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ASYMPTOTIC SUFFICIENCY AND ASYMPTOTICALLY MOST POWERFUL TESTS FOR THE TWO SAMPLE CENSORED SITUATION

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This paper provides a proof of the asymptotic sufficiency of the ranks in the two sample situation when the observations are censored at the rth order statistic. Several important conclusions may then be drawn concerning optimal tests.

0. Introduction and summary. Our objective is to extend the results of Hájek and Šidák [4] by showing that the large sample properties enjoyed by rank tests in the two-sample situation also hold, under appropriate modifications, when only the r smallest observations are available. The key result, Theorem 1.1, states that the set of ranks from one sample, among the smallest r, is asymptotically sufficient. Consequently, for any sequence of tests based on the r smallest observations, there is a sequence of rank tests with the same asymptotic power.

Two special conclusions have immediate practical importance. (1) Any sequence of tests that is asymptotically optimal within the class of rank tests is also optimal within the wider class containing parametric competitors. Specifically, the locally most powerful rank (lmpr) tests derived by Rao–Savage–Sobel [11] and studied by Johnson and Mehrotra [6], are asymptotically most powerful among all tests. (2) Asymptotically most powerful unbiased tests for location differences may be obtained using an equal-tailed test based on the lmpr test statistics. The latter result follows from properties of the exponential family that is constructed when establishing asymptotic sufficiency. Johnson and Roussas [8], [9] discuss this technique in a parametric context.

While our results pertain to contiguous alternatives somewhat weaker although equally interesting results, for complete samples, appeared in a recent paper of Hájek [3] on fixed alternatives. His results, presented in the context of Bahadur efficiency with a finite number of alternatives, complement the local alternative results. However, with censored samples, only Section 1 of [3] seems to follow directly without more work. At this point, we must remind the reader, that, contrary to the terminology, not all the practical information is carried in the ranks even though they can be used for obtaining best tests for equality. The locations and shapes of the two distributions cannot be recovered from the ranks no matter how large the samples.

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The development below is simplified to location parameters but scale parameters can be handled in a parallel manner. In our development, we make an extra assumtion that the pdf f has a derivative f' which is continuous in a neighborhood of the asymptotic censoring point. This extra assumption, not needed in the treatment of the uncensored case, could be relaxed slightly using the technique in Johnson [5].

1. Notations and the main results. Let X_i, $i = 1, \cdots, m$ and $Y_j, j = 1, \cdots, n$ be i.i.d. random variables with cdf's $F_X(x)$ and $G_Y(y)$, respectively. For the combined sample we write $V_1, \cdots, V_N$ where $V_i = X_i$, $i = 1, \cdots, m$ and $V_{i+m} = Y_i$, $i = 1, \cdots, n$; $n + m = N$. Let the random variable m_r equal the number of X-observations among the first ordered r observations and $n_r = r - m_r$ denote the number of observed Y's among first ordered $r(< N)$ observations. These observations are denoted by $X_{(1)} < \cdots < X_{(m_r)}$ and $Y_{(1)} < \cdots < Y_{(n_r)}$. In the combined sample they may be expressed in terms of W's and Z's where $W_1 \leqq \cdots \leqq W_r$ and $Z_i = 1$ if W_i is an X observation $Z_i = 0$ otherwise. As in (1.1) of [6] the likelihood of $X_{(1)}, \cdots, X_{(m_r)}$; $Y_{(1)}, \cdots, Y_{(n_r)}$ can be expressed in terms of the W's and Z's as

$$
\begin{aligned}
q[w_1, \cdots, w_r; \mathbf{Z}_r = \mathbf{z}_r] \\
(1.1) \qquad &= \frac{m!\, n!}{(m - m_r)!\, (n - n_r)!} \prod_1^r f_X{}^{z_i}(w_i) g_Y{}^{1-z_i}(w_i) \\
&\quad \times [1 - F_X(w_r)]^{m-m_r}[1 - G_Y(w_r)]^{n-n_r}, \quad w_1 \leqq \cdots \leqq w_r \\
&= 0, \qquad \text{otherwise}
\end{aligned}
$$

where $f_X(x)$ and $g_Y(y)$ are pdf's corresponding to $F_X(x)$ and $G_Y(y)$, respectively.

Throughout this paper, we study location alternatives with null cdf's $F_X(x) = G_Y(x) = F(x - \bar{\theta})$ and alternative cdf's $F_X(x) = F(x - \theta_1)$ and $G_Y(x) = F(x - \theta_2)$ where $\bar{\theta} = N^{-1}(m\theta_1 + n\theta_2)$. Moreover, we assume that $\lim_N N^{\frac{1}{2}}(\theta_1 - \theta_2) = \Delta$ exists so that the alternatives are contiguous. Substituting these values in (1.1), we get the null and alternative densities presented in (1.2) and (1.3) below.

$$(1.2) \qquad q_{\bar{\theta}}(\mathbf{w}, \mathbf{z}) = \frac{m!\, n!}{(m - m_r)!\, (n - n_r)!} \{\textstyle\prod_{i=1}^r f(w_i - \bar{\theta})\}\{1 - F(w_r - \bar{\theta})\}^{N-r},$$

$$
\begin{aligned}
(1.3) \qquad q_{\theta_1,\theta_2}(\mathbf{w}, \mathbf{z}) &= \frac{m!\, n!}{(m - m_r)!\, (n - n_r)!} \{\textstyle\prod_1^r f^{z_i}(w_i - \theta_1) f^{1-z_i}(w_i - \theta_2)\} \\
&\quad \times \{1 - F(w_r - \theta_1)\}^{m-m_r}\{1 - F(w_r - \theta_2)\}^{n-n_r}.
\end{aligned}
$$

Let L_N denote the ratio of (1.3) to (1.2) and

$$
\begin{aligned}
(1.4) \qquad \phi_i(x) &\equiv \phi(x; \theta_i, \bar{\theta}) = \{f(x - \theta_i)/f(x - \bar{\theta})\}^{\frac{1}{2}}, \\
\psi_i(x) &\equiv \psi(x; \theta_i, \theta) = [\{1 - F(x - \theta_i)\}/\{1 - F(x - \bar{\theta})\}]^{\frac{1}{2}},
\end{aligned}
$$

$i = 1, 2$. Then $\log L_N$ can be expressed as

$$
\begin{aligned}
(1.5) \qquad \log L_N &= 2\{\textstyle\sum_{i=1}^r [z_i \log \phi_1(w_i) + (1 - z_i) \log \phi_2(w_i)] \\
&\quad + (m - m_r) \log \psi_1(w_r) + (n - n_r) \log \psi_2(w_r)\}.
\end{aligned}
$$

Throughout the discussion, we assume that $f^{\frac{1}{2}}(x-\theta)$ has a quadratic mean derivative wrt θ at $\bar{\theta}$. That is, the Fisher information $I(\theta)$ satisfies

$$I(f) = \int_{-\infty}^{\infty} \left(\frac{f'(x)}{f(x)}\right)^2 f(x)\,dx < \infty\,. \tag{1.6}$$

We also assume that the number of observed order statistics r is selected so that $rN^{-1} \to p$; $0 < p < 1$. Suppose ξ_p is uniquely defined so that $F(\xi_p) = p$. Set

$$\sigma_p^2 = \lambda(1-\lambda)\Delta^2 \left\{\int_{-\infty}^{\xi_p} \left(\frac{f'(x)}{f(x)}\right)^2 f(x)\,dx + \frac{f^2(\xi_p)}{1-p}\right\} \tag{1.7}$$

where

$$\begin{aligned} \lambda &= \lim_{N\to\infty} m/N\,, \qquad \theta_1 - \theta_2 = N^{-\frac{1}{2}}\Delta_N \qquad \text{and} \\ \Delta &= \lim_{N\to\infty} \Delta_N\,. \end{aligned} \tag{1.8}$$

The term within brackets in (1.7) is the version of Fisher's information for the censored sample [see Chernoff, Gastwirth and Johns [1]].

Let $||p-q|| = \int |p-q|\,dv$ where v is some σ-finite measure and consider alternatives given by (1.3) which satisfy (1.8) with Δ bounded. Then we have the following theorem establishing the asymptotic sufficiency of ranks in the censored situation.

THEOREM 1.1. *Let f' be continuous in a neighborhood of ξ_p and*

$$\sup_\Delta \frac{mn}{N^2}\Delta^2 \int_{-\infty}^{\xi_p} \left(\frac{f'(x)}{f(x)}\right)^2 f(x)\,dx + \frac{f(\xi_p)}{1-p} < K < \infty \qquad \textit{and}$$

$$\frac{m}{N} \to \lambda\,, \quad 0 < \lambda < 1\,.$$

Consider testing H_0, when density is given by $p_{N\bar{\theta}}$ defined in (1.2) against the alternative $q_{N\theta}$ given by (1.3) subject to (1.8). There exist rank statistics $h_{N\Delta}$ such that for $q_N{}^0 = p_{N\bar{\theta}} h_{N\Delta}$ and $\lim\sup ||q_{N\theta} - q_{N\Delta}^0|| = 0$. That is, the ranks are asymptotically sufficient.

THEOREM. 1.2. *Under the conditions of Theorem* 1.1, *the maximin most powerful rank test has the same power as the maximin most powerful test.*

These results are proved in Section 3 after preliminary results are outlined in Section 2. See [10] for more details.

2. Approximations to the likelihood. To obtain a large sample expansion of $\log L_N$, each term of (1.5) is expanded up to three terms after writing, for example, $\log \phi_j(w_i)$ as $\log[1 + (\phi_j(w_i) - 1)]$. See Roussas [12] for this type of expansion. Thus, we obtain

$$\log L_N = T_{1N} - T_{2N} + T_{3N} \tag{2.1}$$

where

$$
\begin{aligned}
(2.2)\qquad T_{1N} &= 2[\textstyle\sum_1^r \{Z_i(\phi_1(W_i) - 1) + (1 - Z_i)(\phi_2(W_i) - 1)\} \\
&\quad + (m - m_r)(\psi_1(W_r) - 1) + (n - n_r)(\psi_2(W_r) - 1)] \\
T_{2N} &= [\textstyle\sum_1^r \{Z_i(\phi_1(W_i) - 1)^2 + (1 - Z_i)(\phi_2(W_i) - 1)^2\} \\
&\quad + (m - m_r)(\psi_1(W_r) - 1)^2 + (n - n_r)(\psi_2(W_r) - 1)^2] \qquad \text{and} \\
T_{3N} &= [\textstyle\sum_1^r \{C_{N,i,1} Z_i[\phi_1(W_i) - 1]^3 + C_{N,i,2}(1 - Z_i)[\phi_2(W_i) - 1]^3\} \\
&\quad + (m - m_r)C^*_{N,r,1}\{\psi_1(W_i) - 1\}^3 \\
&\quad + (n - n_r)C^*_{N,r,2}\{\psi_2(W_i) - 1\}^3] .
\end{aligned}
$$

In the expression for T_{3N}, $C_{N,i,j}$ and $C^*_{N,i,j}$; $i = 1, 2, \cdots, r$, $j = 1, 2$ are coefficients, uniformly bounded above by 3 for $|\phi_j(W_i) - 1|$, $|\psi_j(W_r) - 1| < \frac{1}{2}$.

LEMMA 2.1. *If* $I(f) < \infty$

$$
\begin{aligned}
&\text{(a)}\quad \max[\max_{1\leq i\leq r} |\phi_j(W_i) - 1|, |\psi_j(W_r) - 1|; j = 1, 2] \to_{P_{N\bar\theta}} 0 . \\
&\text{(b)}\quad \textstyle\sum_1^r \{Z_i[\phi_1(W_i) - 1]^2 + (1 - Z_i)[\phi_2(W_i) - 1]^2\} \\
(2.3)\qquad &\qquad\qquad \to_{P_{N\bar\theta}} \lambda(1 - \lambda)\frac{\Delta^2}{4}\int_{-\infty}^{\xi_p}\left(\frac{f'(x)}{f(x)}\right)^2 f(x)\,dx . \\
&\text{(c)}\quad (m - m_r)[\psi_1(W_r) - 1]^2 + (n - n_r)[\psi_2(W_r) - 1]^2 \\
&\qquad\qquad \to_{P_{N\bar\theta}} \lambda(1 - \lambda)\frac{\Delta^2}{4}\frac{f^2(\xi_p)}{1 - p} .
\end{aligned}
$$

PROOF. The first part of (a) follows from (2.1) of [5] and the method of proof for Lemma 3.3 in [5] establishes (b). Lemmas 2.1 and 3.2 of [5] and the binomial distributions of m_r, n_r imply the remaining claims.

Note. As a consequence of (2.3) we obtain $T_{2N} \to \frac{1}{4}\sigma_p^2$, $T_{3N} \to 0$ in $P_{N\bar\theta}$ and (2.1) gives

$$
(2.4)\qquad (\log L_N - T_{1N} + \tfrac{1}{4}\sigma_p^2) \to_{P_{N\bar\theta}} 0 .
$$

A further approximation is obtained by setting $Q_N(W_r) = \{1 - F(W_r - \bar\theta)\}^{-1}\{F(W_r - \theta_1) - F(W_r - \bar\theta)\}$ and expanding the numerator of Q_N to obtain

$$
\begin{aligned}
N[\psi_1(W_r) - 1] &= N\{(1 + Q_N)^{\frac{1}{2}} - 1\} = NQ_N + o_p(1) \\
(2.5)\qquad &\sim N(\theta_1 - \bar\theta)\frac{f(W_r - \bar\theta)}{1 - F(W_r - \bar\theta)} \\
&\quad - (1 - \lambda)^2\Delta^2\left[\frac{f'(\xi_p)}{2(1 - p)} + \left(\frac{f(\xi_p)}{2(1 - p)}\right)^2\right] .
\end{aligned}
$$

To approximate the remaining terms of T_{1N} we define two statistics Λ_N^p and

T_{1N}^p corresponding to censoring at $(\xi_p - \bar\theta)$ and another two Λ_N^r and T_{1N}^r with the indicator $I_{(-\infty,\xi_p-\bar\theta)}$ replaced by $I_{(-\infty,W_r)}$.

$$\Lambda_N^p = 2\sum_1^m \{\phi_1(V_i) - 1\} I_{(-\infty,\xi_p-\bar\theta)}(V_i) + 2\sum_{m+1}^N \{\phi_2(V_i) - 1\} I_{(-\infty,\xi_p-\bar\theta)}(V_i) \tag{2.6}$$

$$T_{1N}^p = 2\sum_1^m (\theta_1 - \bar\theta)\frac{f'(V_i - \bar\theta)}{f(V_i - \bar\theta)} I_{(-\infty,\xi_p-\bar\theta)}(V_i) + 2\sum_{m+1}^N (\theta_2 - \bar\theta)\frac{f'(V_i - \bar\theta)}{f(V_i - \bar\theta)} I_{(-\infty,\xi_p-\bar\theta)}(V_i)\,.$$

The proof of Lemma 3.5 of [5] shows that $\operatorname{Var}[\Lambda_N{}^p - T_{1N}^p] \to 0$ and

$$E[\Lambda_N^p - T_{1N}^p] \to \lambda(1-\lambda)\Delta^2\left[\tfrac{1}{2}f'(\xi_p) - \tfrac{1}{4}\int_{-\infty}^{\xi_p}\left(\frac{f'(x)}{f(x)}\right)^2 f(x)\,dx\right]. \tag{2.7}$$

Markov's inequality and the continuity of f then imply that (see also [5], Lemma 3.5)

$$|(\Lambda_N^r - \Lambda_N^p) - (T_{1N}^r - T_{1N}^p)| \to 0 \qquad \text{in } P_{N\bar\theta}\,. \tag{2.8}$$

Summarizing this chain of approximations

THEOREM 2.1. *If $I(f) < \infty$, (1.8) holds and $f'(\cdot)$ is continuous in a neighborhood of ξ_p, then*

$$\left(\log L_N - \frac{\Delta_N}{(N)^{\frac{1}{2}}} S_r{}^{*N} + \tfrac{1}{2}\sigma_p{}^2\right) \to 0 \qquad \textit{in } P_{N\bar\theta} \tag{2.9}$$

where

$$\frac{\Delta_N}{(N)^{\frac{1}{2}}} S_r{}^{*N} = -\sum_1^r \{(\theta_1 - \bar\theta)Z_i + (\theta_2 - \bar\theta)(1 - Z_i)\}\frac{f'(W_i - \bar\theta)}{f(W_i - \bar\theta)} + \{(m - m_r)(\theta_1 - \bar\theta) + (n - n_r)(\theta_2 - \bar\theta)\}\frac{f(W_r - \bar\theta)}{1 - F(W_r - \bar\theta)}\,. \tag{2.10}$$

3. Proof of asymptotic normality and asymptotic sufficiency of ranks. In this section, we first show the asymptotic equivalence of the parametric test $S_r{}^{*N}$ defined in (2.10) and the locally most powerful rank test $S_r{}^N$, obtained in Johnson and Mehrotra [6] and defined in (3.1), under the null hypotheses (1.2). In conjunction with Theorem 2.1, asymptotic sufficiency of ranks will be demonstrated.

The lmpr test for the location alternative (1.3) is given by $S_r{}^N = \sum_1^r (1 - Z_i)a_N(i, f) + (n - n_r)/(N - r)\sum_{r+1}^N a_N(i, f)$ where $a_N(i, f) = E[-f'(W_i)/f(W_i)]$. Since $\sum_1^N a_N(i, f) = 0$, whenever $I(f) < \infty$, we can restate $S_r{}^N$ as

$$\frac{\Delta}{(N)^{\frac{1}{2}}} S_r{}^N = \frac{\Delta}{(N)^{\frac{1}{2}}}\left\{\frac{n}{N}\left[\sum_1^r Z_i a_N(i, f) + \frac{m - m_r}{(N - r)}\sum_{r+1}^N a_N(i, f)\right] - \frac{m}{N}\left[\sum_1^r (1 - Z_i)a_N(i, f) + \left(\frac{n - n_r}{N - r}\right)\sum_{r+1}^N a_N(i, f)\right]\right\} \tag{3.1}$$

which is same as (2.10) except $a_N(i, f)$ replaces $\{-f'(W_i - \bar{\theta})/f(W_i - \bar{\theta})\}$ and $(N-r)^{-1} \sum_{r+1}^{N} a_N(i, f)$ replaces $f(W_r - \bar{\theta})/\{1 - F(W_r - \bar{\theta})\}$. Clearly

$$E[S_r^N] = E[S_r^{*N}] = 0 . \tag{3.2}$$

In view of (3.2), the following theorem establishes the asymptotic equivalence of S_r^{*N} and S_r^N.

THEOREM 3.1. *If* $I(f) < \infty$,

$$\frac{1}{N} \operatorname{Var}(S_r^{*N} - S_r^N) \to 0 .$$

PROOF. From (3.2),

$$\operatorname{Var}[S_r^{*N} - S_r^N] = E[S_r^{*N} - S_r^N]^2 = E_{\mathbf{W}}[E_{\mathbf{Z}}(S_r^{*N} - S_r^N)^2 \mid \mathbf{W} = \mathbf{w}] .$$

Since $\mathbf{W} = \mathbf{w}$ in the inner term of the last expression, we apply a well-known theorem of sampling, after writing $(S_r^{*N} - S_r^N)$ in terms of $-[f'(W_i - \bar{\theta})/f(W_i - \bar{\theta}) + a_N(i, f)]$, to get

$$\begin{aligned} &\operatorname{Var}(S_r^{*N} - S_r^N) \\ &\quad \leqq \frac{mn}{N(N-1)} E\Big[\textstyle\sum_1^r \Big\{-\frac{f'(W_i - \bar{\theta})}{f(W_i - \bar{\theta})} - a_N(i, f)\Big\}^2 \\ &\qquad + (N-r)\{f(W_r - \bar{\theta})/[1 - F(W_r - \bar{\theta})] - a_r^*\}^2\Big], \end{aligned} \tag{3.3}$$

where $a_r^* = (N-r)^{-1} \sum_{r+1}^{N} a_N(i, f)$. However, $(N - r) \operatorname{Var}\{f(W_r - \bar{\theta})/[1 - F(W_r - \bar{\theta})]\}$ is less than $\sum_{r+1}^{N} \operatorname{Var}\{f'(W_i - \bar{\theta})/f(W_i - \bar{\theta})\}$ (see Mehrotra [10], page 24). Consequently, the rhs of (3.3) is less than $mn[N(N-1)]^{-1} \sum_1^N E[-f'(W_i - \bar{\theta})/f(W_i - \bar{\theta}) - a_N(i, f)]^2$. Now, following the same argument as [4], we conclude that $N^{-1} \operatorname{Var}(S_r^{*N} - S_r^N) \to 0$.

Since the asymptotic distribution of S_r^N is also normal, see [6], in view of our Theorem 3.1 we obtain

THEOREM 3.2. *If* (1.8) *holds,* $I(f) < \infty$ *and* $f'(\cdot)$ *is continuous in a neighborhood of* ξ_p, *then the asymptotic distribution of* $\log L_N$ is $N(-\frac{1}{2}\sigma_p^2, \sigma_p^2)$.

The result of Theorem 3.2 leads naturally to a new derivation of the limiting distribution of the test statistics under a sequence of local alternatives. One can use any smooth score function and the details are found in Mehrotra [10].

PROOF OF THEOREM 1.1. The proof of the theorem is analogous to the proof of Theorem VII. 1.2 of Hàjek and Šidák [4], with their S_{vd^0} replaced by $N^{-\frac{1}{2}} S_r^N$ defined in (3.1). In case we assume that convergence of L_N to $\exp(N^{-\frac{1}{2}}\Delta S_r^N - \frac{1}{2}\sigma_p^2)$ is not uniform, a contradiction can be obtained as follows. Suppose there exist a sequence of Δ's for which the convergence is not uniform. Then employing the argument of Johnson and Roussas [7], page 1211, it can be shown that any statistic based on $V_1, \cdots, V_N$ has power α. In particular, therefore, the likelihood of the complete sample converges to one. If $P'_{N\bar{\theta}}$ and $Q'_{N_1\theta}$ denote measures corresponding to $V_1, \cdots, V_N$ under the null and alternative hypothesis, then by Proposition 3.1 of Johnson and Roussas [7], we see that $||P'_{N,\bar{\theta}} - Q'_{N\theta}|| = 2 \sup \{|P'_{N\bar{\theta}}(A) - Q'_{N,\theta}(A)|; A \in \mathscr{A}_N\} \to 0$. On the other hand, the sub σ-field generated by $(Z_1, \cdots, Z_r; W_1, \cdots, W_r)$ is contained in $\mathscr{A}_N$ (the σ-field of $V_1, \cdots, V_N$). Since

$$P_{N\bar{\theta}}\left[\left|\frac{Q_{N\theta}}{P_{N\bar{\theta}}} - 1\right| > \varepsilon\right] \leqq \int |q_{N\theta} - p_{N\bar{\theta}}| \leqq \frac{1}{\varepsilon} ||Q_{N\theta} - P_{N\bar{\theta}}|| \to 0 ,$$

we obtain a contradiction.

The remainder of the proof which involves truncating $N^{-\frac{1}{2}}\Delta S_r^N$ remains same as in VII. 1.2 of [4].

PROOF OF THEOREM 1.2. This is the result corresponding to the Theorem VII 1.4 of [4]. We need only change the definition of ψ_{va} in (13) to be defined in terms of observed ranks. The null distribution does not depend upon f or $\bar{\theta}$ and the remainder of the proof remains unchanged.

REMARK. An asymptotically most powerful test can be based on the ranks statistic S_r^N when the right scores are used for then S_r^N is asymptotically sufficient and this test approximates the likelihood ratio test. Another test which is asymptotically as powerful as S_r^N is obtained from the parametric statistic S_r^{*N}. In Gastwirth [2], the test based on S_r^N was shown to be asymptotically most powerful among rank tests only, our results have extended this to any test.

REFERENCES

[1] CHERNOFF, H., GASTWIRTH, J. L. and JOHNS, M. V. (1967). Asymptotic distribution of linear combinations of functions of order statistics with application to estimation. *Ann. Math. Statist.* **38** 52–71.

[2] GASTWIRTH, J. L. (1965). Asymptotically most powerful rank tests for the two sample problem with censored data. *Ann. Math. Statist.* **36** 1243–1246.

[3] HÁJEK, J. (1974). Asymptotic sufficiency of the vector of ranks in the Bahadur sense. *Ann. Statist.* **2** 75–83.

[4] HÁJEK, J. and ŠIDÁK, Z. (1967). *Theory of Rank Tests.* Academic, New York.

[5] JOHNSON, R. A. (1974). Asymptotic result for inference procedures based on the r smallest observations. *Ann. Statist.* **2** 1138–1151.

[6] JOHNSON, R. A. and MEHROTRA, K. G. (1972). Locally most powerful rank tests for the two-sample problems with censored data. *Ann. Math. Statist.* **43** 823–831.

[7] JOHNSON, R. A. and ROUSSAS, G. G. (1969). Asymptotically most powerful tests in Markov processes. *Ann. Math. Statist.* **40** 1207–1215.

[8] JOHNSON, R. A. and ROUSSAS, G. G. (1970). Asymptotically optimal tests in Markov process. *Ann. Math. Statist.* **41** 918–938.

[9] JOHNSON, R. A. and ROUSSAS, G. G. (1971). Applications of contiguity to multiparameter hypothesis testing. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **1** 195–226, Univ. of California Press.

[10] MEHROTRA, K. G. (1970). A nonparametric approach to the problem of ordered alternatives with emphasis on locally most powerful rank tests for two-sample problem with censoring. Ph. D. dissertation, The Univ. of Wisconsin, Madison.

[11] RAO, V., SAVAGE, I. R. and SOBEL, M. (1960). Contribution to the theory of rank order statistics: the two-sample censored case. *Ann. Math. Statist.* **31** 415–426.

[12] ROUSSAS, G. G. (1965). Asymptotic inference in Markov processes. *Ann. Math. Statist.* **38** 978–992.

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