim_{\varepsilon\to 0} Ee^{\lambda\varepsilon Z} = \int_0^\infty e^{\lambda x}\, dH(x)$ for any $\lambda < \frac{1}{2}$ (Theorem 3.3). When we take $\varepsilon' = \varepsilon/2$ and let $Z' = \sup_{n\geqq 1} (S_n - n\varepsilon')^+$, it holds from Theorem 3.2 that $Ee^{\lambda Z'} < \infty$ for some $\lambda > 0$. Therefore, in light of (1.1), it is easily seen that $Ee^{\lambda' M} < \infty$ and $Ee^{\lambda' N} < \infty$ for some $\lambda' > 0$ (e.g., $\lambda' = \lambda\varepsilon/2$). In order to prove for uniform integrability of $Ee^{\lambda\varepsilon Z}$ with respect to ε as $\varepsilon \to 0$ it will be shown in the proof for Theorem 3.3 that, for $\lambda > 0$, $Ee^{\lambda\varepsilon' Z'} I_{\{\varepsilon' Z' \geqq m\}}$ tends to 0 independently of ε' in $[0, \varepsilon_0]$ $(\varepsilon_0 > 0)$ as $m \to \infty$. Since it is clear from (1.1) that

$$e^{\lambda\varepsilon' Z'} I_{\{\varepsilon' Z' \geqq m\}} \geqq e^{\lambda\varepsilon'^2 M} I_{\{\varepsilon'^2 M \geqq m\}} \geqq e^{\lambda\varepsilon^2 M} I_{\{\varepsilon^2 M \geqq 4m\}} \geqq e^{\lambda\varepsilon^2 N} I_{\{\varepsilon^2 N \geqq 4m\}} ,$$

we have uniform convergence for $\{Ee^{\lambda\varepsilon^2 V} I_{\{\varepsilon^2 V \geqq m\}};\ 0 < \varepsilon < 2\varepsilon_0\}$ to 0 as $m \to \infty$, where V stands for any of M, N and $M - N$. This then leads to the result that

$$\lim_{\varepsilon\to 0} Ee^{\lambda\varepsilon^2 M} = \int_0^\infty e^{\lambda x}\, dG(x)$$

and

$$\lim_{\varepsilon\to 0} Ee^{\lambda\varepsilon^2 N} = \lim_{\varepsilon\to 0} Ee^{\lambda\varepsilon^2 (M-N)} = \int_0^\infty e^{\lambda x}\, dF(x)$$

in light of the limiting distributions in Section 2.

THEOREM 3.2. *There exists $\lambda > 0$ such that $Ee^{\lambda Z} < \infty$.*

PROOF. Since $\varepsilon\theta_1 - \log\varphi(\theta_1) = \lambda_0 > 0$, we may take a fixed value λ such that $0 < \lambda < \lambda_0$. It is obvious that $P\{Z \geqq 2m\varepsilon\} = P\{S_n \geqq 2m\varepsilon + n\varepsilon$ for some $n \geqq 1\} \leqq P\{\max_{1\leqq n\leqq m} S_n \geqq 2m\varepsilon\} + P\{S_n \geqq n\varepsilon$ for some $n \geqq m\}$. From the lemma on page 91 of Chow–Robbins–Siegmund [4] we have, for sufficiently large m, that $P\{\max_{1\leqq n< m} S_n \geqq 2m\varepsilon\} \leqq 2P\{S_m \geqq m\varepsilon\}$. Therefore, $P\{Z \geqq 2m\varepsilon\} \leqq 3P\{S_n \geqq n\varepsilon$ for some $n \geqq m\}$. Then by (i) of Theorem 3 in Kao [8], we have $P\{Z \geqq 2m\varepsilon\} \leqq 6e^{-m\lambda_0}$, from which it becomes obvious that $\lim_{k\to\infty} Ee^{\lambda Z} I\{Z \geqq 2k\varepsilon\} \leqq \lim_{k\to\infty} ce^{-k(\lambda_0-\lambda)} = 0$, where c is a constant. This means that $Ee^{\lambda Z} < \infty$. Before presenting the next theorem, we need the following lemma.

LEMMA 3.2. *Let $\lambda_0(\varepsilon) = \varepsilon\theta_1 - \log\varphi(\theta_1)$, then $\lim_{\varepsilon\to 0} \lambda_0(\varepsilon)/\varepsilon^2 = \frac{1}{2}$.*

PROOF. It should be noted that θ_1 and $\varphi(\theta_1)$ are analytic functions of ε on $(0, \delta)$ for some $\delta > 0$. Since $0 \leqq \theta_1 \leqq \theta_0 \to 0$ when $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0} \lambda_0(\varepsilon) = 0 . \tag{3.4}$$

Moreover, since $\varepsilon\theta - \log\varphi(\theta) = 0$ at $\theta = 0$ and θ_0, we have, from the definition of θ_1, that $(d/d\theta)Ee^{\theta(X-\varepsilon)} = 0$ at $\theta = \theta_1$, which is equivalent to

$$Exe^{\theta_1 X} = \varepsilon\varphi(\theta_1) . \tag{3.5}$$

By using this equality, it may be shown that

$$\frac{d}{d\varepsilon}\lambda_0(\varepsilon) = \theta_1 , \tag{3.6}$$

which tends to 0 when ε tends to 0. We defferentiate both sides of (3.5) with respect to ε to obtain that

$$\frac{d\theta_1}{d\varepsilon} = Ee^{\theta_1 X}/(EX^2 e^{\theta_1 X} - \varepsilon EXe^{\theta_1 X}) .$$

So from the assumption of $EX = 0$ and $\operatorname{Var} X = 1$, and the fact that $\lim_{\varepsilon\to 0}\theta_1 = 0$, we have

$$\lim_{\varepsilon\to 0} d\theta_1/d\varepsilon = 1 . \tag{3.7}$$

Therefore, in view of (3.4) and (3.6), we may apply l'Hospital's rule twice to obtain $\lim_{\varepsilon\to 0}\lambda_0(\varepsilon)/\varepsilon^2 = \frac{1}{2}\lim_{\varepsilon\to 0} d\theta_1/d\varepsilon$, which is equal to $\frac{1}{2}$ as seen from (3.7).

THEOREM 3.3. *For any* $\lambda < \frac{1}{2}$, *we have* $\lim_{\varepsilon\to 0} Ee^{\lambda\varepsilon Z} = \int_0^\infty e^{\lambda x}\, dH(x)$.

PROOF. For given $\lambda < \frac{1}{2}$, Lemma 3.2 asserts that there exists ε_0 such that $\varepsilon^{-2}\lambda_0(\varepsilon) > \lambda_1 > \lambda$ for all ε in $[0, \varepsilon_0]$. From the limiting distribution of εZ, we see that the theorem is proved if $\{e^{\lambda\varepsilon Z}; 0 < \varepsilon < \varepsilon_0\}$ are uniformly integrable. Therefore, it is sufficient to prove that $\{Ee^{\lambda\varepsilon Z} I_{\{\varepsilon Z \geqq m\}}; 0 < \varepsilon < \varepsilon_0\}$ converge uniformly to 0 as $m \to \infty$. It is easily seen that $P\{\varepsilon Z \geqq m\} \leqq P\{\max_{1\leqq n\leqq m/\varepsilon^2} S_n \geqq 2m/\varepsilon^2\} + P\{S_n \geqq n\varepsilon \text{ for some } n \geqq m/\varepsilon^2\}$. By the lemma on page 91 of Chow–Robbins–Siegmund [4], we have for $m \geqq 2$ the inequality $P\{\max_{1\leqq n\leqq m/\varepsilon^2} S_n \geqq m/\varepsilon^2\} \leqq 2P\{S_{[m/\varepsilon^2]} \geqq m/\varepsilon^2\}$. Therefore, $P\{\varepsilon Z \geqq m\} \leqq 3P\{S_n \geqq n\varepsilon \text{ for some } n \geqq m/\varepsilon^2\}$. By Theorem 3 in [8], we then have $P\{\varepsilon Z \geqq m\} \leqq 6e^{-m\lambda_0(\varepsilon)/\varepsilon^2}$, which is (upper) bounded by $6e^{-m\lambda_1}$ for all ε in $[0, \varepsilon_0]$. Hence, for each ε in $[0, \varepsilon_0]$,

$$\begin{aligned} Ee^{\lambda\varepsilon Z} I_{\{\varepsilon Z \geqq m\}} &\leqq \textstyle\sum_{n=m}^\infty e^{\lambda m} P\{\varepsilon Z \geqq n\} \leqq 6 \sum_{n=m}^\infty e^{-m(\lambda_1-\lambda)} \\ &\leqq 6(1 - e^{-(\lambda_1-\lambda)})^{-1} e^{-m(\lambda_1-\lambda)}, \end{aligned}$$

which tends to 0 independently of ε as $m \to \infty$. Thus we have the desired uniform convergence of $\{Ee^{\lambda\varepsilon Z} I_{\{\varepsilon Z \geqq m\}}; 0 < \varepsilon < \varepsilon_0\}$ to 0 as $m \to \infty$, and the theorem is proved.

4. Concluding remarks. If we let $Y = -X$ and $Y_n = -X_n$, $1 \leqq n < \infty$, then $Y, Y_1, Y_2, \cdots$ are i.i.d. with $EY = 0$ and $\operatorname{Var} Y = 1$. Define $T_n = -S_n$, $0 \leqq n < \infty$, and for the same given $\varepsilon > 0$ let Z', M' and N' be respectively the largest excess, the last time and the number of crossings of $T_n \geqq n\varepsilon$, $0 \leqq n < \infty$. We are interested in the values a_i' and $\tilde{a}_i$, $1 \leqq i \leqq 5$, defined by $a_1' = E(Y^+)^{2+\lambda}$, $a_2' = EZ'^{1+\lambda}$, $a_3' = EM'^{1+\lambda}$, $a_4' = \sum_{n=1}^\infty n^\lambda P\{T_n \geqq n\varepsilon\}$, $a_5' = EN'^{1+\lambda}$, $\tilde{a}_1 = E|X|^{2+\lambda}$, $\tilde{a}_2 = E\tilde{Z}^{1+\lambda}$, $\tilde{a}_3 = E\tilde{M}^{1+\lambda}$, $\tilde{a}_4 = \sum_{n=1}^\infty n^\lambda P\{|S_n| \geqq n\varepsilon\}$ and $\tilde{a}_5 = E\tilde{N}^{1+\lambda}$. It may be shown that $\max(a_i, a_i') \leqq \tilde{a}_i \leqq 2^{2+\lambda}(a_i + a_i')$ for $1 \leqq i \leqq 5$. Therefore, applying Theorem 3.1 to the X_i's and the Y_i's separately, we conclude that either (i) $\tilde{a}_i < \infty$ for all i, $1 \leqq i \leqq 5$, or (ii) $\tilde{a}_i = \infty$ for all i, $1 \leqq i \leqq 5$, holds (the two-sided analogue of Theorem 3.1).

As for the limiting behavior of $\tilde{M}$ as $\varepsilon \to 0$, we may consult (5.8) on page 329 of Feller [7] to obtain that $P\{\sup_{0\leqq t\leqq 1} |W(t)| \geqq y\} = \tilde{G}(y)$, where

$$\tilde{G}(y) = 2\textstyle\sum_{k=-\infty}^\infty \{\Phi((4k-1)y) - \Phi((4k+1)y)\} - 1 \qquad \text{for} \quad y \geqq 0,$$

to show that $\lim_{\varepsilon\to 0} P\{\varepsilon^2\tilde{M} \leqq x\} = \tilde{G}(x)$ for each $x \geqq 0$.

Finally, since it is clear that $Ee^{\lambda\tilde{V}} \leqq 2\max(Ee^{\lambda V}, Ee^{\lambda V'}) \leqq 2Ee^{\lambda\tilde{V}}$ for $V = Z$, M and N, it may be shown that there exists $\lambda > 0$ such that $Ee^{\lambda\tilde{Z}} < \infty$, $Ee^{\lambda\tilde{M}} < \infty$ and $Ee^{\lambda\tilde{N}} < \infty$ if the moment generating function of X exists in a neighborhood of the origin, and that

$$\lim_{\varepsilon\to 0} Ee^{t\varepsilon\tilde{Z}} = \int_0^\infty e^{tx}\, d\tilde{H}(x), \qquad \lim_{\varepsilon\to 0} Ee^{t\varepsilon\tilde{M}} = \int_0^\infty e^{tx}\, d\tilde{G}(x)$$

and

$$\lim_{\varepsilon\to 0} Ee^{\lambda\varepsilon^2\tilde{N}} = \int_0^\infty e^{\lambda x} f(x)\, dx,$$

where $\tilde{H}(x) = P\{\max_{t \geqq 0} |W(t) - t| \geqq x\}$ ($W(t)$ is the Wiener process) and $f(x)$ is defined at (5.21) in Müller's (1972).

Acknowledgment. Much of this paper is based on the author's Ph. D. dissertation at Columbia University, 1972. The author is indebted to Professor David O. Siegmund for his generous guidance. He wishes to thank Dr. Joseph L. Fleiss for his enthusiasm and encouragement, and the referee for invaluable criticisms and suggestions.

REFERENCES

[1] BAUM, L. E. and KATZ, M. (1965). Convergence rates in the law of large numbers. *Trans. Amer. Math. Soc.* **120** 108–123.

[2] BAXTER, GLEN (1956). Wiener process distributions of the "arcsine law" type. *Proc. Amer. Math. Soc.* **7** 738–741.

[3] CHOW, Y. S. and LAI, T. L. (1975). Some one-sided theorems on the tail distribution of sample sums with application to the last time and largest excess of boundary crossings. *Trans. Amer. Math. Soc.* **208** 51–72.

[4] CHOW, Y. S., ROBBINS, H. and SIEGMUND, D. (1971). *Great Expectations: The Theory of Optimal Stopping.* Houghton Mifflin, Boston.

[5] DONSKER, MONROE D. (1951). An invariance principle for certain probability limit theorems. *Mem. Amer. Math. Soc.* No. 6.

[6] ERDÖS, PAUL (1949). On a theorem of Hsu and Robbins. *Ann. Math. Statist.* **20** 286–291.

[7] FELLER, WILLIAM (1966). *An Introduction to Probability Theory and its Applications*, **2.** Wiley, New York.

[8] KAO, C. S. (1975). Some methods of linear boundary crossings probabilities with results on exponential and normal distributions. *Bull. Inst. Math. Acad. Sinica.* **3** 151–159.

[9] MÜLLER, D. W. (1968). Verteilungs-Invarianzprinzipien fur das starke Gesetz der Grössen Zahl. *Z. Wahrscheinlickeitstheorie und Verw. Gebiete* **10** 173–192.

[10] MÜLLER, D. W. (1972). Randomness and extrapolation. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **2** 1–31. Univ. of California Press.

[11] ROBBINS, H., SIEGMUND, D. and WENDEL, J. (1968). The limiting distribution of the last time $s_n > n_\varepsilon$. *Proc. Nat. Acad. Sci. U.S.A.* **61** 1228–1230.

[12] SLIVKA, J. and SEVERO, N. C. (1970). On the strong law of large numbers. *Proc. Amer. Math. Soc.* **24** 729–734.

[13] STRATTON, HOWARD H. (1972). Moments of oscillations and rule sums. *Ann. Math. Statist.* **43** 1012–1016.

DEPARTMENT OF APPLIED MATHEMATICS
BROOKHAVEN NATIONAL LABORATORY
ASSOCIATED UNIVERSITIES, INC.
UPTON, NEW YORK 11973