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ROBUSTNESS OF BALANCED INCOMPLETE BLOCK DESIGNS

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When a variety is lost or omitted from a balanced incomplete block design, the residual design is not, in general, balanced. A lower bound is obtained for the overall efficiency in this case and it is shown that the overall loss of efficiency even in the unbalanced case is small.

1. Introduction. Hedayat and John (1974) considered the situation in which for some reason one of the varieties in a balanced incomplete block design is missing. This could occur for example in an agricultural experiment in which the 'varieties' are fertilizers and one of them, perhaps through an error in formulation, turns out to be lethal. They showed that only under restrictive conditions is the resulting residual design balanced.

It is well known (Raghavarao (1971), Theorem 4.5.2) that the connected incomplete block designs with the highest intrablock efficiency factors are balanced. It is germane therefore to ask whether the balanced structure can be so fragile that the loss of a single variety can produce a residual design so unbalanced that there is a serious loss of efficiency.

A study of the four nonisomorphic balanced incomplete block designs for $v = 8$, $b = 14$, $r = 7$, $k = 4$, $\lambda = 3$ showed that the thirty-two possible residual designs had only three values of E. In the balanced case $E = 0.83333$; in the other cases $E = 0.83300$ and 0.83311. The loss of efficiency in the unbalanced cases is trivial.

In this paper we consider the general problem of the efficiency of a residual design, and establish a lower bound, $E_{\min}$, for it. This bound is not always attained; in the case of $v = 8$ mentioned in the previous paragraph we have $E_{\min} = 0.83265$. Values of $E_{\min}$ and of E_0 (the maximum efficiency that would be obtained if the residual design were balanced) have been computed for all the balanced incomplete block designs listed by Raghavarao (1971). In no case was $E_{\min}$ less than 99.3% of E_0. This indicates that the loss of efficiency due to unbalance is trivial and that the balanced incomplete block designs are very robust in respect to the omission of one of the varieties.

2. The efficiency of a residual design. Let D be a balanced incomplete block design with parameters v, b, r, k, λ; we shall write t for $v - 1$. Suppose that one of the varieties in D is omitted. The resulting design D' has two subdesigns D_1 and D_2; D_1 consists of those blocks which formerly contained the missing variety and it has parameters t, $b_1 = r$, $r_1 = \lambda$, $k_1 = k - 1$; D_2 consists of the unspoiled

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blocks and has parameters t, $b_2 = b - r$, $r_2 = r - \lambda$, $k_2 = k$. The corresponding incidence matrices are $\mathbf{N}$ for D, $\mathbf{N}_1$ for D_1, and $\mathbf{N}_2$ for D_2. The intrablock matrix for D is

$$\mathbf{A} = r\mathbf{I} - (k-1)^{-1}\mathbf{N}_1\mathbf{N}_1' - k^{-1}\mathbf{N}_2\mathbf{N}_2' .$$

Zero is a simple latent root of $\mathbf{A}$, corresponding to roots $\lambda(k-1)$ of $\mathbf{N}_1\mathbf{N}_1'$, and $k(r-\lambda)$ of $\mathbf{N}_2\mathbf{N}_2'$; in each case the latent vector is $\mathbf{1}$.

THEOREM. *Let θ be a latent root of $\mathbf{N}_1\mathbf{N}_1'$ other than $\lambda(k-1)$ and let $\mathbf{x}$ be a corresponding latent vector; then $\psi = r - (k-1)^{-1}\theta - k^{-1}(r - \lambda - \theta)$ is a latent root of $\mathbf{A}$ with vector $\mathbf{x}$.*

PROOF. We note that $\mathbf{NN}' = (r-\lambda)\mathbf{I}_v + \lambda\mathbf{J}_v$. Hence $\mathbf{N}_1\mathbf{N}_1' + \mathbf{N}_2\mathbf{N}_2' = (r-\lambda)\mathbf{I}_t + \lambda\mathbf{J}$ so that

$$\mathbf{Ax} = r\mathbf{x} - (k-1)^{-1}\mathbf{N}_1\mathbf{N}_1'\mathbf{x} - k^{-1}[(r-\lambda)\mathbf{I} + \lambda\mathbf{J} - \mathbf{N}_1\mathbf{N}_1']\mathbf{x} .$$

Recalling that $\mathbf{Jx} = \mathbf{11}'\mathbf{x} = \mathbf{0}$, we have

$$\mathbf{Ax} = [r - (k-1)^{-1}\theta - k^{-1}(r - \lambda - \theta)]\mathbf{x} = \psi\mathbf{x} .$$

It follows that if θ has multiplicity α then ψ is a root of $\mathbf{A}$ with multiplicity α. We note in passing that the roots of $\mathbf{N}_2\mathbf{N}_2'$, other than $k(r-\lambda)$, are given by $r - \lambda - \theta$. Since $\mathbf{N}_2\mathbf{N}_2'$ is nonnegative definite, this implies that $\theta \leqq r - \lambda$.

The efficiency, E, of an equireplicate incomplete block design may be expressed as

$$E = r^{-1}(t-1)/[\textstyle\sum_i \alpha_i\psi_i^{-1}] .$$

3. A lower bound for E. The minimum value $E_{\min}$ of E occurs when $\operatorname{tr}\mathbf{V} = \sum \alpha_i\psi_i^{-1}$ is maximized. The expression that was obtained earlier for ψ may be rewritten

$$\psi = \frac{\lambda v}{k} - \frac{\theta}{k(k-1)} .$$

We seek, therefore, to maximize the quantity $Q = \sum \alpha_i[\lambda v(k-1) - \theta_i]^{-1}$, subject to these two conditions, $0 \leqq \theta_i \leqq r - \lambda$ and $\sum \alpha_i\theta_i = \lambda(t-k+1) = (r-\lambda)(k-1)$. We may write $\lambda v(k-1) - \theta_i$ as $\lambda v(k-1)[1-\rho_i]$, and, since $0 < \rho < 1$, may expand $[1-\rho_i]^{-1}$ into the infinite geometric series $\sum \rho_i^s$. The problem thus becomes one of maximizing $\sum_s \sum_i \alpha_i\theta_i^s$.

Let $\phi(s) = \sum_i \alpha_i\theta_i^s$. Then $\phi(s+1) \leqq (r-\lambda)\phi(s)$ with equality only if, for each i, $\theta_i^{s+1} = (r-\lambda)\theta_i^s$, i.e., if $\theta_i = 0$ or $\theta_i = r - \lambda$. One of the side conditions states that $\phi(1) = (r-\lambda)(k-1)$. It follows by induction that $\phi(s)$ is maximized for each s, and hence that Q is maximized, when θ_i takes the value $r - \lambda$ with multiplicity $k-1$, and is zero otherwise.

We have therefore only the two values

$$\theta_1 = r - \lambda , \qquad \alpha_1 = k - 1 ,$$
$$\theta_2 = 0 , \qquad \alpha_2 = t - k .$$

The corresponding values of ϕ are $\phi_1 = (\lambda v - r)/(k - 1)$, and $\phi_2 = k^{-1}\lambda v$ so that

$$E_{\min} = r^{-1}(t-1)\left[\frac{(k-1)^2}{\lambda v - r} + \frac{(t-k)k}{\lambda v}\right]^{-1}.$$

The maximum value that E can attain is

$$E_0 = (rt - b)/\{r(t-1)\};$$

this value is attained when the design D' is balanced. We may also, after some simplification, write E_0 as

$$E_0 = \frac{\lambda v}{rk}\left[1 - \frac{r-\lambda}{\lambda v(t-1)}\right] = \mathscr{E}\left[1 - \frac{r-\lambda}{\lambda v(t-1)}\right],$$

where $\mathscr{E}$ is the efficiency of the original balanced incomplete block design with no missing varieties.

The values of $E_{\min}$, E_0 and $\mathscr{E}$ were computed for each of the balanced incomplete block designs listed by Raghavarao (1971). The ratio $E_{\min}/E_0$ exceeded 0.99 for each of the designs. The lowest value of the ratio $E_{\min}/\mathscr{E}$ was 0.933, which occurred for the small design with parameters $v = 6$, $b = 10$, $r = 5$, $k = 3$, $\lambda = 2$. For $v > 7$, $E_{\min}/\mathscr{E} > 0.945$ and for $v > 15$ the ratio exceeds 0.99. The original balanced structure is thus seen to be particularly robust in regard to the overall efficiency when any variety is omitted.

4. Designs in which $E_{\min}$ is attained. The lower bound $E_{\min}$ is always attained if D is a symmetric BIB design.

THEOREM. *Let D be a symmetric* BIB *design. The efficiency of D' is equal to $E_{\min}$, and is independent of the choice of variety to be omitted.*

PROOF. Since D is symmetric, each pair of blocks of D have λ varieties in common; any two blocks of D_1 have $\lambda - 1$ varieties in common. The dual of D_1 is a balanced incomplete block design with parameters

$$v^* = r = k, \qquad r^* = k - 1, \qquad k^* = \lambda, \qquad \lambda^* = \lambda - 1,$$

and $\mathbf{N}_1'\mathbf{N}_1$ has a root $k - \lambda$ with multiplicity $k - 1$. The positive roots of $\mathbf{N}_1'\mathbf{N}_1$ and their multiplicities are the same as those of $\mathbf{N}_1\mathbf{N}_1'$ and so we have

$$\theta_1 = k - \lambda, \qquad \alpha_1 = k - 1,$$
$$\theta_2 = 0, \qquad \alpha_2 = t - k,$$

as in Section 3 above. Thus $E = E_{\min}$ and is independent of the variety omitted.

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