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BALANCED FRACTIONAL 2^m FACTORIAL DESIGNS OF EVEN RESOLUTION OBTAINED FROM BALANCED ARRAYS OF STRENGTH $2l$ WITH INDEX $\mu_l = 0$

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Consider a balanced array T of strength $2l$, size N, m constraints and index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$ with $\mu_l = 0$. Under some conditions T yields a design of even resolution ($2l$, say) with N assemblies such that all the effects involving up to $(l-1)$-factor interactions are estimable provided $(l+1)$-factor and higher order interactions are assumed negligible and that the covariance matrix of their estimates is invariant under any permutation of m factors. The alias structure of the effects of l-factor interactions is explicitly given. Such an array T is called an S-type balanced fractional 2^m factorial design of resolution $2l$. Necessary conditions for the existence of the design T are given. For any given N, there are in general a large number of possible S-type balanced fractional 2^m factorial designs of resolution $2l$. Finally a criterion for comparing these designs is given.

1. Introduction. As an important subclass of irregular fractional designs, the concept of balanced designs was first introduced by Chakravarti (1956). Particularly balanced fractional 2^m factorial (briefly, 2^m-BFF) designs of resolution V have been investigated by Srivastava (1970), Srivastava and Chopra (1971 a, b), Chopra and Srivastava (1973 a, b) and others. It is well known from their results that these designs have close relationships with balanced arrays (B-arrays) of strength 4. A B-array of strength t is defined as follows: A (0, 1) matrix T of size $m \times N$ is called a B-array of strength t, size N, m constraints and index set $\{\mu_0, \mu_1, \cdots, \mu_t\}$ if for every t-rowed submatrix $T^{(t)}$ of T, every vector with weight (or number of nonzero elements) j occurs exactly μ_j times ($j = 0, 1, \cdots, t$) as a column of $T^{(t)}$. For the B-array defined above, it is easily shown that $N = \sum_{i=0}^{t} \binom{t}{i}\mu_i$. Thus the term "size N" will be omitted, if it is not necessary.

Recently, Yamamoto, Shirakura and Kuwada (1975) have established a general connection between a 2^m-BFF design of resolution $2l+1$ and a B-array of strength $2l$, m constraints and index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$. These authors have discussed some properties of a triangular type multidimensional partially balanced (TMDPB) association scheme which is defined among the effects up to l-factor interactions. Furthermore using the decomposition of the TMDPB association algebra into its two-sided ideals, Yamamoto, Shirakura and Kuwada (1974) have succeeded in obtaining an explicit formula for the characteristic polynomial of the information matrix M_T of a 2^m-BFF design T of resolution $2l+1$. This polynomial is useful for comparing designs by the popular criteria such as the

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trace, the determinant and the largest root of M_T^{-1}. (For well-known studies on optimal designs using various criteria, see for example, Kiefer (1959).) Shirakura (1975 b) has given optimal (w.r.t. the trace criterion) 2^m-BFF designs of resolution VII for $m = 6$, 7 and 8.

However those investigations have been restricted only to designs of odd resolution. The term "resolution" of a design was introduced by Box and Hunter (1961), as one means of classifying fractional factorial designs. In general it is difficult to obtain a design of even resolution. For work on designs of resolution IV, see for example, Anderson and Srivastava (1972), Margolin (1969), Srivastava and Anderson (1970), Webb (1968). It is shown here that under some conditions, a B-array T of strength $2l$, m constraints and index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$ with $\mu_l = 0$ yields a fractional 2^m factorial design of resolution $2l$ such that all the effects involving up to $(l-1)$-factor interactions are estimable and the covariance matrix of their estimates is invariant under any permutation of m factors. At the same time it is shown that the mean of the effects of l-factor interactions and $\{\binom{m}{l-1} - 1\}$ independent contrasts between these effects are made estimable by the B-array T. Such a B-array T shall be called an S-type balanced fractional 2^m factorial (briefly, 2^m-SBFF) design of resolution $2l$. However the design T is no more of resolution $2l + 1$, since a necessary condition for a 2^m-BFF design to be of resolution $2l + 1$ is that $\mu_l \neq 0$ (see Shirakura and Kuwada (1975)). Also necessary conditions for the existence of the design T are given. For B-arrays of strength $2l$ with $\mu_l = 0$, their combinational properties have been already studied by Shirakura (1975 a). There are, in general, a large number of possible 2^m-SBFF designs of resolution $2l$ with N assemblies. Finally a criterion for comparing these designs is given.

2. Preliminaries. Consider a 2^m factorial design with m factors $F_1, F_2, \cdots, F_m$. An assembly or treatment combination will be represented by $(j_1, j_2, \cdots, j_m)$, where j_k, the level of F_k, equals 0 or 1. Consider the situation where $(l+1)$-factor and higher order interactions are assumed negligible for any fixed integer $l\,(1 \leqq l \leqq m/2)$ throughout this paper. Then the total number of unknown parameters is $\nu_l = 1 + \binom{m}{1} + \cdots + \binom{m}{l}$. The vector of unknown parameters $\boldsymbol{\theta}(\nu_l \times 1)$ will be written as

$$\begin{aligned}\boldsymbol{\theta}' &= (\theta_\phi; \theta_1, \cdots, \theta_m; \theta_{12}, \theta_{13}, \cdots, \theta_{m-1\cdots m}; \cdots; \theta_{12\cdots l}, \cdots, \theta_{m-l+1\cdots m}) \\ &= (\{\theta_\phi\}; \{\theta_{t_1}\}; \{\theta_{t_1 t_2}\}; \cdots; \{\theta_{t_1\cdots t_l}\}),\end{aligned}$$

where θ_ϕ denotes the general mean, θ_{t_1} denotes the main effect of the factor F_{t_1} and, in general, $\theta_{t_1 t_2 \cdots t_k}$ denotes the k-factor interaction of the factors $F_{t_1}, F_{t_1}, \cdots, F_{t_k}$.

Let T be a fraction with N assemblies, then T can be expressed as a (0, 1) matrix of size $m \times N$ whose columns denote assemblies. Consider the $N \times 1$ observation vector $\mathbf{y}(T')$ of T with the covariance matrix $\sigma^2 I_N$ (σ^2 is the unknown variance and I_N denotes the identity matrix of size N). Then $\mathbf{y}(T')$ can be expressed as

$$\mathscr{E}(\mathbf{y}(T')) = E_T \boldsymbol{\theta}\,, \tag{2.1}$$

where $\mathscr{E}$ stands for the expected value and E_T is the $N \times \nu_l$ design matrix of T whose elements are -1 or 1 (see, e.g., Yamamoto, Shirakura and Kuwada 1975)). The normal equations for estimating $\boldsymbol{\theta}$ are

$$M_T \hat{\boldsymbol{\theta}} = E_T' \mathbf{y}(T') , \tag{2.2}$$

where M_T $(= E_T' E_T)$ is the information matrix of T. A fractional design T is of resolution $2l + 1$ if and only if M_T is nonsingular. For any design T of resolution $2l + 1$, the best linear unbiased estimate (BLUE) of $\boldsymbol{\theta}$ and the covariance matrix of its estimate are given by $\hat{\boldsymbol{\theta}} = V_T E_T' \mathbf{y}(T')$ and Var $(\hat{\boldsymbol{\theta}}) = \sigma^2 V_T$, respectively, where $V_T = M_T^{-1}$.

When V_T is invariant under any permutation of m factors, T is called a 2^m-BFF design of resolution $2l + 1$. In other words, V_T is such that the covariance Cov $(\hat{\theta}_{t_1 \cdots t_u}, \hat{\theta}_{t_1' \cdots t_v'})$ of any two estimates $\hat{\theta}_{t_1 \cdots t_u}$ and $\hat{\theta}_{t_1' \cdots t_v'}$ in $\hat{\boldsymbol{\theta}}$ is a function of u, v and $|\{t_1, \cdots, t_u\} \cap \{t_1', \cdots, t_v'\}|$, and the variance Var $(\hat{\theta}_{t_1 \cdots t_u})$ is a function of u only. Here, $|\cdot|$ denotes the cardinality of a set. For a design T of resolution $2l + 1$, it has been shown in Yamamoto, Shirakura and Kuwada (1975) that a necessary and sufficient condition for T to be a 2^m-BFF design is that T is a B-array of strength $2l$, m constraints and index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$.

Consider $l + 1$ sets $\{\theta_\phi\}$, $\{\theta_{t_1}\}$, $\{\theta_{t_1 t_2}\}$, $\cdots$, and $\{\theta_{t_1 t_2 \cdots t_l}\}$ of effects, the cardinalities of these sets being $1, m, \binom{m}{2}, \cdots$, and $\binom{m}{l}$, respectively. Among these sets, an $l + 1$ sets TMDPB association scheme is defined in a way such that $\theta_{t_1 \cdots t_u}$ and $\theta_{t_1' \cdots t_v'}$ are the αth associates if

$$|\{t_1, \cdots, t_u\} \cap \{t_1', \cdots, t_v'\}| = \min(u, v) - \alpha ,$$

where min (u, v) denotes the minimum of the integers u and v. For this association scheme, we shall use the same matrix notations $A_\beta^{(u,v)\#}$, $D_\alpha^{(u,v)}$, $D_\beta^{(u,v)\#}$ and $B_\alpha^{(u,v)}$ as in Yamamoto, Shirakura and Kuwada (1974, 1975). Therefore the reader is referred to those papers for properties of these matrices used here.

Now consider the information matrix M_T of a B-array T of strength $2l$, m constraints and index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$. Then it is easily shown that M_T can be expressed as

$$M_T = \sum_{\beta=0}^{l} \sum_{i=0}^{l-\beta} \sum_{j=0}^{l-\beta} \kappa_\beta^{i,j} D_\beta^{(\beta+i,\beta+j)\#} , \tag{2.3}$$

where for $0 \leqq i \leqq j \leqq l - \beta$; $\beta = 0, 1, \cdots, l$,

$$\kappa_\beta^{i,j} = \kappa_\beta^{j,i} = \sum_{\alpha=0}^{\beta+i} \gamma_{j-i+2\alpha} z_{\beta\alpha}^{(\beta+i,\beta+j)} . \tag{2.4}$$

Here

$$\gamma_i = \sum_{j=0}^{2l} \sum_{p=0}^{i} (-1)^p \binom{i}{p} \binom{2l-i}{j-i+p} \mu_j \qquad \text{for} \quad i = 0, 1, \cdots, 2l ,$$

$$z_{\beta\alpha}^{(u,v)} = \sum_{b=0}^{\alpha} (-1)^{\alpha-b} \frac{\binom{u-\beta}{b}\binom{u-b}{u-\alpha}\binom{m-u-\beta+b}{b}\left\{\binom{m-u-\beta}{v-u}\binom{v-\beta}{v-u}\right\}^{\frac{1}{2}}}{\binom{v-u+b}{b}} \tag{2.5}$$

$$\text{for} \quad 0 \leqq \alpha , \quad \beta \leqq u \leqq v \leqq l .$$

We assume throughout this paper that $\binom{a}{b} = 0$ if and only if $b > a \geqq 0$ or $b < 0$.

Let $\mathfrak{A}$ be the $l+1$ sets TMDPB association algebra generated by all $\binom{l+3}{3}$ association matrices $B_\alpha^{(u,v)}$. It is known that T is a 2^m-BFF design of resolution $2l+1$ if and only if $V_T\ (= M_T^{-1}) \in \mathfrak{A}$. Algebraic details of $\mathfrak{A}$ will be stated in Appendix. It follows from (2.3) and Appendix that the irreducible representations of M_T are given by the $(l-\beta+1)\times(l-\beta+1)$ matrices $\mathbf{K}_\beta$ such that

$$\text{(2.6)} \qquad \mathbf{K}_\beta = \begin{bmatrix} \kappa_\beta^{0,0} & \kappa_\beta^{0,1} & \cdots & \kappa^{0,l-\beta} \\ \vdots & \vdots & & \vdots \\ \kappa_\beta^{l-\beta,0} & \kappa_\beta^{l-\beta,1} & \cdots & \kappa_\beta^{l-\beta,l-\beta} \end{bmatrix} \qquad \text{for } \beta = 0, 1, \cdots, l\,.$$

3. 2^m-SBFF designs of resolution $2l$. Consider the vector of unknown parameters $\boldsymbol{\theta}$ in the following partitioned form:

$$\boldsymbol{\theta}' = (\boldsymbol{\theta}_1'; \boldsymbol{\theta}_2')\,,$$

where $\boldsymbol{\theta}_1(\nu_{l-1}\times 1)$ is the vector whose elements are effects involving up to $(l-1)$-factor interactions and $\boldsymbol{\theta}_2(\binom{m}{l}\times 1)$ is the vector whose elements are effects of l-factor interactions only, i.e., $\boldsymbol{\theta}_1' = (\{\theta_\phi\}, \{\theta_{t_1}\}, \cdots, \{\theta_{t_1\cdots t_{l-1}}\})$ and $\boldsymbol{\theta}_2' = (\{\theta_{t_1\cdots t_l}\})$.

DEFINITION. A design T is said to be a 2^m-BFF design of resolution $2l$ if all the parameters in $\boldsymbol{\theta}_0 = (\{\theta_{t_1}\}, \{\theta_{t_1t_2}\}, \cdots, \{\theta_{t_1\cdots t_{l-1}}\})'$ are estimable and the covariance matrix Var $(\hat{\boldsymbol{\theta}}_0)$ of its BLUE $\hat{\boldsymbol{\theta}}_0$ is invariant under any permutation of m factors.

2^m-BFF designs of resolution $2l+1$, of course, are also of resolution $2l$. We are interested in 2^m-BFF designs of resolution $2l$ which are not of resolution $2l+1$. In the following we shall obtain a 2^m-BFF design of resolution $2l$ such that all the parameters in $\boldsymbol{\theta}_1$ are estimable and the covariance matrix Var $(\hat{\boldsymbol{\theta}}_1)$ of its BLUE $\hat{\boldsymbol{\theta}}_1$ is invariant under any permutation of m factors.

In (2.6) consider a B-array T of strength $2l$ and m constraints with index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$ such that the following condition is satisfied:

$$\text{(3.1)} \qquad |\mathbf{K}_\beta| \neq 0 \quad \text{for all} \quad \beta = 0, 1, \cdots, l-1\,, \qquad \mathbf{K}_l = 0\,.$$

This condition implies that the matrices $\mathbf{K}_\beta$ $(\beta = 0, 1, \cdots, l-1)$ are positive definite, since M_T is positive semidefinite.

EXAMPLE 1. The following is a B-array with $m = 8$, $t = 6$ (i.e., $l = 3$), index set $\{3, 3, 1, 0, 1, 2, 2\}$ and $N = 65$;

$$\left[\begin{array}{c:c:c:c} \Omega(1;8) & \Omega(2;8) & \Omega(6;8) & \begin{matrix}1\\1\\ \vdots\\1\end{matrix} \end{array}\right],$$

where $\Omega(j; m)$ is the (0, 1) matrix of size $m\times\binom{m}{j}$ whose columns are all the distinct vectors of weight j $(0 \leqq j \leqq m)$. Then it can be easily checked that this array satisfies Condition (3.1). (An explicit expression of $\mathbf{K}_\beta$ ($\beta = 0, 1, 2, 3$) for the case $l = 3$ has been given by Shirakura (1975 b).)

Let C be a $\nu_l \times \nu_l$ matrix such that

$$C = \operatorname{diag}[I_{\nu_{l-1}}, H],$$

where $H = h_0 A_0^{(l,l)\#} + h_1 A_1^{(l,l)\#} + \cdots + h_{l-1} A_{l-1}^{(l,l)\#}$ (h_β is any real number). Then the matrix C is also expressed as

$$\begin{aligned} C &= \sum_{u=0}^{l-1} \sum_{\beta=0}^{u} D_\beta^{(u,u)\#} + \sum_{\beta=0}^{l-1} h_\beta D_\beta^{(l,l)\#} \\ &= \sum_{\beta=0}^{l-1} \{\sum_{u=0}^{l-\beta-1} D_\beta^{(u+\beta,u+\beta)\#} + h_\beta D_\beta^{(l,l)\#}\}. \end{aligned}$$

LEMMA 3.1. *For a B-array T satisfying Condition* (3.1), *there exists a $\nu_l \times \nu_l$ matrix X such that $XM_T = C$.*

PROOF. It follows from Appendix that the irreducible representations of C are given by the $(l - \beta + 1) \times (l - \beta + 1)$ matrices Γ_β such that

$$\Gamma_\beta = \operatorname{diag}[I_{l-1}, h_\beta] \qquad \text{for } \beta = 0, 1, \cdots, l-1.$$

From Condition (3.1), we have the $(l - \beta + 1) \times (l - \beta + 1)$ matrix $\boldsymbol{\chi}_\beta = \Gamma_\beta \cdot \mathbf{K}_\beta^{-1}$ for each $\beta = 0, 1, \cdots, l-1$. Let

$$X = \sum_{\beta=0}^{l-1} \sum_{i=0}^{l-\beta} \sum_{j=0}^{l-\beta} \chi_\beta^{i,j} D_\beta^{(\beta+i,\beta+j)\#}, \tag{3.2}$$

where $\chi_\beta^{i,j}$ are (i, j) elements of $\boldsymbol{\chi}_\beta$, then X satisfies $XM_T = C$ clearly.

THEOREM 3.1. *Let T be a B-array satisfying Condition* (3.1). *Then a parametric function*

$$\boldsymbol{\psi} = C\boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\psi}_2 \end{bmatrix}, \tag{3.3}$$

where $\boldsymbol{\psi}_2 = H\boldsymbol{\theta}_2$ is an estimable function of $\boldsymbol{\theta}$. The BLUE *$\hat{\boldsymbol{\psi}}$ of $\boldsymbol{\psi}$ is given by*

$$\hat{\boldsymbol{\psi}} = XE_T'\mathbf{y}(T'),$$

where X is the matrix in (3.2).

PROOF. From (2.1) and Lemma 3.1, $\mathscr{E}(\hat{\boldsymbol{\psi}}) = XE_T'\mathscr{E}(\mathbf{y}(T')) = XE_T'E_T\boldsymbol{\theta} = XM_T\boldsymbol{\theta} = C\boldsymbol{\theta} = \boldsymbol{\psi}$. Hence $\boldsymbol{\psi}$ is an estimable function of $\boldsymbol{\theta}$. On the other hand, from Gauss-Markov theorem it follows that the BLUE $\hat{\boldsymbol{\psi}}$ of $\boldsymbol{\psi}$ is uniquely given by $\hat{\boldsymbol{\psi}} = C\hat{\boldsymbol{\theta}}$ where $\hat{\boldsymbol{\theta}}$ is a solution of the normal equations in (2.2). Hence we have $\hat{\boldsymbol{\psi}} = XM_T\hat{\boldsymbol{\theta}} = XE_T'\mathbf{y}(T')$.

The estimability of $\boldsymbol{\psi}_2$ implies that $A_\beta^{(l,l)\#}\boldsymbol{\theta}_2$ are estimable for all $\beta = 0, 1, \cdots, l-1$. From the properties of $A_\beta^{(l,l)\#}$, it follows that (i) every element of the vector $A_0^{(l,l)\#}\boldsymbol{\theta}_2$ represents the mean of effects of l-factor interactions, (ii) the elements of $A_\beta^{(l,l)\#}\boldsymbol{\theta}_2$ ($\beta \neq 0$) represent contrasts between these effects, (iii) any two contrasts, one belonging to $A_\alpha^{(l,l)\#}\boldsymbol{\theta}_2$ and the other to $A_\beta^{(l,l)\#}\boldsymbol{\theta}_2$ ($\alpha \neq \beta$), are orthogonal, and (iv) there are ϕ_β independent parametric functions of $\boldsymbol{\theta}_2$ in $A_\beta^{(l,l)\#}\boldsymbol{\theta}_2$ where $\phi_\beta = \binom{m}{\beta} - \binom{m}{\beta-1}$.

THEOREM 3.2. *For a B-array T satisfying Condition* (3.1), *the covariance matrix* Var $(\hat{\boldsymbol{\phi}})$ *of* $\hat{\boldsymbol{\phi}}$ *is given by*

$$\text{Var}(\hat{\boldsymbol{\phi}}) = XC\sigma^2$$

$$(3.4) \qquad = [\textstyle\sum_{\beta=0}^{l-1} \{\sum_{i=0}^{l-\beta-1} \sum_{j=0}^{l-\beta-1} \kappa_{i,j}^{\beta} D_{\beta}^{(\beta+i,\beta+j)\#} + \sum_{i=0}^{l-\beta} h_{\beta} \kappa_{i,l-\beta}^{\beta} D_{\beta}^{(\beta+i,l)\#} + \sum_{j=0}^{l-\beta} h_{\beta} \kappa_{l-\beta,j}^{\beta} D_{\beta}^{(l,\beta+j)\#} + h_{\beta}^2 \kappa_{l-\beta,l-\beta}^{\beta} D_{\beta}^{(l,l)\#}\}]\sigma^2 ,$$

where $\kappa_{i,j}^{\beta}$ *are* (i, j) *elements of* $\mathbf{K}_{\beta}^{-1}$ *for each* $\beta = 0, 1, \cdots, l-1$.

PROOF. Clearly,

$$\text{Var}(\hat{\boldsymbol{\phi}}) = \text{Var}(XE_T'\mathbf{y}(T')) = XE_T' \text{Var}(\mathbf{y}(T'))E_T X' = XM_T X'\sigma^2 = XC\sigma^2 .$$

From Lemma 3.1 and Appendix, we have the irreducible representations of XC, i.e., for $\beta = 0, 1, \cdots, l-1$.

$$\boldsymbol{\chi}_{\beta}\Gamma_{\beta} = \Gamma_{\beta}\mathbf{K}_{\beta}^{-1}\Gamma_{\beta} = \begin{bmatrix} \kappa_{0,0}^{\beta} & \cdots & \kappa_{0,l-\beta-1}^{\beta} & h_{\beta}\kappa_{0,l-\beta}^{\beta} \\ & \ddots & \vdots & \vdots \\ & & \kappa_{l-\beta-1,l-\beta-1}^{\beta} & h_{\beta}\kappa_{l-\beta-1,l-\beta}^{\beta} \\ (\text{Sym.}) & & & h_{\beta}^2\kappa_{l-\beta,l-\beta}^{\beta} \end{bmatrix} .$$

Clearly, this leads to (3.4).

From Theorem 3.2, we have

$$(3.5) \qquad X_1 = \text{diag}[X_{11}, 0_{\binom{m}{l}}] = \textstyle\sum_{\beta=0}^{l-1} \sum_{i=0}^{l-\beta-1} \sum_{j=0}^{l-\beta-1} \kappa_{i,j}^{\beta} D_{\beta}^{(\beta+i,\beta+j)\#} ,$$

where X_{11} is the $\nu_{l-1} \times \nu_{l-1}$ submatrix of X and 0_k denotes the $k \times k$ matrix whose elements are all 0. Furthermore

$$(3.6) \qquad \text{Var}(\hat{\boldsymbol{\phi}}_2) = [\textstyle\sum_{\beta=0}^{l-1} h_{\beta}^2 \kappa_{l-\beta,l-\beta}^{\beta} A_{\beta}^{(l,l)\#}]\sigma^2 .$$

Since $X_1 \in \mathfrak{A}$, it follows that $\text{Var}(\hat{\boldsymbol{\theta}}_1) = X_{11}\sigma^2$ is invariant under any permutation of m factors. Thus we have

THEOREM 3.3. *B-arrays satisfying Condition* (3.1) *yield* 2^m-BFF *designs of resolution* $2l$ *such that the covariance matrices* Var $(\hat{\boldsymbol{\theta}}_1)$ *are invariant under any permutation of m factors and that the vectors* $A_{\beta}^{(l,l)\#}\boldsymbol{\theta}_2$ $(\beta = 0, 1, \cdots, l-1)$ *are estimable.*

Such designs can be regarded as a subclass of 2^m-BFF designs of resolution $2l$. Thus we make the following definition:

DEFINITION. B-arrays satisfying Condition (3.1) are called S-type balanced fractional 2^m factorial (2^m-SBFF) designs of resolution $2l$.

It is easily seen that the covariance matrix Var $(\hat{\boldsymbol{\theta}}_1)$ has at most $\binom{l+2}{3}$ distinct elements. We shall express these elements explicitly, using elements $\kappa_{i,j}^{\beta}$ of inverse matrices $\mathbf{K}_{\beta}^{-1}$ $(\beta = 0, 1, \cdots, l-1)$.

THEOREM 3.4. *For a* 2^m-SBFF *design of resolution* $2l$, *let* $c_{\alpha}^{(u,v)}$ *be the element of* X_{11} *corresponding to* $\theta_{t_1\cdots t_u}$ *and* $\theta_{t_1'\cdots t_v'}$ *which are the* αth *associates* (i.e., $c_{\alpha}^{(u,v)}\sigma^2$ *is the covariance of their* BLUEs *and, particularly,* $c_0^{(u,u)}\sigma^2$ *is the variance of* $\hat{\theta}_{t_1\cdots t_u}$).

Then

$$c_\alpha^{(u,v)} = \sum_{\beta=0}^{u} \kappa_{u-\beta,v-\beta}^{\beta} z_{(u,v)}^{\beta\alpha} \qquad \text{for } 0 \leqq \alpha \leqq u \leqq v \leqq l-1\,, \tag{3.7}$$

where

$$z_{(u,v)}^{\beta\alpha} = \frac{\phi_\beta z_{\beta\alpha}^{(u,v)}}{\binom{m}{u}\binom{u}{\alpha}\binom{m-u}{v-u+\alpha}}\,.$$

PROOF. It has been shown in Shirakura and Kuwada (1976) that $D_\beta^{(u,v)\#} = (D_\beta^{(v,u)\#})' = \sum_{\alpha=0}^{u} z_{(u,v)}^{\beta\alpha} D_\alpha^{(u,v)}$ hold for all $\beta = 0, 1, \cdots, u$; $0 \leqq u \leqq v \leqq l$. Hence the matrix X_1 in (3.5) can be also expressed as

$$\begin{aligned} X_1 &= \sum_{u=0}^{l-1} \sum_{v=0}^{l-1} \sum_{\beta=0}^{\min(u,v)} \kappa_{u-\beta,v-\beta}^{\beta} D_\beta^{(u,v)\#} \\ &= \sum_{u=0}^{l-1} \sum_{v=u}^{l-1} \sum_{\alpha=0}^{u} \{\sum_{\beta=0}^{u} \kappa_{u-\beta,v-\beta}^{\beta} z_{(u,v)}^{\beta\alpha}\} B_\alpha^{(u,v)}\,. \end{aligned}$$

This leads to (3.7).

4. Constructions of 2^m-SBFF designs of resolution $2l$. In this section, we make certain investigations on B-arrays satisfying Condition (3.1).

THEOREM 4.1. *The rank of the information matrix M_T of a 2^m-SBFF design T of resolution $2l$ is $\nu_l^* = \nu_l - \phi_l$.*

PROOF. The proof of this theorem follows from (2.3), (2.6) and Appendix.

Theorem 4.1 implies that the number of distinct columns in T must be at least ν_l^*. For example, we have $\nu_l^* = 65$ for $m = 8$ and $l = 3$. The B-array in Example 1 is just a 2^8-SBFF design of resolution VI with the smallest number of N assemblies.

It has been shown in Shirakura and Kuwada (1975) that $\mathbf{K}_l = 2^{2l}\mu_l$. Thus we have

THEOREM 4.2. *For a B-array of strength $2l$ with index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$, $\mu_l = 0$ is equivalent to $\mathbf{K}_l = 0$.*

This theorem indicates that in order to construct 2^m-SBFF designs of resolution $2l$, we need to investigate B-arrays of strength $2l$ with index $\mu_l = 0$. We now consider an array obtained by juxtaposing each $\Omega(j; m)$ $(j = 0, 1, \cdots, m)$ α_j $(\geqq 0)$ times, where $\Omega(j; m)$ are illustrated in Example 1. Such an array is called a simple array with parameters $(m; \alpha_0, \alpha_1, \cdots, \alpha_m)$. For example, the B-array given in Example 1 is a simple array with parameters $(m = 8; \alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 0, \alpha_4 = 0, \alpha_5 = 0, \alpha_6 = 1, \alpha_7 = 0, \alpha_8 = 1)$. The following theorem has been given by Shirakura (1975a):

THEOREM 4.3. *T is a B-array of strength $2l$, m constraints and index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$ with $\mu_l = 0$ if and only if T is a simple array with parameters $(m; \alpha_0, \alpha_1, \cdots, \alpha_{l-1}, 0, \cdots, 0, \alpha_{m-l+1}, \cdots, \alpha_m)$. A connection between the indices μ_i and the parameters α_j is given as follows: For $j = 0, 1, \cdots, l-1$,*

$$\alpha_j = \sum_{i=0}^{l-1} (-1)^{i+j}\binom{m-2l-1+i-j}{i-j}\mu_i\,, \qquad \alpha_{m-l+1+j} = \sum_{i=0}^{l-1} (-1)^{i+j}\binom{m-2l-1+j-i}{j-1}\mu_{l+1+i}\,,$$

or for $i = 0, 1, \cdots, l-1$,

$$\mu_i = \sum_{j=0}^{l-1} \binom{m-2l}{j-i} \alpha_j\,, \qquad \mu_{l+1+i} = \sum_{j=0}^{l-1} \binom{m-2l}{i-j} \alpha_{m-l+1+j}\,.$$

This theorem makes the construction of 2^m-SBFF designs of resolution $2l$ much easier. Moreover, as a by-product, we can obtain an important result that a necessary and sufficient condition for the existence of a B-array of strength $2l$, m constraints and index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$ with $\mu_l = 0$ is that $\sum_{i=0}^{l-1} (-1)^{i+j} \times \binom{m-2l-1+i-j}{i-j}\mu_i \geqq 0$ and $\sum_{i=0}^{l-1}(-1)^{i+j}\binom{m-2l-1+j-i}{j-i}\mu_{l+1+i} \geqq 0$ hold for all $j = 0, 1, \cdots, l-1$.

THEOREM 4.4. *If there exists a* 2^m-SBFF *design* T *of resolution* $2l$ *with* $N\,(\geqq \nu_l^*)$ *assemblies, then for every* $\tilde{N} > N$, *there exist* 2^m-SBFF *designs of resolution* $2l$.

PROOF. Let $\tilde{T}$ be an array obtained from T by adding $(\tilde{N} - N)$ columns, each being $(0, 0, \cdots, 0)'$ or $(1, 1, \cdots, 1)'$. Then it is clear that $\tilde{T}$ is a B-array of strength $2l$ with $\mu_l = 0$. Also from Theorem 4.1 and (2.1), we have $\nu_l^* = \text{rank}\, M_T = \text{rank}\, E_T \leqq \text{rank}\, E_{\tilde{T}} = \text{rank}\, M_{\tilde{T}}$. Let $\mathbf{K}_\beta$ $(\beta = 0, 1, \cdots, l-1)$ be the matrices corresponding to $\tilde{T}$. Now assume that $\mathbf{K}_\beta$ for some β is singular. Then from (2.3), (2.6) and Appendix, we have rank $M_{\tilde{T}} < \nu_l^*$. This implies a contradiction. Thus $\tilde{T}$ satisfies Condition (3.1). This completes the proof.

From Theorem 4.4 and Example 1, we can obtain 2^8-SBFF designs of resolution VI for any $N \geqq 65$. Particularly for designs of resolution IV (i.e., $l = 2$), we find that there exist 2^m-SBFF designs for any m $(\geqq 4)$ and any $N \geqq \nu_l^*$ $(= 2m + 1)$. In fact consider a simple array T with parameters $(m;\, \alpha_0 = 1, \alpha_1 = 1, 0, \cdots, 0, \alpha_{m-1} = 1, \alpha_m = 0)$, which is equivalent to a B-array of strength 4, size $N = \nu_l^*$, m constraints and index set $\{\mu_0 = (m-3), \mu_1 = 1, \mu_2 = 0, \mu_3 = 1, \mu_4 = (m-4)\}$. Then it can be easily checked that T satisfies Condition (3.1).

THEOREM 4.5. *Let* T *be a* B*-array of strength* $2l$, m *constraints and index set* $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$ *with* $\mu_l = 0$. *Then a necessary condition for* T *to be a* 2^m-SBFF *design of resolution* $2l$ *is that* $\mu_{l-1} \neq 0$ *and* $\mu_{l+1} \neq 0$ *hold.*

PROOF. From (2.4) and (2.5), we have $\kappa_{l-1}^{0,0} = 2^{2l-2}(\mu_{l-1} + \mu_{l+1})$, $\kappa_{l-1}^{0,1} = \kappa_{l-1}^{1,0} = 2^{2l-2}(m - 2l + 2)^{\frac{1}{2}}(\mu_{l+1} - \mu_{l-1})$ and $\kappa_{l-1}^{1,1} = 2^{2l-2}(m - 2l + 2)(\mu_{l-1} + \mu_{l+1})$ (see Shirakura and Kuwada (1975)). Since $\mathbf{K}_{l-1}$ is positive definite, it follows that $|\mathbf{K}_{l-1}| = 2^{4l-2}(m - 2l + 2)\mu_{l-1}\mu_{l+1} > 0$. This completes the proof.

This theorem is very useful for constructing 2^m-SBFF designs of resolution $2l$. In the same way, we can obtain a result similar to Theorem 4.5 from conditions for $\mathbf{K}_\beta$ $(\beta = 0, 1, \cdots, l-2)$ to be positive definite. However it is very complicated and will make this paper unduly lengthy.

5. The optimality of 2^m-SBFF designs of resolution $2l$. For any two B-arrays T_1 and T_2, T_1 is said to be isomorphic to T_2 if there exist permutation matrices P and Q of appropriate size such that $T_1 = PT_2Q$. In general, for a fixed number of N assemblies, there are more than one nonisomorphic 2^m-SBFF designs of resolution $2l$. (For example, it follows from Theorem 4.4 that for $N > 65$, we can obtain $(N-64)$ nonisomorphic 2^8-SBFF designs of resolution VI from the B-array of Example 1.) Among these, we must choose one which maximizes information in some sense. For this purpose, we shall consider the sum of the variances of the estimates in $\hat{\boldsymbol{\theta}}_1$ and the estimates of $\binom{m}{l-1}$ normalized independent parameters in $A_\beta{}^{(l,l)\#}\boldsymbol{\theta}_2$ $(\beta = 0, 1, \cdots, l-1)$ corresponding to the trace criterion.

Consider $\boldsymbol{\psi}_2{}^\beta = (z^{\beta 0}_{(l,l)})^{-\frac{1}{2}} A_\beta{}^{(l,l)\#}\boldsymbol{\theta}_2$ corresponding to $\boldsymbol{\psi}_2$ in (3.3) when $h_\beta = (z^{\beta 0}_{(l,l)})^{-\frac{1}{2}}$ and $h_0 = h_1 = \cdots = h_{\beta-1} = h_{\beta+1} = \cdots = h_{l-1} = 0$ for each $\beta = 0, 1, \cdots, l-1$. Then from the properties of $A_\beta{}^{(l,l)\#}$, it is easily seen that all elements of $\boldsymbol{\psi}_2{}^\beta$ are normalized parametric functions of $\boldsymbol{\theta}_2$. Also it follows from (3.6) that every estimate in the BLUE $\hat{\boldsymbol{\psi}}_2{}^\beta$ of $\boldsymbol{\psi}_2{}^\beta$ has the same variance $\kappa^\beta_{l-\beta,l-\beta}\sigma^2$. Since the number of independent parameters in $A_\beta{}^{(l,l)\#}\boldsymbol{\theta}_2$ is ϕ_β, the sum of the variances of the BLUEs of $\binom{m}{l-1}$ normalized independent parametric functions of $\boldsymbol{\theta}_2$ is given by

$$(\kappa^0_{l,l} + \phi_1\kappa^1_{l-1,l-1} + \cdots + \phi_{l-1}\kappa^{l-1}_{1,1})\sigma^2 .$$

On the other hand, from Theorem 3.4 it follows that the sum of the variances of the estimates in $\hat{\boldsymbol{\theta}}_1$ is

$$\begin{aligned} \operatorname{tr} \operatorname{Var}(\hat{\boldsymbol{\theta}}_1) &= \textstyle\sum_{u=0}^{l-1} \binom{m}{u} c_0{}^{(u,u)}\sigma^2 \\ &= \textstyle\sum_{\beta=0}^{l-1} \phi_\beta(\kappa^\beta_{0,0} + \kappa^\beta_{1,1} + \cdots + \kappa^\beta_{l-\beta-1,l-\beta-1})\sigma^2 . \end{aligned}$$

Thus we have

THEOREM 5.1. *For a* 2^m-SBFF *design* T *of resolution* $2l$, *the sum of the variances of the* $\nu_l{}^*$ BLUEs *of the parameters in* $\boldsymbol{\theta}_1$ *and normalized independent parameters in* $\boldsymbol{\psi}_2{}^\beta$ $(\beta = 0, 1, \cdots, l-1)$ *is given as follows*:

$$\begin{aligned} (5.1) \qquad S_T\sigma^2 &= \textstyle\sum_{\beta=0}^{l-1} \phi_\beta(\kappa^\beta_{0,0} + \kappa^\beta_{1,1} + \cdots + \kappa^\beta_{l-\beta-1,l-\beta-1} + \kappa^\beta_{l-\beta,l-\beta})\sigma^2 \\ &= \textstyle\sum_{\beta=0}^{l-1} \phi_\beta \operatorname{tr} \mathbf{K}_\beta{}^{-1}\sigma^2 . \end{aligned}$$

From (2.3), (2.6), (5.1) and Appendix, we find that S_T denotes the trace of the generalized inverse matrix of M_T. Thus we define

DEFINITION. Let T_1 and T_2 be two 2^m-SBFF designs of resolution $2l$. Then T_1 is said to be better than T_2 if $S_{T_1} < S_{T_2}$. Such a criterion is said to be the generalized trace (GT) criterion.

EXAMPLE 2. For a 2^m-SBFF design T of resolution VI, we have $S_T = \operatorname{tr}\mathbf{K}_0{}^{-1} + (m-1)\operatorname{tr}\mathbf{K}_1{}^{-1} + m(m-3)/2 \cdot \operatorname{tr}\mathbf{K}_2{}^{-1}$. Now, for $N = 65$ and $m = 8$, let us compare the design T of Example 1 and another B-array T_1 (as a 2^m-SBFF design of resolution VI) with index set $\{4, 3, 1, 0, 1, 2, 1\}$, using the GT criterion. Then we have $S_T = 2.10130$ and $S_{T_1} = 4.26375$. Thus the design T is better than T_1 with respect to the GT criterion.

In Table 1, an optimal (w.r.t. the GT criterion) 2^8-SBFF design of resolution VI for each N with $65 \leqq N < 93\ (= \nu_l)$ is given with the distinct elements $c_\alpha{}^{(u,v)}$ of $\operatorname{Var}(\hat{\boldsymbol{\theta}}_1)$ and the parameters α_i of the corresponding simple array. It may be

TABLE 1

Optimal 2^8-SBFF *designs of resolution* VI

N	μ_0	μ_1	μ_2	μ_4	μ_5	μ_6	S_T	$c_0^{(0,0)}$	$c_0^{(1,1)}$	$c_0^{(2,2)}$	$c_0^{(0,1)}$	$c_0^{(0,2)}$	$c_1^{(1,1)}$	$c_0^{(1,2)}$	$c_1^{(1,2)}$	$c_1^{(2,2)}$	$c_2^{(2,2)}$	α_0	α_1	α_2	α_6	α_7	α_8
65	3	3	1	1	2	2	2.10130	0.04253	0.10681	0.02515	−0.00045	−0.00483	−0.01428	−0.00276	0.00114	−0.00220	0.00171	0	1	1	1	0	1
66	4	3	1	1	2	2	2.08583	0.03638	0.10681	0.02493	−0.00047	−0.00368	−0.01428	−0.00276	0.00115	−0.00241	0.00149	1	1	1	1	0	1
67	4	3	1	1	2	3	2.07524	0.03323	0.10677	0.02474	−0.00012	−0.00291	−0.01432	−0.00284	0.00106	−0.00260	0.00131	1	1	1	1	0	2
68	5	3	1	1	2	3	2.07009	0.03112	0.10677	0.02467	−0.00011	−0.00251	−0.01432	−0.00284	0.00106	−0.00268	0.00123	2	1	1	1	0	2
69	5	3	1	1	2	4	2.06641	0.03000	0.10676	0.02460	0.00001	−0.00224	−0.01434	−0.00287	0.00103	−0.00274	0.00116	2	1	1	1	0	3
70	6	3	1	1	2	4	2.06383	0.02893	0.10676	0.02456	0.00001	−0.00203	−0.01434	−0.00287	0.00103	−0.00278	0.00113	3	1	1	1	0	3
71	6	3	1	1	2	5	2.06195	0.02835	0.10675	0.02453	0.00008	−0.00189	−0.01434	−0.00289	0.00102	−0.00281	0.00109	3	1	1	1	0	4
72	3	3	1	1	3	3	1.60852	0.04500	0.06906	0.02455	0.	−0.00500	−0.00906	0.	0.	−0.00244	0.00182	0	1	1	1	1	0
73	4	3	1	1	3	3	1.59556	0.04085	0.06906	0.02440	−0.00005	−0.00424	−0.00906	0.00001	0.00001	−0.00258	0.00168	1	1	1	1	1	0
74	4	3	1	1	3	4	1.58058	0.03492	0.06906	0.02420	0.	−0.00314	−0.00906	0.	0.	−0.00279	0.00148	1	1	1	1	1	1
75	5	3	1	1	3	4	1.57592	0.03307	0.06906	0.02414	−0.00002	−0.00280	−0.00906	0.00000	0.00000	−0.00285	0.00141	2	1	1	1	1	1
76	5	3	1	1	3	5	1.57094	0.03101	0.06906	0.02407	0.	−0.00242	−0.00906	0.	0.	−0.00292	0.00134	2	1	1	1	1	2
77	6	3	1	1	3	5	1.56851	0.03000	0.06906	0.02404	−0.00001	−0.00224	−0.00906	0.00000	0.00000	−0.00295	0.00131	3	1	1	1	1	2
78	6	3	1	1	3	6	1.56599	0.02893	0.06906	0.02400	0.	−0.00204	−0.00906	0.	0.	−0.00299	0.00127	3	1	1	1	1	3
79	7	3	1	1	3	6	1.56449	0.02830	0.06906	0.02398	−0.00000	−0.00192	−0.00906	0.00000	0.00000	−0.00301	0.00125	4	1	1	1	1	3
80	5	4	1	1	3	3	1.48889	0.04250	0.06067	0.02427	−0.00031	−0.00438	−0.00791	−0.00057	0.00030	−0.00264	0.00170	0	2	1	1	1	0
81	5	4	1	1	3	4	1.47527	0.03788	0.06067	0.02411	−0.00028	−0.00351	−0.00791	−0.00058	0.00029	−0.00280	0.00154	0	2	1	1	1	1
82	6	4	1	1	3	4	1.46767	0.03449	0.06066	0.02403	−0.00014	−0.00299	−0.00791	−0.00060	0.00027	−0.00288	0.00146	1	2	1	1	1	1
83	6	4	1	1	3	5	1.46305	0.03266	0.06066	0.02397	−0.00012	−0.00265	−0.00791	−0.00060	0.00026	−0.00294	0.00140	1	2	1	1	1	2
84	7	4	1	1	3	5	1.45944	0.03096	0.06066	0.02393	−0.00005	−0.00239	−0.00792	−0.00062	0.00025	−0.00298	0.00136	2	2	1	1	1	2
85	7	4	1	1	3	6	1.45704	0.02996	0.06066	0.02389	−0.00003	−0.00220	−0.00792	−0.00062	0.00025	−0.00302	0.00132	2	2	1	1	1	3
86	8	4	1	1	3	6	1.45491	0.02893	0.06066	0.02387	0.00001	−0.00204	−0.00792	−0.00062	0.00024	−0.00304	0.00130	3	2	1	1	1	3
87	8	4	1	1	3	7	1.45343	0.02830	0.06066	0.02385	0.00001	−0.00193	−0.00792	−0.00063	0.00024	−0.00306	0.00128	3	2	1	1	1	4
88	5	4	1	1	4	5	1.40625	0.04000	0.05531	0.02393	0.	−0.00375	−0.00719	0.	0.	−0.00286	0.00161	0	2	1	1	2	0
89	6	4	1	1	4	5	1.39943	0.03748	0.05530	0.02387	0.00015	−0.00336	−0.00720	−0.00002	−0.00002	−0.00292	0.00155	1	2	1	1	2	0
90	6	4	1	1	4	6	1.39184	0.03409	0.05530	0.02379	0.	−0.00284	−0.00720	0.	0.	−0.00300	0.00147	1	2	1	1	2	1
91	7	4	1	1	4	6	1.38851	0.03260	0.05530	0.02375	0.00006	−0.00261	−0.00720	−0.00001	−0.00001	−0.00303	0.00143	2	2	1	1	2	1
92	7	4	1	1	4	7	1.38495	0.03091	0.05529	0.02371	0.	−0.00235	−0.00721	0.	0.	−0.00307	0.00139	2	2	1	1	2	2

(0. *denotes zero exactly*.)

remarked here that since there exist always 2^8-BFF designs of resolution VII with $N(\geqq \nu_l)$ assemblies (see Shirakura (1975 b)), we need not consider 2^8-SBFF designs of resolution VI for larger N assemblies. Note that for a 2^m-SBFF design T of resolution $2l$, we have $S_T = S_{\bar{T}}$ (see Shirakura and Kuwada (1975)), where $\bar{T}$ is the complementary design obtained from T by an interchange of 0 and 1. This means that so far as optimal (w.r.t. the GT criterion) 2^8-SBFF designs of resolution VI are concerned, we may restrict to B-arrays such that $\mu_2 > \mu_4$ if $\mu_2 \neq \mu_4$, $\mu_1 > \mu_5$ if $\mu_2 = \mu_4$ and $\mu_1 \neq \mu_5$, or $\mu_0 \geqq \mu_6$ if $\mu_2 = \mu_4$ and $\mu_1 = \mu_5$.

6. Remark. B-arrays of strength t reduce to orthogonal arrays of strength t when $\mu_0 = \mu_1 = \cdots = \mu_t$. It is well known that orthogonal fractional 2^m factorial designs of resolution $2l$ are obtained from orthogonal arrays of strength $2l - 1$. However it is, in general, unknown whether a 2^m-BFF design of resolution $2l$ can be obtained from a B-array of strength $2l - 1$ which is not an orthogonal array of strength $2l - 1$. In fact, for such an array T (as a design), the information matrix M_T cannot be expressed in terms of association matrices $B_\alpha^{(u,v)}$ of an $l + 1$ sets TMDPB association schemes (i.e., $M_T \notin \mathfrak{A}$). Therefore it is very difficult to know whether there exists a matrix X of size $\nu_l \times \nu_l$ (or $\nu_l \times N$) such that XM_T (or XE_T) $= \text{diag}[0, I_p, 0_q]$, where $p = \nu_{l-1} - 1$ and $q = \binom{m}{l}$. Moreover, even if it exists, it is very difficult to show that the design T has the property of balanced designs. This problem has been partly solved for designs of resolution IV. Srivastava and Anderson (1970) have shown that some 2^m-BFF designs of resolution IV are obtained from B-arrays of strength 3, m constraints and index set $\{\mu_0, \mu_1, \mu_2, \mu_3\}$ with $\mu_0 = \mu_3$ and $\mu_1 = \mu_2$. However, in comparison with B-arrays of strength 4 with $\mu_2 = 0$, they are unavailable for an odd number of N and cannot explicitly express the alias structure of the effects of two-factor interactions.

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APPENDIX

It has been shown in Yamamoto, Shirakura and Kuwada (1974) that the $l + 1$ sets TMDPB association algebra $\mathfrak{A}$ is represented by the linear closure of all $(l+1)(l+2)(2l+3)/6$ matrices $D_\beta^{(u,v)\#}$, i.e., $\mathfrak{A} = [D_\beta^{(u,v)\#} \mid \beta = 0, 1, \cdots, \min(u, v); u, v = 0, 1, \cdots, l]$. The matrices $D_\beta^{(u,v)\#}$ have the following properties:

$$\text{(A.1)} \qquad (D_\beta^{(u,v)\#})' = D_\beta^{(v,u)\#}, \qquad \text{rank } D_\beta^{(u,v)\#} = \phi_\beta,$$
$$D_\alpha^{(u,s)\#} D_\beta^{(w,v)\#} = \delta_{sw}\delta_{\alpha\beta} D_\beta^{(u,v)\#},$$

where $\phi_\beta = \binom{m}{\beta} - \binom{m}{\beta-1}$ and δ_{ij} is equal to 1 or 0 according as $i = j$ or not.

THEOREM. *For every B ($= \sum_{\beta=0}^{l} \sum_{i=0}^{l-\beta} \sum_{j=0}^{l-\beta} \lambda_\beta^{i,j} D_\beta^{(\beta+i,\beta+j)\#}$, say) belonging to $\mathfrak{A}$,* *there exists a* $\nu_l \times \nu_l$ *orthogonal matrix* P *such that*

$$P'BP = \operatorname{diag}[\Lambda_0; \underbrace{\Lambda_1, \cdots, \Lambda_1}_{\phi_1}; \cdots; \underbrace{\Lambda_l, \cdots, \Lambda_l}_{\phi_l}], \tag{A.2}$$

where Λ_β *are the* $(l-\beta+1) \times (l-\beta+1)$ *matrices with* (i, j) *elements* $\lambda_\beta^{i,j}$.

PROOF. The matrices $D_\beta^{(u,u)\#}$ are idempotent, so that their characteristic roots are 1 and 0. For any fixed β $(0 \leqq \beta \leqq l)$ and some u $(\beta \leqq u \leqq l)$, consider ϕ_β characteristic vectors $\mathbf{d}_{\beta,i}^{(u)}$ $(i = 1, 2, \cdots, \phi_\beta)$ of $D_\beta^{(u,u)\#}$ such that $D_\beta^{(u,u)\#}\mathbf{d}_{\beta,i}^{(u)} = \mathbf{d}_{\beta,i}^{(u)}$ and $(\mathbf{d}_{\beta,i}^{(u)})'\mathbf{d}_{\beta,j}^{(u)} = \delta_{ij}$. Let $\mathbf{d}_{\beta,i}^{(v)} = D_\beta^{(v,u)\#}\mathbf{d}_{\beta,i}^{(u)}$ for any v with $\beta \leqq v \leqq l$. Now we shall show that $\mathbf{d}_{\beta,i}^{(v)}$ $(i = 0, 1, \cdots, \phi_\beta)$ are those of $D_\beta^{(v,v)\#}$ such that $D_\beta^{(v,v)\#}\mathbf{d}_{\beta,i}^{(v)} = \mathbf{d}_{\beta,i}^{(v)}$ and $(\mathbf{d}_{\beta,i}^{(u)})'\mathbf{d}_{\beta,j}^{(v)} = \delta_{uv}\delta_{ij}$. From (A.1), we have $D_\beta^{(v,v)\#}\mathbf{d}_{\beta,i}^{(v)} = D_\beta^{(v,u)\#}D_\beta^{(u,v)\#} \times D_\beta^{(v,u)\#}\mathbf{d}_{\beta,i}^{(u)} = D_\beta^{(v,u)\#}D_\beta^{(u,u)\#}\mathbf{d}_{\beta,i}^{(u)} = D_\beta^{(v,u)\#}\mathbf{d}_{\beta,i}^{(u)} = \mathbf{d}_{\beta,i}^{(v)}$. Also $(\mathbf{d}_{\beta,j}^{(u)})'\mathbf{d}_{\beta,i}^{(v)} = (\mathbf{d}_{\beta,j}^{(u)})' \times D_\beta^{(u,u)\#}D_\beta^{(v,u)\#}\mathbf{d}_{\beta,i}^{(u)} = \delta_{uv}(\mathbf{d}_{\beta,j}^{(u)})'\mathbf{d}_{\beta,i}^{(u)} = \delta_{uv}\delta_{ij}$. Of course, $(\mathbf{d}_{\alpha,i}^{(v)})'\mathbf{d}_{\beta,j}^{(u)} = 0$ for $\alpha \neq \beta$. Let

$$P = [\mathbf{d}_{0,1}^{(0)}, \mathbf{d}_{0,1}^{(1)}, \cdots, \mathbf{d}_{0,1}^{(l)}; \mathbf{d}_{1,1}^{(1)}, \mathbf{d}_{1,1}^{(2)}, \cdots, \mathbf{d}_{1,1}^{(l)}, \cdots, \mathbf{d}_{1,\phi_1}^{(1)}, \mathbf{d}_{1,\phi_1}^{(2)}, \cdots, \mathbf{d}_{1,\phi_1}^{(l)}; \cdots; \mathbf{d}_{l,1}^{(l)}, \cdots, \mathbf{d}_{l,\phi_l}^{(l)}].$$

Then the above statement shows that P is a $\nu_l \times \nu_l$ orthogonal matrix satisfying (A.2).

The matrices Λ_β $(\beta = 0, 1, \cdots, l)$ in (A.2) are called the irreducible representations of B.

REFERENCES

[1] ANDERSON, D. A. and SRIVASTAVA, J. N. (1972). Resolution IV designs of the $2^m \times 3$ series. *J. Roy. Statist. Soc. Ser. B* **34** 377–384.

[2] BOX, G. E. P. and HUNTER, J. S. (1961). The 2^{k-p} fractional factorial designs, I and II. *Technometrics* **3** 311–351, 449–458.

[3] CHAKRAVARTI, I. M. (1956). Fractional replication in asymmetrical factorial designs and partially balanced arrays. *Sankhyā* **17** 143–164.

[4] CHOPRA, D. V. and SRIVASTAVA, J. N. (1973a). Optimal balanced 2^7 fractional factorial designs of resolution V, with $N \leqq 42$. *Ann. Inst. Statist. Math.* **25** 587–604.

[5] CHOPRA, D. V. and SRIVASTAVA, J. N. (1973b). Optimal balanced 2^7 fractional factorial designs of resolution V, $49 \leqq N \leqq 55$. *Commun. Statist.* **2** 59–84.

[6] KIEFER, J. (1959). Optimum experimental designs. *J. Roy. Statist. Soc. Ser. B* **21** 272–319.

[7] MARGOLIN, B. H. (1969). Results on factorial designs of resolution IV for the 2^n and 2^n3^m series. *Technometrics* **10** 431–444.

[8] SHIRAKURA, T. (1975). On balanced arrays of 2 symbols, strength $2l$, m constraints and index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$ with $\mu_l = 0$. *J. Japan Statist. Soc.* **5** 53–56.

[9] SHIRAKURA, T. (1976). Optimal balanced fractional 2^m factorial designs of resolution VII, $6 \leqq m \leqq 8$. *Ann. Statist.* **4** 515–531.

[10] SHIRAKURA, T. and KUWADA, M. (1975). Note on balanced fractional 2^m factorial designs of resolution $2l+1$. *Ann. Inst. Statist. Math.* **27** 377–386.

[11] SHIRAKURA, T. and KUWADA, M. (1976). Covariance matrices of the estimates for balanced fractional 2^m factorial designs of resolution $2l+1$. *J. Japan Statist. Soc.* To appear.

[12] SRIVASTAVA, J. N. (1970). Optimal balanced 2^m fractional factorial designs. S. N. Roy Memorial Volume. Univ. of North Carolina and Indian Statist. Inst. 689–706.

[13] SRIVASTAVA, J. N. and ANDERSON, D. A. (1970). Optimal fractional factorial plans for

main effects orthogonal to two-factor interactions: 2^m series. *J. Amer. Statist. Assoc.* **65** 828–843.

[14] SRIVASTAVA, J. N. and CHOPRA, D. V. (1971 a). Balanced optimal 2^m fractional factorial designs of resolution V, $m \leqq 6$. *Technometrics* **13** 257–269.

[15] SRIVASTAVA, J. N. and CHOPRA, D. V. (1971 b). On the characteristic roots of the information matrix of 2^m balanced factorial designs of resolution V, with applications. *Ann. Math. Statist.* **42** 722–734.

[16] WEBB, S. R. (1968). Nonorthogonal designs of even resolution. *Technometrics* **10** 291–300.

[17] YAMAMOTO, S., SHIRAKURA, T. and KUWADA, M. (1974). Characteristic polynomials of the information matrices of balanced fractional 2^m factorial designs of higher $(2l + 1)$ resolution. To appear in *Essays in Probability* and *Statist*. Presented in honor of Professor J. Ogawa on his 60th birthday.

[18] YAMAMOTO, S., SHIRAKURA, T. and KUWADA, K. (1975). Balanced arrays of strength $2l$ and balanced fractional 2^m factorial designs. *Ann. Inst. Statist. Math.* **27** 143–157.

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