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COMPARISON OF SOME BOUNDS IN ESTIMATION THEORY

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Conditions are given for the attainment of the Hammersley-Chapman-Robbins bound for the variance of an unbiased estimator, in both regular and nonregular cases. Comparisons are made between this bound and the Bhattacharyya system of bounds for a wide class of distributions and parametric functions. Sufficient conditions are provided to determine wher one bound is sharper than the other one.

1. Introduction. Let $(\mathscr{X}, \mathscr{A}, \mu)$ be an arbitrary measure space with μ sigma-finite. Let X be a random variable (rv) taking values in $\mathscr{X}$ with probability distribution $P_\theta(dx) = f_\theta(x)\mu(dx)$ for $x \in \mathscr{X}$ and $\theta \in \Theta$, where Θ is a known subset of the real line. In the sequel, it is taken that $\mathscr{X}$ is the n-dimensional Euclidean space ($n \geqq 1$) and $\mathscr{A}$ is the Borel field on $\mathscr{X}$.

Let τ be a real-valued estimable function on Θ, not identically constant. Let $t(X)$ be an unbiased estimator of $\tau(\theta)$ with $E_\theta(t^2(X)) < \infty$. We are primarily concerned with the following three well-known lower bounds for the variance of $t(X)$.

(i) The classical *Cramér-Rao inequality* states that, under certain regularity assumptions A (see [6]),

$$\text{Var}_\theta\,(t(X)) \geqq \{(d/d\theta)\tau(\theta)\}^2/\text{Var}_\theta\,((\partial/\partial\theta)\log f_\theta(X)) = A(\theta)\,, \quad \text{say.} \tag{1.1}$$

Assumptions A include, among others, that Θ be an open interval and τ be differentiable. The strict equality in (1.1) holds for all $\theta \in \Theta$ iff $f_\theta(x)$ is of the form (1.4) with $g(x)$ replaced by $t(x)$.

(ii) The *Bhattacharyya system of inequalities* states that, under more stringent (than A) regularity assumptions B_k for $k \geqq 1$ (see [2]),

$$\text{Var}_\theta\,(t(X)) \geqq \boldsymbol{\tau}_\theta{}'\mathbf{V}_\theta{}^{-1}\boldsymbol{\tau}_\theta = B_k(\theta)\,, \quad \text{say,} \tag{1.2}$$

where $\boldsymbol{\tau}_\theta = (\tau^{(1)}(\theta), \cdots, \tau^{(k)}(\theta))'$, $\mathbf{V}_\theta = ((V_{ij}(\theta)))_{i,j=1,\cdots,k}$ with

$$\tau^{(i)}(\theta) = (d^i/d\theta^i)\tau(\theta) \qquad \text{and} \tag{1.3}$$
$$V_{ij}(\theta) = E_\theta\{f_\theta{}^{-2}(X)\cdot(\partial^i/\partial\theta^i)f_\theta(X)\cdot(\partial^j/\partial\theta^j)f_\theta(X)\}\,.$$

It is well known that $B_1(\theta) = A(\theta)$ and $B_k(\theta) \geqq B_{k-1}(\theta)$ for all $k > 1$ and $\theta \in \Theta$. A sufficient condition for the attainment of the strict equality in (1.2) for all θ

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is that $t(x)$ be a polynomial of degree k in some real-valued function g on $\mathscr{X}$ and $f_\theta(x)$ be of the form

(1.4) $$f_\theta(x) = \alpha(\theta)h(x)\exp\{\gamma(\theta)g(x)\}, \qquad \text{for all} \quad x \in \mathscr{X} \quad \text{and} \quad \theta \in \Theta,$$

in which $\alpha > 0$ and γ (monotonic) are continuously differentiable, and h is positive except, perhaps, on a μ-null set in $\mathscr{X}$ which is independent of $\theta \in \Theta$. Fend (1959) also showed that, under (1.4), any t achieving the equality in (1.2) is necessarily a polynomial in g.

(iii) Finally, the *Hammersley–Chapman–Robbins inequality* ([1, 3]) gives a bound without any regularity assumptions. Define $\mathscr{X}_\theta \subseteqq \mathscr{X}$ by

(1.5) $$f_\theta(x) > 0 \quad \text{a.e.} \quad x \in \mathscr{X}_\theta, \qquad f_\theta(x) = 0 \quad \text{a.e.} \quad x \in \mathscr{X} - \mathscr{X}_\theta,$$

and let $\Phi_\theta \subset \Theta$ be the set of all $\phi \in \Theta$ satisfying

(1.6) $$\tau(\phi) \neq \tau(\theta), \qquad \mathscr{X}_\phi \subseteqq \mathscr{X}_\theta.$$

Then

(1.7) $$\mathrm{Var}_\theta(t(X)) \geqq \sup_{\phi \in \Phi_\theta} \{\tau(\phi) - \tau(\theta)\}^2 / \mathrm{Var}_\theta(f_\phi(X)/f_\theta(X)) = C(\theta), \quad \text{say}.$$

If Φ_θ is empty for some $\theta \in \Theta$, we define $C(\theta) = 0$. Chapman and Robbins (1951) proved that, when assumptions A hold, $C(\theta) \geqq A(\theta)$ for all θ. No relation is known to exist between $C(\theta)$ and $B_k(\theta)$ when $B_k(\theta) > B_1(\theta)$ for some $k > 1$.

The purpose of this paper is to explore some further properties of $C(\theta)$. Section 2 deals with conditions under which the equality in (1.7) holds. The results provide a method of recognizing the UMVU estimator in many situations where $B_k(\theta)$ fails to provide tne answer for every $k \geqq 1$. In Section 3, we investigate the relative status of $C(\theta)$ and $B_k(\theta)$ when assumptions B_k hold for some $k \geqq 1$. Sufficient conditions are given under which $C(\theta)$ is greater or less than $B_k(\theta)$. Finally, Section 4 deals with certain aspects of $C(\theta)$ when assumptions A (and therefore B_k for any $k \geqq 1$) do not hold.

We conclude this section with the following example illustrating the scope of our results in a simple situation. Suppose $X = (X_1, \cdots, X_n)$, $n \geqq 1$, where the X_i are i.i.d. rv's with the normal distribution with 0 mean and an unknown variance $\theta > 0$, and let $s = (\sum_{i=1}^n X_i^2)/n$. (i) If $\tau(\theta) = \theta$, it is well known [1] that s is the UMVU estimator and $B_1(\theta) = C(\theta) = \mathrm{Var}_\theta(s)$ for all θ. (ii) If $\tau(\theta) = (1 + \theta)^{-n/2}$, then $\exp(-ns/2)$ is unbiased and its variance equals $C(\theta)$ for all θ, so that $\exp(-ns/2)$ is the UMVU estimator (Theorem 2.2). Moreover, $C(\theta) > B_k(\theta)$ for all θ and $k \geqq 1$ (Theorems 3.1, 3.2). (iii) If $\tau(\theta) = \theta^2$, then $ns^2/(n + 2)$ is unbiased and its variance equals $B_2(\theta)$ which is greater than $C(\theta)$ for all θ (Theorem 3.3). Moreover, $C(\theta) > B_1(\theta)$ for all θ (Theorem 3.1). (iv) If $\tau(\theta) = \theta^{\frac{3}{2}}$, then $B_1(\theta) < C(\theta) < B_2(\theta)$ for all θ (Theorems 3.1, 3.4). In this case, $\{\Gamma(n/2)/\Gamma((n + 3)/2)\}(ns/2)^{\frac{3}{2}}$ is the UMVU estimator [5] and its variance exceeds $B_k(\theta)$ for all $k \geqq 1$ and θ [2]. (v) If $\tau(\theta) = \theta$ and $\theta \in \Theta = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, then $\mathrm{Var}_\theta(t(X)) > C(\theta)$ for any unbiased $t(X)$ (Theorem 4.2). Assumptions A do not hold in this case, but it can be shown by the Lehmann–Scheffé [5] theorem that s is still the UMVU estimator. As $n \to \infty$, $C(\theta)/\mathrm{Var}_\theta(s) \to 0$ for every $\theta \in \Theta$, so that the bound $C(\theta)$ does not serve any useful purpose. (vi) Finally, if $\tau(\theta) = \theta^{\frac{1}{2}}$ (see Example 2 in [1]), then the variance of the UMVU estimator $\{\Gamma(n/2)/\Gamma((n+1)/2)\}(ns/2)^{\frac{1}{2}}$ exceeds both $C(\theta)$ and $B_k(\theta)$ for all $\theta > 0$ and $k \geqq 1$ (Theorem 2.1 and [2]), and $C(\theta) > B_1(\theta)$ for all $\theta > 0$ (Theorem 3.1). However, in this case, our general results fail to provide any information about the relative magnitudes of $C(\theta)$ and $B_k(\theta)$ for any $k > 1$. In fact, direct computations show that for all $\theta > 0$, $C(\theta) > B_3(\theta)$ when $n = 1, 2$ and $C(\theta) < B_3(\theta)$ when $n \geqq 3$.

2. Attainment of $C(\theta)$. We state without proof the following theorem which is a simple extension of the results in [1, 3].

THEOREM 2.1. *Given any fixed $\theta \in \Theta$,*

$$\begin{aligned} \mathrm{Var}_\theta(t(X)) &\geqq \{\tau(\phi) - \tau(\theta)\}^2/\mathrm{Var}_\theta(f_\phi(X)/f_\theta(X)) \\ &\geqq C(\phi, \theta), \quad \textit{say}, \end{aligned} \tag{2.1}$$

for all $\phi \in \Phi_\theta$, and the strict equality in (2.1) holds for any $\phi \in \Phi_\theta$ iff

$$\begin{aligned} &\{f_\phi(x)/f_\theta(x) - 1\}/\{\tau(\phi) - \tau(\theta)\} \\ &\quad = \{t(x) - \tau(\theta)\}/\mathrm{Var}_\theta(t(X)) \qquad \text{a.e. } x \in \mathscr{X}_\theta. \end{aligned} \tag{2.2}$$

COROLLARY 2.1.1. *If (2.2) holds for some $\phi = \phi^*(\theta) \in \Phi_\theta$, then $\mathrm{Var}_\theta(t(X)) = C(\theta)$ and the supremum in (1.7) is achieved by $\phi = \phi^*(\theta)$.*

COROLLARY 2.1.2. *If $\mathrm{Var}_\theta(t(X)) = C(\theta)$ and $C(\theta) = C(\phi^*(\theta), \theta)$ for some $\phi^*(\theta) \in \Phi_\theta$, then (2.2) holds for $\phi = \phi^*(\theta)$.*

COROLLARY 2.1.3. *Suppose assumptions A hold. Then $\mathrm{Var}_\theta(t(X)) = C(\theta, \theta)$ implies (2.2) at $\phi = \theta$ and the equality in (1.7). Conversely, if (2.2) holds at $\phi = \theta$ then $\mathrm{Var}_\theta(t(X)) = C(\theta, \theta) = C(\theta)$.*

COROLLARY 2.1.4. *Suppose $t(X)$ is sufficient for the family P_θ, $\theta \in \Theta$, and, for all $\theta \in \Theta$, $t(X)$ can assume only two distinct values, t_1 and t_2 with probabilities p_θ and $1 - p_\theta$ respectively ($0 < p_\theta < 1$). Then $\mathrm{Var}_\theta(t(X)) = C(\theta)$ for all θ.*

PROOF OF THE COROLLARIES. Corollaries 2.1.1 and 2.1.2 are immediate consequences of the theorem and the definition of $C(\theta)$ in (1.7). In either case, $\phi^*(\theta)$ may not be unique. Next, assumptions A imply (see [1]) $C(\theta, \theta) = \lim_{\phi \to \theta} C(\phi, \theta)$ exists and equals $A(\theta)$. Moreover, by L'Hospital's rule, the left side of (2.2) yields $\lim_{\phi \to \theta}\{f_\phi(x)/f_\theta(x) - 1\}/\{\tau(\phi) - \tau(\theta)\} = \{(\partial/\partial\theta)\log f_\theta(x)\}/\{(d/d\theta)\tau(\theta)\}$ a.e. $x \in \mathscr{X}$. The assertions of Corollary 2.1.3 now follow from the well-known necessary and sufficient condition, $(\partial/\partial\theta)\log f_\theta(x) = a(\theta)t(x) + b(\theta)$ a.e. $x \in \mathscr{X}$, for the attainment of (1.1). Finally, to prove Corollary 2.1.4, observe that $\tau(\theta) = t_2 + (t_1 - t_2)p_\theta$ and $\mathrm{Var}_\theta(t(X)) = (t_1 - t_2)^2 p_\theta(1 - p_\theta)$. If $\mathscr{X}_\theta^{(1)} \subset \mathscr{X}_\theta$ denotes the set in which $t(x) = t_1$, then sufficiency of $t(X)$ implies that, for all $\phi \in \Phi_\theta$, both sides of (2.2) equal $(t_1 - t_2)^{-1}$

a.e. $x \in \mathscr{X}_\theta^{(1)}$ and $(t_2 - t_1)^{-1}$ a.e. $x \in \mathscr{X}_\theta - \mathscr{X}_\theta^{(1)}$. Hence, by Corollary 2.1.1, we get that $\operatorname{Var}_\theta(t(X)) = C(\theta)$ for all θ. □

The assumption $C(\theta) = C(\phi^*(\theta), \theta)$ in Corollary 2.1.2 holds trivially when Φ_θ is a finite set. On the other hand, Example 3.1 of Section 3 cites a case where $\operatorname{Var}_\theta(t(X)) = C(\theta)$ for all $\theta \in \Theta$ but relation (2.2) does not hold for any $\phi \in \Phi_\theta$. Assumptions A in Corollary 2.1.3 always hold for the family (1.4) (see [5]), provided of course $\tau(\theta)$ is differentiable. Theorem 2.1 and its corollaries give a method of finding MVU estimators. An obvious consequence of Corollary 2.1.1 is that, if $\tau(\theta) = E_\theta(f_{\theta_1}(X)/f_{\theta_2}(X))$ for a given pair $\theta_1 \neq \theta_2$ in Θ, then $t(X) = f_{\theta_1}(X)/f_{\theta_2}(X)$ is a locally (at $\theta = \theta_2$) MVU estimator of $\tau(\theta)$. The following theorem exploits this fact more fully to give a general characterization of $f_\theta(x)$ and $\tau(\theta)$ for finding UMVU estimators.

THEOREM 2.2. *Suppose that $f_\theta(x)$ is given by* (1.4), *and let $\theta_1 \neq \theta_2$ be a specified pair in Θ satisfying $2\gamma(\theta_1) - 2\gamma(\theta_2) + \gamma(\theta) \in \Gamma$ for all $\theta \in \Theta$, where Γ is the range of $\gamma(\theta)$. Then an unbiased estimator of*

$$\tau(\theta) = \alpha(\theta)/\alpha(\gamma^{-1}\{\gamma(\theta_1) - \gamma(\theta_2) + \gamma(\theta)\}) \tag{2.3}$$

is $t(X) = \exp(\{\gamma(\theta_1) - \gamma(\theta_2)\}g(X))$ with $\operatorname{Var}_\theta(t(X)) = C(\theta)$ for all θ.

PROOF. The monotonicity of $\gamma(\theta)$ implies that Γ is an interval, and therefore conditions $\gamma(\theta) \in \Gamma$ and $2\gamma(\theta_1) - 2\gamma(\theta_2) + \gamma(\theta) \in \Gamma$ imply that $\gamma(\theta_1) - \gamma(\theta_2) + \gamma(\theta) \in \Gamma$. Consequently, $E_\theta(t(X)) = \alpha(\theta) \int h(x) \exp\{[\gamma(\theta_1) - \gamma(\theta_2) + \gamma(\theta)]g(x)\}\mu(dx) = \tau(\theta)$, and similarly $\operatorname{Var}_\theta(t(X)) = \alpha(\theta)/\alpha(\gamma^{-1}\{2\gamma(\theta_1) - 2\gamma(\theta_2) + \gamma(\theta)\}) - \tau^2(\theta)$. For any $\theta \in \Theta$, let $\phi^*(\theta) = \gamma^{-1}\{\gamma(\theta_1) - \gamma(\theta_2) + \gamma(\theta)\}$ which clearly belongs to Φ_θ. Substituting $f_\theta(x)$ of (1.4) and $\phi = \phi^*(\theta)$ in (2.2), one easily verifies that the relation (2.2) holds. It follows from Corollary 2.1.1 that $\operatorname{Var}_\theta(t(X)) = C(\theta)$ for all $\theta \in \Theta$. Note that, since $\gamma(\theta)$ is monotonic, $\phi^*(\theta)$ is unique for every θ. □

If one reparametrizes (1.4) by γ (instead of θ), Theorem 2.2 essentially states that $\exp\{cg(X)\}$ is the UMVU estimator of the parametric function $\tau(\gamma) = \alpha(\gamma)/\alpha(\gamma + c)$, $c \neq 0$ being a given constant. In any practical situation one of course makes an objective judgment as to whether such a parametric function has any statistical motivation. This is also true, we may add, when one speaks of $g(X)$ in (1.4) being the UMVU estimator of $\tau(\gamma) = -\alpha'(\gamma)/\alpha(\gamma)$.

We now give four examples as applications of Theorem 2.2. It will be shown in Section 3 that, in all these cases (with $n > 1$ in Example 2.2), $\operatorname{Var}_\theta(t(X)) > B_k(\theta)$ for all $k \geqq 1$ and $\theta \in \Theta$.

EXAMPLE 2.1. Let $X = (X_1, \cdots, X_n)$, $n > 1$, where the X_i are i.i.d. Poisson rv's with mean $\theta > 0$, and let $\tau(\theta) = \exp(-\theta) = P_\theta(X_1 = 0)$. In (1.4), we now have $\alpha(\theta) = \exp(-n\theta)$, $\gamma(\theta) = \log\theta$ and $g(X) = \sum_{i=1}^n X_i$. Choosing $\theta_1 = (1 - n^{-1})\theta_2$ and θ_2 as any positive number in Theorem 2.2, we conclude that $t(X) = (1 - n^{-1})^{\Sigma X_i}$ is the UMVU estimator of $\tau(\theta)$ with $\operatorname{Var}_\theta(t(X)) = \{\exp(-2\theta)\}\{\exp(\theta/n) - 1\} = C(\theta)$. If $n = 1$, the theorem does not work because one can not choose $\theta_1 \neq \theta_2$ such that $\tau(\theta) = \exp(-\theta)$ (but the theorem still applies to find the UMVU estimator of $\exp((c-1)\theta)$ for and positive $c \neq 1$). If $n = 1$, one verifies directly that the only unbiased estimator of $\exp(-\theta)$ is $t(X) = 1$ if $X = 0$ and $t(X) = 0$, otherwise.

EXAMPLE 2.2. Let $X = (X_1, \cdots, X_n)$, $n \geqq 1$, where the X_i are i.i.d. Bernoulli variables with mean $\theta \in (0, 1)$, and let $\tau(\theta) = (1 - \theta + c\theta)^n$, $c \neq 1$ being a given positive number. Clearly, $\tau(\theta)$ is the moment-generating function $M_\theta(t)$ of $\sum X_i$ at $t = \log c$. Here, $\alpha(\theta) = (1-\theta)^n$, $\gamma(\theta) = \log\{\theta/(1-\theta)\}$ and $g(X) = \sum X_i$. Choosing $\theta_1 = c\theta_2/(1 - \theta_2 + c\theta_2)$ and θ_2 as any number on $(0, 1)$, we get $t(X) = c^{\sum X_i}$ as the UMVU estimator with $\operatorname{Var}(t(X)) = (1 - \theta + c^2\theta)^n - (1 - \theta + c\theta)^{2n} = C(\theta)$.

EXAMPLE 2.3. Let $X = (X_1, \cdots, X_n)$, $n \geqq 1$, the X_i being i.i.d. normal rv's with mean $\theta \in (-\infty, \infty)$ and variance 1, and let $\tau(\theta) = \exp(c\theta)$, $c \neq 0$. Here, $\alpha(\theta) = \exp(-n\theta^2/2)$, $\gamma(\theta) = \theta$ and $g(X) = \sum X_i$. Choosing $\theta_1 = \theta_2 + c/n$ and θ_2 as any real number, we get $t(X) = \exp\{cn^{-1} \sum X_i - c^2/(2n)\}$ as the UMVU estimator with $\operatorname{Var}_\theta(t(X)) = \{\exp(2c\theta)\}\{\exp(c^2/n) - 1\} = C(\theta)$.

EXAMPLE 2.4. Let $X = (X_1, \cdots, X_n)$, $n \geqq 1$, the X_i being i.i.d. exponential variables with mean $\theta > 0$, and let $\tau(\theta) = (1 + c\theta)^{-n}$, where c is any given positive number. Here, $\alpha(\theta) = \theta^{-n}$, $\gamma(\theta) = \theta^{-1}$ and $g(X) = -\sum X_i$. Choosing $\theta_1 = \theta_2/(1 + c\theta_2)$ and θ_2 as any positive number, we get $t(X) = \exp(-c \sum X_i)$ as the UMVU estimator with $\operatorname{Var}_\theta(t(X)) = (1 + 2c\theta)^{-n} - (1 + c\theta)^{-2n} = C(\theta)$.

Since $t(X)$ of Theorem 2.2 is a complete sufficient statistic (see [5]) for the family (1.4), the UMVU character of the estimators in Examples 2.1–2.4 also follows from the Lehmann–Scheffé theorem. However, the following example shows that $C(\theta)$ can be attained even in the absence of a complete sufficient statistic (and when $B_k(\theta)$ is unattainable for any $k \geqq 1$, see Example 3.2).

EXAMPLE 2.5. Let X be a nonnegative integer-valued rv with density $f_\theta(x) = \delta_{0x}\theta + (1 - \delta_{0x})(1-\theta)^2\theta^{x-1}$, $\theta \in [0, 1]$ and δ_{ij} is the usual Kronecker delta. Let $\tau(\theta) = (1-\theta)^2 = P_\theta(X = 1)$. Here $f_\theta(x)$ can not be expressed in the form (1.4). Moreover, the sufficient statistic X is not complete, for $E_\theta(X - 1) = 0$ but $P_\theta(X \neq 1) > 0$ for all $\theta \in (0, 1)$. Hence, the Lehmann–Scheffé theorem does not apply. Consider now $t(X) = \delta_{1X}$, $X \geqq 0$, so that $E_\theta(t(X)) = (1-\theta)^2 = \tau(\theta)$ and $\operatorname{Var}_\theta(t(X)) = \theta(1-\theta)^2(2-\theta)$ for all $\theta \in [0, 1]$. If $\theta = 0$ or 1, then $\operatorname{Var}_\theta(t(X)) = 0 = C(\theta)$. For any $\theta \in (0, 1)$, Corollary 2.1.1 with $\phi^*(\theta) = 0$ shows that relation (2.2) holds for all $x \geqq 0$. Consequently, $\operatorname{Var}_\theta(t(X)) = C(\theta)$ for all $\theta \in [0, 1]$, and $t(X)$ is the UMVU estimator of $\tau(\theta)$.

3. Comparison between $C(\theta)$ and $B_k(\theta)$. We assume throughout this section that regularity assumptions B_k hold for some $k \geqq 1$. As mentioned earlier, we always have $C(\theta) \geqq B_1(\theta) = A(\theta)$ for all $\theta \in \Theta$, and $C(\theta) = B_1(\theta)$ for all $\theta \in \Theta$ in densities of the form (1.4) with $g(x) = t(x)$. To formulate general conditions under which $C(\theta) > B_1(\theta)$ we define the $\tau^{(i)}$ and V_{ij} as in (1.3), and let

$$\beta_i(\theta) = \tau^{(i)}(\theta)/\tau^{(1)}(\theta)\,, \quad i \geqq 1\,, \qquad \nu_1(\theta) = V_{12}(\theta)/V_{11}(\theta)\,, \tag{3.1}$$
$$\nu_2(\theta) = \{V_{11}(\theta)V_{22}(\theta) - V_{12}^2(\theta)\}/V_{12}^2(\theta)\,.$$

THEOREM 3.1. *If assumptions* B_2 *hold, then* $\beta_2(\theta) \neq \nu_1(\theta)$ *implies* $C(\theta) > B_1(\theta)$. *If* $\beta_2(\theta) = \nu_1(\theta)$ *but assumptions* B_3 *hold, then* $\beta_3(\theta) > \frac{3}{4}\nu_2(\theta) + V_{13}(\theta)/V_{11}(\theta)$ *implies* $C(\theta) > B_1(\theta)$.

PROOF. To prove the first part, if assumptions B_2 hold, then for sufficiently small $|h| > 0$ we have

$$h^{-2}\{\tau(\theta + h) - \tau(\theta)\}^2 = \{\tau^{(1)}(\theta)\}^2\{1 + h\beta_2(\theta) + o(h)\}\,,$$
$$h^{-2}\operatorname{Var}_\theta(f_{\theta+h}(X)/f_\theta(X)) = V_{11}(\theta)\{1 + h\nu_1(\theta) + o(h)\}\,.$$

It follows from (1.2) and (1.7) that $C(\theta) \geqq B_1(\theta)\{1 + h[\beta_2(\theta) - \nu_1(\theta)] + o(h)\}$, which shows that $C(\theta) > B_1(\theta)$ whenever $\beta_2(\theta) \neq \nu_1(\theta)$. The second part is proved similarly by expanding $\tau(\theta + h)$ and $f_{\theta+h}(X)$ up to order h^3 under assumptions B_3. □

COROLLARY 3.1.1. *Assumptions* B_2, $\tau^{(2)}(\theta) \neq 0$ *and* $V_{12}(\theta) = 0$ *imply that* $C(\theta) > B_1(\theta)$.

Note that if $f_\theta(x) = f(x - \theta)$ where f is symmetric about 0, then, under assumptions B_2, f is twice differentiable and $f(x) = f(-x)$, $f'(x) = -f'(-x)$ and $f''(x) = f''(-x)$, for all $x \geqq 0$. Consequently, $V_{12}(\theta) = E_\theta(f_\theta^{-2}(X)f_\theta'(X)f_\theta''(X)) = -\int f^{-2}(x)f'(x)f''(x)\mu(dx) = 0$, and hence from Corollary 3.1.1 we arrive at the following

COROLLARY 3.1.2. *If* $\tau^{(2)}(\theta) \neq 0$ *and* $f_\theta(x) = f(x - \theta)$ *where* f *is symmetric about* 0, *then under assumptions* B_2, $C(\theta) > B_1(\theta)$.

It is clear from the proof of Theorem 3.1 that conditions involving higher order derivatives of $\tau(\theta)$ and $f_\theta(x)$ can also be framed to assert $C(\theta) > B_1(\theta)$. One easily verifies that Corollary 3.1.1 applies to our Examples 2.1–2.5 and to the example of [1] for all $\theta \in \Theta$ ($\theta = 0, 1$ excluded in Example 2.5). In fact, in these examples as well as in many others (e.g., Cauchy distribution with scale or location parameter θ, logistic distribution with parameter θ etc.), whenever $\tau(\theta)$ has a nonzero second derivative, $C(\theta) > B_1(\theta)$.

The method of Theorem 3.1 cannot be used to compare $C(\theta)$ and $B_k(\theta)$ when $k \geqq 2$. However, for the family (1.4), we are able to specify simple conditions on $\tau(\theta)$ such that $C(\theta) >$ (or $<$) $B_k(\theta)$ for all $k \geqq 1$ and $\theta \in \Theta$.

THEOREM 3.2. *Under the assumptions of Theorem* 2.2, $\operatorname{Var}_\theta(t(X)) = C(\theta) > B_k(\theta)$ *for all* $k \geqq 1$ *and* $\theta \in \Theta$.

PROOF. Defining $t(X)$ and $\tau(\theta)$ as in Theorem 2.2, we have from there $E_\theta(t(X)) = \tau(\theta)$ and $\operatorname{Var}_\theta(t(X)) = C(\theta)$ for all $\theta \in \Theta$. On the other hand, $t(x)$ is not a polynomial in $g(x)$ but $f_\theta(x)$ is of the form (1.4), and hence it follows from Fend (1959) that $\operatorname{Var}_\theta(t(X)) > B_k(\theta)$ for all $k \geqq 1$ and $\theta \in \Theta$. □

The conclusion of Theorem 3.2 applies to Examples 2.1–2.4 but not to Example 2.5 where $f_\theta(x)$ is not of the form (1.4).

THEOREM 3.3. *Suppose that* X *has the density of the form* (1.4), *and let*

$$\tau(\theta) = E_\theta(\textstyle\sum_{i=0}^k a_i g^i(X)) = E_\theta(t(X)), \quad say, \qquad k > 1 \tag{3.2}$$

where $a_0, \cdots, a_k \neq 0$ *are arbitrary constants. Assume that, for all* $\theta \in \Theta$, (i) $C(\theta) = C(\phi^*(\theta), \theta)$ *for some* $\phi^*(\theta) \in \Theta$ *and* (ii) $P_\theta(g(X) \in A) < 1$ *for all Borel sets containing* $k + 1$ *elements. Then* $\operatorname{Var}_\theta(t(X)) = B_k(\theta) > C(\theta)$ *for all* $\theta \in \Theta$.

PROOF. Since $f_\theta(x)$ is of the form (1.4) and $t(x)$ is a polynomial in $g(x)$ of degree k, it follows from Fend (1959) that $\operatorname{Var}_\theta(t(X)) = B_k(\theta)$ for all $\theta \in \Theta$. We shall now show that $\operatorname{Var}_\theta(t(X)) > C(\theta)$ for all $\theta \in \Theta$ by forcing a contradiction. Suppose $\operatorname{Var}_\theta(t(X)) = C(\theta)$ and $\phi^*(\theta) = \theta$. Then, by Corollary 2.1.3 and (1.4), we must have

$$K_1(\theta)g(x) + K_2(\theta) = \textstyle\sum_{i=0}^k a_i g^i(x) \qquad \text{a.e.} \quad x \in \mathscr{X}, \tag{3.3}$$

where K_1 and K_2 are nonzero functions on Θ. But, since a polynomial in g of degree $k > 1$ can have at most k zeros, the identity (3.3) is impossible to hold under assumption (ii). Suppose, next, $\operatorname{Var}_\theta(t(X)) = C(\theta)$ and $\phi^*(\theta) \neq \theta$. Then, by Corollary 2.1.2 and (1.4), we must have

$$\begin{aligned} K_1(\phi^*, \theta) \exp(\{\gamma(\phi^*) - \gamma(\theta)\}g(x)) \\ = K_2(\phi^*, \theta) + \textstyle\sum_{i=0}^k a_i g^i(x) \qquad \text{a.e.} \quad x \in \mathscr{X}, \end{aligned} \tag{3.4}$$

where K_1 and K_2 are nonzero functions on Θ. But, since the equation $e^y = \sum_{j=0}^k b_j y^j$ has at most $k + 1$ real roots for every set $(b_0, \cdots, b_k)$ of constants, the identity (3.4) is impossible to hold under assumption (ii). Hence, we must have $\operatorname{Var}_\theta(t(X)) > C(\theta)$ for all θ. □

In Examples 2.1–2.4, $\tau(\theta)$ of (3.2) turns out to be a polynomial in θ of degree k (Example 1 in [2] is a special case of our Example 2.4). There are, of course, many special cases of (1.4) where this may not be true (e.g., $f_\theta(x) = \theta \exp(-\theta x)$, $x \geqq 0$). Assumption (ii) of Theorem 3.3 holds, in particular, for every $k \geqq 1$ when (1.4) is a Lebesgue density and g is continuous everywhere in $\mathscr{X}$ (e.g., Examples 2.3, 2.4). It also holds in Example 2.1 for every $k \geqq 1$ and in Example 2.2 when $1 \leqq k < n$. It is easily verified that assumption (i) holds in Example 2.3 and 2.4, so that Theorem 3.3 applies in estimating any second or higher degree polynomial in the mean of a normal or exponential distribution. On the other hand, the validity of assumption (i) in Examples 2.1 and 2.2 depends on the choice of the coefficients a_i in (3.2). This aspect and the fact that the converse of Theorem 3.3 is not true are clarified in the following example.

EXAMPLE 3.1. In Example 2.2, let $n = 2$ and $\tau(\theta) = b_0 + b_1\theta + b_2\theta^2$, where b_0, b_1 and $b_2 \neq 0$ are given constants. Here $k = 2$, $g(X) = X_1 + X_2$, and the estimator $t(X) = b_0 + \frac{1}{2}(b_1 - b_2)g(X) + \frac{1}{2}b_2 g^2(X)$ is easily verified to be unbiased for $\tau(\theta)$

with $\text{Var}_\theta(t(X)) = \frac{1}{2}\theta(1-\theta)(b_1^2 + 2b_2(2b_1+b_2)\theta + 2b_2^2\theta^2) = B_2(\theta)$ for all $\theta \in \Theta = (0, 1)$. Now, assumption (ii) does not hold for $A = \{0, 1, 2\}$ and therefore the conclusion $\text{Var}_\theta(t(X)) > C(\theta)$ does not follow from Theorem 3.3 for any $\theta \in \Theta$. Note that here $C(\phi, \theta) = \theta^2(1-\theta)^2(b_1 + b_2\phi + b_2\theta)^2/\{(\phi-\theta)^2 + 2\theta(1-\theta)\}$, which has a unique absolute maximum at $\phi^* = (b_1 + 2b_2)\theta/(b_1 + 2b_2\theta)$, and $C(\phi^*, \theta) = B_2(\theta)$. However, depending on the choice of b_1 and b_2, ϕ^* may be an inner point of Θ (assumption (ii) holds), or a boundary or external point of Θ (assumption (ii) does not hold). Thus, if $\tau(\theta) = \theta(1+\theta)$, then $\phi^* = 3\theta/(1+2\theta) \in (0, 1)$ and $C(\theta) = B_2(\theta)$ for all θ (Corollary 2.1.1 applies here). If $\tau(\theta) = \theta^2$, then $\phi^* = 1$ is a boundary point and $C(\theta) = B_2(\theta)$ for all θ, which incidentally shows why the second condition of Corollary 2.1.2 is needed. Finally, if $\tau(\theta) = \theta(1-\theta)$, then $\phi^* = \theta/(2\theta - 1)$ is exterior to Θ and $C(\theta) < B_2(\theta)$ for all θ, which also shows that assumptions (i) and (ii) are not necessary for the assertion of Theorem 3.3.

When the form of $f_\theta(x)$ is different from (1.4), or the form of $t(X)$ is different from those in Theorems 3.2 and 3.3, it seems difficult to formulate general conditions under which $C(\theta) <$ (or $>$) $B_k(\theta)$ for $k > 1$. Nevertheless, the following theorem provides a partial answer.

THEOREM 3.4. *Suppose that assumptions B_k hold for some $k > 1$, $\mathbf{V}_\theta$ is continuous in $\theta \in \Theta$, and for ϕ in the neighbourhood of ϕ^*, where the supremum in* (1.7) *occurs, we have*

$$(3.5)\qquad |\tau(\phi) - \tau(\theta)| \leqq |\textstyle\sum_{i=1}^k (\phi-\theta)^i \tau^{(i)}(\theta)/i!|$$

and

$$(3.6)\qquad \text{Var}_\theta(f_\phi(X)/f_\theta(X)) \geqq \textstyle\sum_{i=1}^k \sum_{j=1}^k (\phi-\theta)^{i+j} V_{ij}(\theta)/(i!\,j!)\,.$$

Then $B_k(\theta) \geqq C(\theta)$ and the strict inequality holds if (3.5) *or* (3.6) *is a strict inequality.*

PROOF. Using (3.5) and (3.6) in (1.7), we obtain that

$$(3.7)\qquad \begin{aligned} C(\theta) &\leqq \sup_{\phi\in\Phi_\theta}\left(\frac{\{\sum_{i=1}^k (\phi-\theta)^i\tau^{(i)}(\theta)/i!\}^2}{\sum_{i=1}^k\sum_{j=1}^k (\phi-\theta)^{i+j}V_{ij}(\theta)/(i!\,j!)}\right) \\ &\leqq \sup_{\mathbf{\Delta}}\{(\mathbf{\Delta}'\boldsymbol{\tau}_\theta)^2/(\mathbf{\Delta}'\mathbf{V}_\theta\mathbf{\Delta})\}\,, \quad \mathbf{\Delta} = ((\phi-\theta)/1!, \cdots, (\phi-\theta)^k/k!)'\,, \end{aligned}$$

where $\boldsymbol{\tau}_\theta$ and $\mathbf{V}_\theta$ are defined in (1.3), and the supremum in (3.7) extends over the range $\{\mathbf{\Delta} : \phi \in \Theta_\theta\}$. Obviously, (3.7) becomes a strict inequality if either (3.5) or (3.6) is the same. Since $\{\mathbf{\Delta} : \phi \in \Phi_\theta\} \subseteqq R^k$, (3.7) yields

$$(3.8)\qquad C(\theta) \leqq \sup_{\mathbf{\Delta}\in R^k}\{(\mathbf{\Delta}'\boldsymbol{\tau}_\theta)^2/(\mathbf{\Delta}'\mathbf{V}_\theta\mathbf{\Delta})\} = \boldsymbol{\tau}_\theta'\mathbf{V}_\theta^{-1}\boldsymbol{\tau}_\theta = B_k(\theta)\,.$$

The penultimate equality in (3.8) follows from the well-known fact that, if $\mathbf{A} = \mathbf{aa}'$ and $\mathbf{B}$ (positive definite) are two $p \times p$ matrices, then $\sup_{\mathbf{x}}\{(\mathbf{x}'\mathbf{aa}'\mathbf{x})/(\mathbf{x}'\mathbf{Bx})\} =$ largest characteristic root of $\mathbf{aa}'\mathbf{B}^{-1} = \mathbf{a}'\mathbf{B}^{-1}\mathbf{a}$. □

Assumption (3.5) holds for all ϕ when $\tau(\theta)$ is a polynomial in θ. It also holds for some ϕ for functions like $\tau(\theta) = \theta^m$, $\theta > 0$, $0 < m < 1$, and $\tau(\theta) = \theta\log\theta$,

$\theta > 1$. Unfortunately, it does not apply to Example 2 in [1] (i.e., our Example (vi) of Section 1) nor to Example 2 in [2] (i.e., our Example 2.4 with $\tau(\theta) = \theta^m$, $0 < m < 1$) for ϕ in the neighbourhood of ϕ^*. Assumption (3.6) holds for all ϕ when $f_\phi(x)/f_\theta(x)$ possesses an orthogonal expansion (e.g., Examples 2.3, 2.4). Example (iv) of Section 1 is an application of Theorem 3.4.

There are situations where Theorems 3.2–3.4 do not apply, but we may have $C(\theta) > B_k(\theta)$ for all $k \geqq 1$ and $\theta \in \Theta$. The following is a case in point.

EXAMPLE 3.2. Consider the problem of Example 2.5 where we showed that $\mathrm{Var}_\theta\,(t(X)) = \mathrm{C}(\theta)$ for all $\theta \in [0, 1]$. We shall now show that $\mathrm{Var}_\theta\,(t(X)) > B_k(\theta)$ for all $k \geqq 1$ and $\theta \in (0, 1)$. It is easily verified that assumptions B_k hold for each $k \geqq 1$ and all $\theta \in (0, 1)$. Now, the strict equality in (1.2) holds (see [2]) iff $t(X) = \tau(\theta) + \sum_{i=1}^k a_i(\theta) f_\theta^{-1}(X)(\partial^i/\partial\theta^i) f_\theta(X)$ with probability one, where $a_k(\theta) \neq 0$ for some $k \geqq 1$. In our case, a necessary condition for this identity to hold is easily shown to be

$$\sum_{i=1}^k a_i^*(\theta)\{\tbinom{x+1}{i}\theta^2 - 2\tbinom{x}{i}\theta + \tbinom{x-1}{i}\} = (1-\theta)^2 a_1^*(\theta)\,, \qquad \text{for } x = 2, 3, \cdots, \tag{3.9}$$

where $a_i^*(\theta) = i!\, a_i(\theta)/\theta^i$, $i \geqq 1$. But, for any $\theta \in (0, 1)$ and $k \geqq 1$, (3.9) is a polynomial in x of degree k and can therefore hold for at most k values of x. It follows that $a_k(\theta) = 0$ for all $k \geqq 1$ and $\theta \in (0, 1)$, and, by contradiction, the equality in (1.2) cannot hold for any $k \geqq 1$ and $\theta \in (0, 1)$.

4. $C(\theta)$ in nonregular families.

One chief advantage of $C(\theta)$ over $B_k(\theta)$, $k \geqq 1$, is that the former applies to many situations where the Cramér–Rao regularity assumptions (and hence B_k for $k \geqq 1$) do not hold. This is true, in particular, when Θ is a countable set or when the range $\mathscr{X}_\theta$ depends on θ. Hammersley (1950) proposed $C(\theta)$ in the context of former possibilities, while Kiefer (1952) proposed a refinement of $C(\theta)$ to handle the latter possibilities. We shall now show that, for a wide class of such nonregular families, the lower bound $C(\theta)$ is unattainable by the UMVU estimator.

As an extreme example, suppose that $X = (X_1, \cdots, X_n)$, $n \geqq 1$, where the X_i are i.i.d. uniform rv's on $[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$, $\theta \in (-\infty, \infty)$ being the unknown location parameter. Then Φ_θ defined by (1.5) and (1.6) is empty for every θ, so that $C(\theta) = 0$ for all θ. On the other hand, any unbiased estimator $t(X)$ of every estimable (nonconstant) $\tau(\theta)$ has $\mathrm{Var}_\theta\,(t(X)) > 0$ for all θ. The following theorem provides an answer in the same direction for less extreme cases (where Φ_θ is not empty).

THEOREM 4.1. *Suppose there exists a* $\phi^* = \phi^*(\theta) \in \Phi_\theta$ *such that* (i) $C(\theta) = C(\phi^*, \theta)$, (ii) $P_\theta(X \in \mathscr{X}_{\phi^*}) < 1$, *and* (iii) $P_\theta(t(X) = \textit{constant}, X \in \mathscr{X}_\theta - \mathscr{X}_{\phi^*}) = 0$. *Then* $\mathrm{Var}_\theta\,(t(X)) > C(\theta)$. *Moreover, if* $t(X)$ *is sufficient for the family* P_θ, $\theta \in \Theta$, *then* $\mathrm{Var}_\theta\,(t^*(X)) > C(\theta)$ *for every unbiased estimator* $t^*(X)$ *of* $\tau(\theta)$.

PROOF. Condition $\phi^* \in \Phi_\theta$ and the definition of Φ_θ imply $\mathscr{X}_{\phi^*} \subseteqq \mathscr{X}_\theta$ and

$f_{\phi^*}(x) = 0$ a.e. $x \in \mathscr{X}_\theta - \mathscr{X}_{\phi^*}$. We shall prove the first part of the theorem by forcing a contradiction. If $\text{Var}_\theta(t(X)) = C(\theta)$, then by assumption (i) and Corollary 2.1.2 we must have

$$(4.1) \qquad t(x) = \tau(\theta) + \{\tau(\theta) - \tau(\phi^*)\}^{-1} \text{Var}_\theta(t(X)) \qquad \text{a.e.} \quad x \in \mathscr{X}_\theta - \mathscr{X}_{\phi^*}.$$

By assumption (ii), relation (4.1) has positive probability under P_θ, which contradicts assumption (iii). Hence, we must have $\text{Var}_\theta(t(X)) > C(\theta)$. The second part is a direct consequence of the Blackwell–Rao theorem (see [5]). □

COROLLARY 4.1.1. *Suppose that* $f_\theta(x) = \alpha(\theta)h(x)$ *for all* $x \in \mathscr{X}_\theta$ *and* $\theta \in \Theta$. *If* (i) $C(\theta) = C(\phi^*, \theta)$ *and* (ii) *either* $P_\theta(t(X) = \text{constant}, X \in \mathscr{X}_{\phi^*}) = 0$ *or* $P_\theta(t(X) = \text{constant}, X \in \mathscr{X}_\theta - \mathscr{X}_{\phi^*}) = 0$, *then* $\text{Var}_\theta(t(X)) > C(\theta)$.

PROOF. If $\text{Var}_\theta(t(X)) = C(\theta)$, following (4.1), we must have $t(x) = K_1(\theta)$ a.e. $x \in \mathscr{X}_{\phi^*}$ and $t(x) = K_2(\theta)$ a.e. $x \in \mathscr{X}_\theta - \mathscr{X}_{\phi^*}$, contradicting assumption (ii) of the corollary. □

Note that, if $\mathscr{X}_\theta$ is not monatomic and $\mathscr{X}_\phi$ is not a subset of $\mathscr{X}_\theta$ for any $\phi \neq \theta$, then we have trivially $\text{Var}_\theta(t(X)) > C(\theta) = 0$. We now give two applications of Corollary 4.1.1.

EXAMPLE 4.1. Let $X = (X_1, \cdots, X_n)$, where the X_i are i.i.d. rv's with the uniform $[0, \theta]$ distribution, and let $\tau(\theta) = \theta^m$, $m > -\frac{1}{2}$ being specified. Here, $\alpha(\theta) = \theta^{-n}$, $h(x) = 1$, $\Theta = (0, \infty)$ and $\mathscr{X}_\theta = \{(x_1, \cdots, x_n): 0 \leqq x_i \leqq \theta, i = 1, \cdots, n\}$, so that (1.5) and (1.6) imply $\Phi_\theta = (0, \theta)$ for and $\theta > 0$. It is easily verified from $C(\phi, \theta) = (\phi^m - \theta^m)^2/\{(\theta/\phi)^n - 1\}$ that $C(\phi, \theta)$ attains a maximum at some $\phi^* \in (0, \theta)$ (ϕ^* depends of course on θ, m and n). Thus assumption (i) of Corollary 4.1.1 holds for all $\theta > 0$. The unbiased estimator $t(X) = n^{-1}(n + m)\{\max_{1 \leqq i \leqq n} X_i\}$ is sufficient and satisfies assumption (ii) for all $\theta > 0$ ($t(X)$ is in fact UMVU). Consequently, $\text{Var}_\theta(t^*(X)) > C(\theta)$ for all $\theta > 0$, $m > -\frac{1}{2}$, $n \geqq 1$ and unbiased $t^*(X)$. As a matter of added interest, it can be shown that $\{\text{Var}_\theta(t(X))\}/C(\theta) \to 1.54$ as $n \to \infty$, for any $\theta > 0$ and $m > -\frac{1}{2}$. Taking $m = 1$, one gets Kiefer's (1952) Example 1. His Example 2 (i.e., same $f_\theta(x)$ but $\tau(\theta) = -\log\theta$ and $t(X) = -n^{-1} - \log(\max_{1 \leqq i \leqq n} X_i)$) can be treated similarly, and it has the same features as above.

EXAMPLE 4.2. Consider $f_\theta(x) = \binom{\theta}{n}^{-1}\binom{x-1}{n-1}$, $n \geqq 1$ being a given integer, $x \in \mathscr{X}_\theta = \{n, n + 1, \cdots, \theta\}$, $\theta \in \Theta = \{n, n + 1, \cdots\}$, and let $\tau(\theta) = \theta$. If $\theta = n$, then $\mathscr{X}_\theta$ becomes monoatomic so that $C(\theta) = 0 = \text{Var}_\theta(t(X))$ for any unbiased estimator. If $\theta \geqq n + 1$, then (1.5) and (1.6) imply $\Phi_\theta = \{n, \cdots, \theta - 1\}$, and assumption (i) holds obviously. The unbiased estimator $t(X) = n^{-1}(n + 1)X - 1$ satisfies the second part of assumption (ii) so that $\text{Var}_\theta(t(X)) > C(\theta)$ for all $\theta \geqq n + 1$ (here $t(X)$ is unique).

It is well known that, when $\tau(\theta) = \theta$ in Examples 2.1.–2.4, the bound $C(\theta)$ is actually attained for every $\theta \in \Theta$ by the corresponding UMVU estimator. However, the picture may change quite drastically if the natural parameter spaces in these situations are reduced to a countable set. Hammersley (1950) showed this for $\tau(\theta) = \theta$ in our Example 2.3. The following theorem formalizes this aspect and it complements Theorems 2.2 and 4.1.

THEOREM 4.2. *Suppose that* $f_\theta(x)$ *is given by* (1.4) *but* Θ *is a countable set, and assume that* $C(\theta) = C(\phi^*, \theta)$ *for some* $\phi^* = \phi^*(\theta) \neq \theta$ *in* Θ. *Then* $\mathrm{Var}_\theta(t(X)) = C(\theta)$ *only if* $t(x) = A(\theta) + B(\theta)\exp\{[\gamma(\phi^*) - \gamma(\theta)]g(x)\}$ a.e. $x \in \mathscr{X}$ *for some* $A(\theta)$ *and* $B(\theta)$, *and then* $\tau(\theta) = A(\theta) + B(\theta)\alpha(\theta)/\alpha(\phi^*)$.

PROOF. The result follows by substituting (1.4) in Corollaries 2.1.1 and 2.1.2, and we have, in fact, $B(\theta) = [\alpha(\phi^*)/\alpha(\theta)][\tau(\phi^*) - \tau(\theta)]^{-1}\mathrm{Var}_\theta(t(X))$. □

EXAMPLE 4.3. Suppose $\tau(\theta) = \theta$ in Example 2.2 and $\Theta = \{0/M, 1/M, \cdots, M/M\}$, where $M(> 1)$ is a given integer. If $\theta = 0$ or 1, $\mathscr{X}$ becomes monatomic so that $C(\theta) = 0 = \mathrm{Var}_\theta(t(X))$ for any unbiased estimator. If $\theta \in \Theta' = \{1/M, \cdots, (M-1)/M\}$, then $f_\theta(x)$ is of the form (1.4) as shown earlier. From $C(\phi, \theta) = (\phi - \theta)^2/[(\phi^2 - 2\phi\theta + \theta)^n\theta^{-n}(1-\theta)^{-n} - 1]$ we find, for any $\theta \in \Theta'$, $C(\theta) = C(\phi^*, \theta)$ with $\phi^* = \theta + M^{-1}$. Consider now the unbiased estimator $t(X) = n^{-1}\sum_{i=1}^n X_i$. If $n = 1$, we can write $t(x)$ as stipulated in Theorem 4.2 by taking $A(\theta) = \theta(1 - M + M\theta) = -B(\theta)$, so that $\mathrm{Var}_\theta(t(X)) = C(\theta)$ for all $\theta \in \Theta'$. On the other hand, if $n > 1$, $t(x)$ can not be expressed this way for any $\theta \in \Theta'$, so that $\mathrm{Var}_\theta(t(X)) > C(\theta)$ for all $\theta \in \Theta'$. It can be shown that $t(X)$ is in fact the UMVU estimator when $M \geqq n$, and interestingly enough one finds $\{C(\theta)\}^{-1}\cdot\mathrm{Var}_\theta(t(X)) \to 1$ as $M \to \infty$ for any $n > 1$ and $\theta \in \Theta'$. This asymptotic property is comparable to the standard result that, in the case of unrestricted $\Theta' = (0, 1)$, $\mathrm{Var}_\theta(t(X)) = C(\theta)$ for all $n \geqq 1$ and θ.

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