

## ASYMPTOTIC COMPARISON OF RANK TESTS FOR THE REGRESSION PROBLEM WHEN TIES ARE PRESENT

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Without assumptions on the underlying distributions we prove the asymptotic normality of averaged scores rank statistics under all product distributions which are contiguous to the null hypothesis, and find a very simple form of the centering constants. In the one-sided two-sample and trend situations this enables us to show that monotonicity of the scores generating function is equivalent to the asymptotic unbiasedness of the corresponding averaged scores rank test. For such asymptotically unbiased tests we prove simple necessary and sufficient conditions for having bounds for their asymptotic relative efficiency under all contiguous alternatives of the model. As a by-product, we get the local asymptotic superiority of averaged scores rank tests to the associated tests with randomized ranks not only for shift but for general alternatives.

In addition we prove that every one-sided averaged scores rank test is asymptotically most powerful (asymptotically equivalent to likelihood ratio test) for a suitable nonparametric subclass of alternatives, provided the test and the associated subclass of alternatives are generated by a non-decreasing, square-integrable function defined on the unit interval.

**1. Introduction.** There is a substantial literature on the problem of ties with respect to rank tests (cf. the bibliography in Hájek and Šidák (1967)). However, most of the papers are concerned with special properties of special tests. The first paper dealing with more general properties in a rather general class of rank tests is by Vorličková (1970). The disadvantages of this paper are the heavy assumptions on the underlying distributions and the restrictive results gotten under alternatives caused by the methods applied. The same is true for a similar paper by Vorličková (1972) on the symmetry case. Especially, it is assumed that the underlying distributions are purely discrete with mass only at equidistant points. Moreover, the asymptotic normality and efficiency results under contiguous alternatives are proved only for special subclasses of such alternatives. The reason for the latter restriction is the application of Hájek and Šidák (1967), Lemma VI.1.4 (Le Cam's third lemma). Conover (1973) dropped the assumptions on the underlying distributions, but since his proofs follow the lines of Vorličková the restrictions on the alternatives still are there.

In this paper we shall prove asymptotic normality and power results under all product distributions which are contiguous to the null hypothesis, without any assumption on the underlying distributions besides the trivial one of not having

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one-point measures. As a consequence we are able to derive results on comparison of rank tests by means of asymptotic relative efficiency, which extend the results for the continuous case of Behnen (1972).

Let  $X_{n1}, \dots, X_{nn}$  be independent real valued random variables and let  $F_{nj}(x) = \Pr(X_{nj} \leq x)$  be the (arbitrary) cumulative distribution function of  $X_{nj}, j = 1, \dots, n, n \in \mathbb{N}$ . As usual, the statistic

$$R_{ni} = \sum_{j=1}^n I_{[0,\infty)}(X_{ni} - X_{nj})$$

is called the rank of  $X_{ni}$  in the ordered sample

$$X_n^{(\cdot)} = (X_n^{(1)}, \dots, X_n^{(n)}), \quad X_n^{(1)} \leq \dots \leq X_n^{(n)}, \quad X_n^{(R_{ni})} = X_{ni}.$$

Following the notation of Hájek and Šidák (1967) and Vorličková (1970) we call the  $g_n$  groups of equal observations in  $X_n^{(\cdot)}$  ties and denote the number of observations in the  $j$ th tie ( $1 \leq j \leq g_n$ ) by  $\tau_{nj}$ ,

$$\tau_n = (\tau_{n1}, \dots, \tau_{ng_n}), \quad T_{nj} = \sum_{i=1}^j \tau_{ni}, \quad j = 1, \dots, g_n,$$

$T_{n0} = 0$ . Thus

$$X_n^{(T_{nj-1}+1)} = \dots = X_n^{(T_{nj})}, \quad X_n^{(T_{nj})} < X_n^{(T_{nj+1})}.$$

For convenience we will suppress the first subscript  $n$  throughout this paper, and limits are taken for  $n \rightarrow \infty$  if not otherwise stated.

*Randomization of ranks.* From the definitions it is clear that exactly the observations in one tie get equal ranks. Therefore the randomization is done in the most convenient way: Let  $U_1, U_2, \dots$  be i.i.d. random variables (also independent from  $(X_1, \dots, X_n)$  for each  $n \in \mathbb{N}$ ) each having a rectangular distribution on  $(0, 1)$ . Put

$$R_i^* = \sum_{j=1}^n I_{[0,\infty)}((R_i - U_i) - (R_j - U_j)), \quad R^* = (R_1^*, \dots, R_n^*).$$

This implies

$$\begin{aligned} \{R_1^*, \dots, R_n^*\} &= \{1, \dots, n\} \quad \text{a.e.}, \\ R_i < R_j &\Rightarrow R_i^* < R_j^*, \\ R_i^* &\leq R_i, \quad \forall i, j = 1, \dots, n. \end{aligned}$$

For the definition of scores let  $b$  be a square-integrable function defined on  $(0, 1)$  such that (for convenience)

$$(1.1) \quad \int b \, d\lambda = 0, \quad \int b^2 \, d\lambda = 1,$$

where  $\lambda$  denotes Lebesgue-measure on  $(0, 1)$  throughout the paper. Then scores  $b_i$  are defined by

$$b_i = Eb(U_n^{(i)}), \quad i = 1, \dots, n,$$

or in some other way such that (cf. Hájek and Šidák (1967) or Vorličková (1970))

$$(1.2) \quad \int (b_{[nu+1]} - b(u))^2 \lambda(du) \rightarrow 0, \quad \sum_{i=1}^n b_i = 0.$$

Moreover, let regression coefficients  $c_i$  be given such that

$$(1.3) \quad \sum_{i=1}^n c_i = 0, \quad \sum_{i=1}^n c_i^2 = 1, \quad \max_{1 \leq i \leq n} c_i^2 \rightarrow 0.$$

With these assumptions and notations we shall consider *randomized linear rank statistics*  $S_b^*$  of the form

$$(1.4) \quad S_b^* = \sum_{i=1}^n c_i b_{R_i^*},$$

and for given  $\alpha \in (0, 1)$  *randomized ranks tests*  $\varphi_b^*$  of the form

$$(1.5) \quad \varphi_b^* = I_{\{S_b^* > k_n\}},$$

with  $k_n \rightarrow u_\alpha$ ,  $u_\alpha$  the  $(1 - \alpha)$ -quantile of  $N(0, 1)$ .

Now we introduce another method of treating ties (cf. Hájek (1969)) which will be shown to be generally better than randomization. Nevertheless randomization is a convenient tool.

*Averaged scores.* In the general notation stated above average the scores  $b_i$  over the length of the ties, i.e.,

$$\bar{b}_i = b_i(\tau) = \frac{1}{\tau_k} \sum_{j=T_{k-1}+1}^{T_k} b_j \quad \text{if } T_{k-1} < i \leq T_k.$$

Define *averaged scores rank statistics*  $\bar{S}_b$  by

$$(1.6) \quad \bar{S}_b = \sum_{i=1}^n c_i \bar{b}_{R_i}$$

and *averaged scores rank tests*  $\bar{\varphi}_b$  by

$$(1.7) \quad \bar{\varphi}_b = I_{\{\bar{S}_b > k_n(1/(n-1) \sum_{i=1}^n \bar{b}_i^2)^{1/2}\}}.$$

EXAMPLE. Consider the two-sample situation and the averaged scores versions of the normal scores test, the Wilcoxon test, and the rank median test, with centered and standardized scores generating functions  $\Phi^{-1}(u)$ ,  $(12)^{1/2}(u - \frac{1}{2})$ ,  $\text{sign}(u - \frac{1}{2})$ , respectively. Fix any point  $\vartheta_0$  of the hypothesis of randomness  $H_0$ . Then by Theorem 2.2.(c) and Remark 3.1 for each one of the three tests there exist sequences from the alternative  $K_0$  (first sample stochastically larger than second sample) which are contiguous to  $\vartheta_0$  and for which the respective tests have optimal asymptotic power. On the other hand, Theorem 3.3 gives the range of the asymptotic relative efficiency (ARE) of two such tests for the alternatives from  $K_0$  which are contiguous to  $\vartheta_0$ . If  $\vartheta_0$  corresponds to a distribution which is not purely discrete, then the ARE of rank median to Wilcoxon and normal scores test ranges down to zero. If  $\vartheta_0$  corresponds to a distribution function  $F$  such that  $F^{-1}$  is strictly increasing in a neighborhood of  $\frac{1}{2}$ , then the ARE of Wilcoxon and normal scores to rank median test ranges down to zero, too. If  $\vartheta_0$  corresponds to a continuous distribution function, then the range of the ARE of normal scores to Wilcoxon test is the interval  $[\pi/6, \infty]$ . With a lot of computations one can prove the existence of a constant  $c > 0$  such that the ARE of normal scores to Wilcoxon test is not less than  $c$ , uniformly in  $\vartheta_0 \in H_0$ . Any such constant is strictly less than  $\pi/6$ .

**2. Statement of the general theorems.** In this section we state the theorems concerning asymptotic normality under general contiguous alternatives and optimality of averaged scores rank tests. The proofs are given in Section 5. For the formulation of the statements we first have to introduce some general notation.

For each one-dimensional distribution function  $F$  and each  $\lambda$ -integrable function  $h$  defined on the interval  $(0, 1)$  we define a function  $h_F$  on  $(0, 1)$  by

$$(2.1) \quad \begin{aligned} h_F(u) &= h(u) && \text{if } u \in (0, 1) - \bigcup_{x \in D(F)} (F(x-0), F(x)], \\ &= \int_{F(x-0)}^{F(x)} h \, d\lambda / (F(x) - F(x-0)) && \text{if } u \in (F(x-0), F(x)], \quad x \in D(F), \end{aligned}$$

where  $D(F)$  denotes the countable set of points where  $F$  is not continuous. For continuous  $F$  the function  $h_F$  is equal to  $h$ . For general  $F$

$$(2.2) \quad h_F = E_\lambda(h | \mathfrak{B}_F) \quad \lambda\text{-a.e.},$$

where  $E_\lambda(h | \mathfrak{B}_F)$  denotes the conditional expectation of  $h$  with respect to  $\lambda$  and the sub- $\sigma$ -algebra  $\mathfrak{B}_F$  of the Borel- $\sigma$ -algebra on  $(0, 1)$ , which is generated by the intervals  $(F(x-0), F(x)]$ ,  $x \in D(F)$ , and the Borel-subsets of the set  $(0, 1) - \bigcup_{x \in D(F)} (F(x-0), F(x)]$ .

In the sequel we denote the distribution of  $(X_{n1}, \dots, X_{nn})$  by  $\prod_{j=1}^n F_{nj}$ , the product measure of distributions corresponding to the distribution functions  $F_{nj}$ . In case of  $F_{nj} = F$  we use  $F^{(n)}$  instead of  $\prod_{j=1}^n F_{nj}$ . Moreover, we identify distribution functions with the corresponding probability measures, if convenient.

Let  $H_0$  denote the hypothesis of randomness, i.e.,

$$H_0 = \{ \{ \prod_{j=1}^n F_{nj} \} : F_{nj} = F \quad \forall j, n, F(x) - F(x-0) < 1 \quad \forall x \in \mathbb{R} \}.$$

As in Hájek and Šidák (1967) call a sequence of distributions  $\{Q_n\}$  contiguous to the sequence  $\{P_n\}$  if  $P_n(A_n) \rightarrow 0$  implies  $Q_n(A_n) \rightarrow 0$ . Then define  $K_0$  to be the set of all sequences of product distributions which are contiguous to some sequence from  $H_0$ ,

$$K_0 = \{ \{ \prod_{j=1}^n F_{nj} \} : \{ \prod_{j=1}^n F_{nj} \} \text{ contiguous to some } \{ F^{(n)} \} \in H_0 \}.$$

**REMARK 2.1.** Contiguity of  $\mathcal{D} \in K_0$  to  $\mathcal{D}_1 \in H_0$  and  $\mathcal{D}_2 \in H_0$  implies  $\mathcal{D}_1 = \mathcal{D}_2$ , since  $F_1 \neq F_2$  implies the existence of measurable sets  $B_n$  such that  $F_1^{(n)}(B_n) \rightarrow 0$ ,  $F_2^{(n)}(B_n) \rightarrow 1$  (cf. Behnen and Neuhaus (1975), Proof of the Proposition) which contradicts contiguity of  $\mathcal{D} \in K_0$  to  $\{F_1^{(n)}\}$  and  $\{F_2^{(n)}\}$ .

Now we are in the position to formulate the results. Throughout the section we assume (1.1), (1.2), and (1.3).

**THEOREM 2.1.** (a)  $\{ \prod_{j=1}^n F_j \} \in H_0$  implies  $\mathfrak{S}(S_b^*) \rightarrow N(0, 1)$ .

(b)  $\{ \prod_{j=1}^n F_j \} \in K_0$  implies  $\mathfrak{S}(S_b^* - \delta_n(b, \{ \prod_{j=1}^n F_j \})) \rightarrow N(0, 1)$ , with

$$(2.3) \quad \delta_n(b, \{ \prod_{j=1}^n F_j \}) = \sum_{i=1}^n c_i \int (\bar{b}_{nF} \circ F) \, dF_i,$$

where  $F$  is associated to  $\{ \prod_{j=1}^n F_j \}$  according to the definition of  $K_0$  (cf. Remark 2.1),

and where  $\{\bar{b}_n\}$  is any sequence of Lebesgue-measurable functions defined on the unit interval and having the properties

$$(2.4) \quad \int \bar{b}_n d\lambda = 0, \quad \sup_x |\bar{b}_n(x)| < \infty \quad \forall n \in \mathbb{N}, \\ \int (b - \bar{b}_n)^2 d\lambda \rightarrow 0, \quad \max_i c_i^2 \sup_x \bar{b}_n^4(x) \rightarrow 0.$$

(c) If  $b = b_F$   $\lambda$ -a.e. for  $\{F^{(n)}\} \in H_0$  and if

$$K_{b_F} = \{ \{X_1^n F_{id}\} : F_{id} \text{ has } F\text{-density } 1 + dc_i(\bar{b}_{n_F} \circ F), n \in \mathbb{N}, d > 0, \\ \{\bar{b}_n\} \text{ any approximation (2.4)} \},$$

then the randomized rank test  $\{\varphi_b^*\}$  according to (1.5) is asymptotically most powerful for  $H_0$  against  $K_{b_F}$  at level  $\alpha$ . Moreover, each sequence from  $K_{b_F}$  is contiguous to  $\{F^{(n)}\}$ .

REMARK 2.2. For any two choices of centering constants we have

$$\delta_n(b, \vartheta) - \delta_n'(b, \vartheta) \rightarrow 0.$$

Moreover, contiguity and asymptotic normality imply

$$(2.5) \quad \limsup_n |\delta_n(b, \vartheta)| < \infty \quad \forall \vartheta \in K_0.$$

REMARK 2.3. Obviously, assumptions (1.1) and (1.3) ensure the existence of approximations (2.4) of  $b$ . In case of bounded  $b$  we may choose  $\bar{b}_n$  equal to  $b$ . In general we could choose a recentered truncation, where the truncation points suitably depend on  $\max_i |c_i|$ .

THEOREM 2.2. (a)  $\{X_1^n F_j\} = \{F^{(n)}\} \in H_0$  implies  $\mathcal{Q}(\bar{S}_b) \rightarrow N(0, \int b_F^2 d\lambda)$  and  $\text{Var}(\bar{S}_b | \tau) = 1/(n-1) \sum_{i=1}^n (\bar{b}_i)^2 \rightarrow \int b_F^2 d\lambda$  in probability.

(b)  $\{X_1^n F_j\} \in K_0$  implies  $\mathcal{Q}(\bar{S}_b - \delta_n(b, \{X_1^n F_j\})) \rightarrow N(0, \int b_F^2 d\lambda)$ , with  $\delta_n(b, \{X_1^n F_j\})$  and  $F$  defined as in Theorem 2.1. (b).

(c) Assume  $b$  to be  $\lambda$ -a.e. monotone. Then  $\{\bar{\varphi}_b\}$  is an asymptotically most powerful level  $\alpha$  test for  $H_0$  against

$$K_b \equiv \{ \{X_1^n F_j\} \in K_{b_F} : F(x) - F(x-0) < 1 \quad \forall x \in \mathbb{R} \},$$

where  $K_{b_F}$  is defined as in Theorem 2.1. (c). Moreover,  $K_b \subset K_0$ .

REMARK 2.4. On one hand the monotonicity condition on  $b$  is assumed to ensure  $\int b_F^2 d\lambda > 0$  on all of  $H_0$ , and thus to ensure that  $\{\bar{\varphi}_b\}$  is asymptotically level  $\alpha$  for  $H_0$ . On the other hand we know from Behnen (1972), Theorem 2.2, that in the special applications the monotonicity condition is necessary for having asymptotically unbiased tests, even under the restrictive assumption of continuous distributions. Therefore it seems natural to pose this condition in part (c) instead of restricting the hypothesis  $H_0$  to  $F$ 's with  $\int b_F^2 d\lambda > 0$ , which depends on  $b$  if  $b$  is not monotone  $\lambda$ -a.e.

REMARK 2.5. Part (b) is a generalization of Conover (1973), Theorem 8.3, to all contiguous alternatives  $K_0$  without any parametrization and without any

regularity assumptions, whereas the following corollary corresponds in a certain sense to the result of Conover.

**COROLLARY 2.3.** *Let  $h$  fulfill  $\int h d\lambda = 0$ ,  $\int h^2 d\lambda = 1$  and define  $K_{h_F}$  according to part (c) of Theorem 2.1. Then, for*

$$\begin{aligned} & \{\mathbf{X}_1^n F_j\} = \{\mathbf{X}_1^n F_{jd}\} \in K_{h_F}, \\ (a) \quad & \delta_n(b, \{\mathbf{X}_1^n F_{jd}\}) \rightarrow d \int b_F h_F d\lambda, \\ (b) \quad & E\varphi_b^* \rightarrow \Phi(-u_\alpha + d \int b_F h_F d\lambda), \\ (c) \quad & E\bar{\varphi}_b \rightarrow \Phi(-u_\alpha + d \int b_F h_F d\lambda / (\int b_F^2 d\lambda)^{1/2}), \end{aligned}$$

where  $\Phi$  denotes the distribution function of  $N(0, 1)$ .

**3. Efficiency of two-sample tests.** In this section we assume (1.1), (1.2),

$$(3.1) \quad n = n_1 + n_2, \quad c_{ni} = (n_1 n_2 / n)^{1/2} / n_i \quad \text{if } i = 1, \dots, n_1, \\ = -(n_1 n_2 / n)^{1/2} / n_2 \quad \text{if } i = n_1 + 1, \dots, n,$$

and  $n_1 \rightarrow \infty, n_2 \rightarrow \infty$ , which means that condition (1.3) is fulfilled. As (one-sided) alternative we take

$$K'_0 = \{\{\mathbf{X}_1^n F_j\} \in K_0 : F_1 = \dots = F_{n_1}, F_{n_1+1} = \dots = F_n, F_1 \leq F_n, F_1 \neq F_n\},$$

i.e., we consider the alternative that the first sample is stochastically larger than the second sample, which is more general and more useful than shift alternatives.

**REMARK 3.1.** If  $b$  is  $\lambda$ -a.e. nondecreasing and if we take  $\lambda$ -a.e. nondecreasing approximations (2.4) in the definition of  $K_{b_F}$  and  $K_b$ , then we have  $K_b \subset K'_0$ . Thus the optimality result of Theorem 2.2. (c) really is concerned with subclasses of  $K'_0$ . More generally we have  $K_b \subset K'_0$  if  $\int_0^t b d\lambda \leq 0 \forall t \in (0, 1)$ , and  $K_b \cap K'_0 = \emptyset$  if  $\int_0^t b d\lambda \geq 0 \forall t \in (0, 1)$ , provided the approximations (2.4) in the definition of  $K_b$  are suitably chosen.

**THEOREM 3.1.** *The averaged scores rank test  $\{\bar{\varphi}_b\}$  according to (1.7) is asymptotically unbiased on all of  $K'_0$  if and only if  $b$  is  $\lambda$ -a.e. nondecreasing.*

**PROOF.** Because of the asymptotic normality of  $S_b^*$  (Theorem 2.1) and  $\bar{S}_b$  (Theorem 2.2) under  $H_0$  and  $K'_0$  with the same centering constants  $\delta_n(\cdot, \cdot)$  and because of  $\int b_F^2 d\lambda > 0 \forall \{F^{(n)}\} \in H_0$  if  $b$  is  $\lambda$ -a.e. nondecreasing, the proof is immediate from Lemma 5.1, Lemma 5.3, and Behnen (1972), Theorem 2.2 (applied to  $\{\varphi_b^*\}$ ).  $\square$

**REMARK 3.2.** For  $\lambda$ -a.e. nondecreasing  $b$  we have as a consequence

$$(3.2) \quad \liminf_n \delta_n(b, \mathcal{J}) \geq 0 \quad \forall \mathcal{J} \in K'_0.$$

In case of

$$E_{H_0} \varphi_{nj} \rightarrow \alpha, \quad E_{\mathcal{J}} \varphi_{nj} - \Phi(-u_\alpha + \delta_{nj}) \rightarrow 0, \quad j = 1, 2,$$

we define asymptotic relative efficiency of  $\{\varphi_{n1}\}$  relative to  $\{\varphi_{n2}\}$  with respect to

$\mathcal{G}$  by

$$\text{ARE}(\{\varphi_{n1}\} : \{\varphi_{n2}\} | \mathcal{G}) = \liminf_n (\delta_{n1}^2 / \delta_{n2}^2)$$

$$\text{iff } 0 \leq \liminf_n \delta_{n1}, \quad 0 < \liminf_n \delta_{n2}, \quad \limsup_n \delta_{n1} < \infty.$$

REMARK 3.3. This definition is consistent with the usual definition of Pitman-efficiency. Obviously any definition of ARE doesn't make sense without the assumption of unbiasedness, i.e.,  $\liminf_n \delta_{ni} \geq 0, i = 1, 2$ . Since we do not assume convergence of  $\delta_{ni}$  we have to impose some other conditions on the behavior of the  $\delta_{ni}$ .

THEOREM 3.2. Let  $b$  be  $\lambda$ -a.e. nondecreasing. Then for each  $\mathcal{G} \in K'_0$  either

$$\liminf_n E_{\mathcal{G}} \hat{\varphi}_b = \liminf_n E_{\mathcal{G}} \varphi_b^* = \alpha$$

or

$$\text{ARE}(\{\hat{\varphi}_b\} : \{\varphi_b^*\} | \mathcal{G}) \geq 1,$$

where equality iff  $b$  is  $\lambda$ -a.e. constant over each interval  $(F(x - 0), F(x)), x \in D(F)$ , where  $F$  is the distribution function such that  $\mathcal{G}$  is contiguous to  $\{F^{(n)}\}$ .

REMARK 3.4. This theorem is an extension of Conover (1973), formula (10.8), to all contiguous alternatives  $K'_0$ . It should be mentioned that even in the restricted case Conover doesn't clarify the validity of his formula (10.8) (cf. Remark 3.3 or Hájek and Šidák (1967), Remark VII.2.1.a).

PROOF. Because of  $\int b_F^2 d\lambda > 0$  we get from Theorems 2.1 and 2.2 either the first statement or

$$\text{ARE}(\dots) = \int b^2 d\lambda / \int b_F^2 d\lambda.$$

But formula (2.2) implies

$$\int b_F^2 d\lambda = E_{\lambda}(E_{\lambda}(b | \mathfrak{B}_F))^2 \leq E_{\lambda} E_{\lambda}(b^2 | \mathfrak{B}_F) = \int b^2 d\lambda,$$

with equality iff  $(E_{\lambda}(b | \mathfrak{B}_F))^2 = E_{\lambda}(b^2 | \mathfrak{B}_F)$   $\lambda$ -a.e. Because of (2.1) this means

$$(\int_{F(x-0)}^{F(x)} b d\lambda / (F(x) - F(x - 0)))^2 = \int_{F(x-0)}^{F(x)} b^2 d\lambda / (F(x) - F(x - 0)) \quad \forall x \in D(F),$$

and this is equivalent to

$$b = \int_{F(x-0)}^{F(x)} b d\lambda / (F(x) - F(x - 0))$$

$$\lambda\text{-a.e. on } (F(x - 0), F(x)) \quad \forall x \in D(F). \quad \square$$

THEOREM 3.3. Let  $b_1$  and  $b_2$  be  $\lambda$ -a.e. nondecreasing scores generating functions which fulfill (1.1), (1.2), (3.1). Then, for each  $F$  different from one-point measures and for each  $c > 0$ , the following two statements are equivalent:

(i)  $b_{1F} / (\int b_{1F}^2 d\lambda)^{1/2} - c b_{2F} / (\int b_{2F}^2 d\lambda)^{1/2}$  is  $\lambda$ -a.e. nondecreasing.

(ii)  $\text{ARE}(\{\hat{\varphi}_{b_1}\} : \{\hat{\varphi}_{b_2}\} | \mathcal{G}) \geq c^2$  for each  $\mathcal{G} \in K'_0$  which is contiguous to  $\{F^{(n)}\}$  and which fulfills

$$\liminf_n E_{\mathcal{G}} \hat{\varphi}_{b_2} > \alpha.$$

REMARK 3.5. Continuity of  $F$  implies  $b_{1F} = b_1, b_{2F} = b_2$ , and  $\int b_{1F}^2 d\lambda = \int b_1^2 d\lambda = 1$ . Thus Theorem 3.3 contains Theorem 2.3 of Behnen (1972) as a

special case. (Notice the misprinted inequality sign in statement (II) of that theorem.)

REMARK 3.6. Theorem 2.2, formula (2.5), and formula (3.2) imply the existence of ARE  $(\{\tilde{\varphi}_{b_1}\} : \{\tilde{\varphi}_{b_2}\} | \mathcal{D})$  for each  $\mathcal{D} \in K'_0$  with  $\liminf_n E_{\mathcal{D}} \tilde{\varphi}_{b_2} > \alpha$ .

PROOF. Given  $F$  and  $c > 0$  we define  $b_F$  to be the difference stated in (i). If  $b_F$  is not  $\lambda$ -a.e. nondecreasing we define similar to the proof of Theorem 2.2 in Behnen (1972) a  $b_0$  such that

$$\int b_0 d\lambda = 0, \quad \int b_0^2 d\lambda = 1, \quad \int_0^t b_0 d\lambda \leq 0 \quad \forall t \in (0, 1),$$

and

$$\int b_F b_0 d\lambda < 0, \quad b_0 = b_{0F}, \quad \sup_x |b_0(x)| \leq M < \infty.$$

Therefore, with the definition

$$K_{b_0F} = \{ \{X_1^n F_{jd}\} : F_{jd} \text{ has } F\text{-density } 1 + dc_j(b_0 \circ F), n \in \mathbb{N}, d > 0 \},$$

we have (cf. Remark 3.1)

$$\emptyset \neq K_{b_0F} \subset K'_0,$$

and  $\mathcal{D}(d) = \{X_1^n F_{jd}\} \in K_{b_0F}$  implies (cf. Corollary 2.3 and Remark 3.2)

$$\delta_n(b_i, \mathcal{D}(d)) \rightarrow d \int b_{iF} b_0 d\lambda \geq 0, \quad i = 1, 2,$$

$$\delta_n(b_1, \mathcal{D}(d)) / (\int b_{1F}^2 d\lambda)^{\frac{1}{2}} - c \delta_n(b_2, \mathcal{D}(d)) / (\int b_{2F}^2 d\lambda)^{\frac{1}{2}} \rightarrow d \int b_F b_0 d\lambda < 0.$$

Because of Theorem 2.2 and formula (2.5) this implies for each  $\mathcal{D} \in K_{b_0F} \subset K'_0$  the existence of

$$\text{ARE}(\{\tilde{\varphi}_{b_1}\} : \{\tilde{\varphi}_{b_2}\} | \mathcal{D}) < c^2 \quad \text{and} \quad \liminf_n E_{\mathcal{D}} \tilde{\varphi}_{b_2} > \alpha,$$

which contradicts (ii).

For the other direction of the proof we assume in a first step that  $b_F$  equals a constant  $a \in \mathbb{R}$   $\lambda$ -a.e. This implies

$$a = 0, \quad c = 1, \quad \text{and} \quad b_{1F} / (\int b_{1F}^2 d\lambda)^{\frac{1}{2}} = b_{2F} / (\int b_{2F}^2 d\lambda)^{\frac{1}{2}} \quad \lambda\text{-a.e.},$$

which proves (ii) because of Theorem 2.2.

To complete the proof we may now assume  $0 < \int b_F^2 d\lambda$ . Then we proceed the same way as in the corresponding part of the proof of Theorem 2.3 in Behnen (1972) using Theorem 2.2 and formula (3.2) here instead of Theorem 2.1 and Theorem 2.2 there, and taking into account the variances  $\int b_{iF}^2 d\lambda$  of the limiting distributions.  $\square$

COROLLARY 3.4. Let  $b_1$  and  $b_2$  be  $\lambda$ -a.e. nondecreasing scores generating functions which fulfill (1.1), (1.2), (3.1). If for some  $c_1 > 0, c_2 > 0$  the functions  $b_1 - c_1 b_2$  and  $b_2 - c_2 b_1$  are  $\lambda$ -a.e. nondecreasing, then for each  $\mathcal{D} \in K'_0$  either

$$\liminf_n E_{\mathcal{D}} \tilde{\varphi}_{b_1} = \liminf_n E_{\mathcal{D}} \tilde{\varphi}_{b_2} = \alpha$$

or

$$(c_1 c_2)^2 \leq \text{ARE}(\{\tilde{\varphi}_{b_1}\} : \{\tilde{\varphi}_{b_2}\} | \mathcal{D}) \leq (c_1 c_2)^{-2}.$$



PROOF. From (2.1) and the assumptions on  $b_1$  and  $b_2$  we get for each  $F$  that  $b_{1F} - c_1 b_{2F}$  and  $b_{2F} - c_2 b_{1F}$  are  $\lambda$ -a.e. nondecreasing. Moreover, Theorem 3.1 implies  $\liminf_n E_{\mathcal{G}} \hat{\varphi}_{b_i} \geq \alpha$ ,  $i = 1, 2$ . Assume  $\liminf_n E_{\mathcal{G}} \hat{\varphi}_{b_2} > \alpha$ . Then Theorem 3.3 with  $c_1(\int b_{2F}^2 d\lambda)^{\frac{1}{2}}/(\int b_{1F}^2 d\lambda)^{\frac{1}{2}}$  instead of  $c$  implies

$$\text{ARE}(\{\hat{\varphi}_{b_1}\} : \{\hat{\varphi}_{b_2}\} | \mathcal{G}) \geq c_1^2 \int b_{2F}^2 d\lambda / \int b_{1F}^2 d\lambda > 0,$$

which implies  $\liminf_n E_{\mathcal{G}} \hat{\varphi}_{b_1} > \alpha$ , too. Therefore we may again apply Theorem 3.3 in order to get

$$\text{ARE}(\{\hat{\varphi}_{b_2}\} : \{\hat{\varphi}_{b_1}\} | \mathcal{G}) \geq c_2^2 \int b_{1F}^2 d\lambda / \int b_{2F}^2 d\lambda > 0,$$

or, because of Theorem 2.2 and formula (2.5),

$$c_2^{-2} \int b_{2F}^2 d\lambda / \int b_{1F}^2 d\lambda \geq \text{ARE}(\{\hat{\varphi}_{b_1}\} : \{\hat{\varphi}_{b_2}\} | \mathcal{G}).$$

Obviously, the same inequalities can be proved under the assumption  $\liminf_n E_{\mathcal{G}} \hat{\varphi}_{b_1} > \alpha$ . Since  $\int (b_{1F} - c_1 b_{2F}) d\lambda = 0$  and  $b_{1F} - c_1 b_{2F}$ ,  $b_{1F} + c_1 b_{2F}$  are  $\lambda$ -a.e. nondecreasing we get from Behnen (1972), formulas (3.1.C), (3.4), (3.9.C), (3.12.C), the inequality

$$\begin{aligned} \int b_{1F}^2 d\lambda &= c_1^2 \int b_{2F}^2 d\lambda + \int (b_{1F} - c_1 b_{2F})(b_{1F} + c_1 b_{2F}) d\lambda \\ &\geq c_1^2 \int b_{2F}^2 d\lambda, \end{aligned}$$

and similarly

$$\int b_{2F}^2 d\lambda \geq c_2^2 \int b_{1F}^2 d\lambda.$$

This completes the proof.  $\square$

4. Efficiency of trend tests. To get the standardization assumptions of the  $c$ 's, we take here for each  $n \in \mathbb{N}$

$$c_j = (12/n)^{\frac{1}{2}}(j/(n+1) - \frac{1}{2})((n+1)/(n-1))^{\frac{1}{2}}, \quad j = 1, \dots, n.$$

As the (one sided) alternative we take

$$K'_0 = \{ \{X_1^n F_j\} \in K_0 : F_1 \geq F_2 \geq \dots \geq F_n, \text{ at least one inequality different from equality} \}.$$

Similar to Section 3 we have to assume here that  $b$  is  $\lambda$ -a.e. increasing or, more generally, that  $b$  fulfills the condition

$$\int_0^t b d\lambda \leq 0 \quad \forall t \in (0, 1).$$

This implies

$$K'_b \subset K'_0$$

NOTE. Compared with Section 3 we have here reversed signs and reversed inequalities in the definition of the  $c$ 's and of  $K'_0$ , respectively.

For this trend case we get by a slight modification of the proofs of Section 3 results which are completely similar to the ones stated in Section 3.

With respect to the proof of the counterpart of formula (3.2) one should note that  $\delta_n(\cdot, \cdot)$  can be written in the following form:

$$\delta_n(b, \{X_1^n F_j\}) = \sum_{j=1}^{\lfloor n/2 \rfloor} c_j \int (\tilde{b}_{nF} \circ F)(dF_j - dF_{n+1-j}),$$

with  $c_j \leq 0$ ,  $F_j \geq F_{n+1-j}$ , for all  $j = 1, \dots, \lfloor n/2 \rfloor$ .

**5. Proofs of the theorems of Section 2.** Let  $\mathcal{D}_0 = \{F^{(n)}\}$  and  $\mathcal{D} = \{\bigtimes_{j=1}^n F_{nj}\}$  be given, and let  $D$  denote a countable subset of the line which contains the points of discontinuity of  $F$  and  $F_{nj}$ ,  $j = 1, \dots, n$ ,  $n \in \mathbb{N}$ ,

$$\bigcup_{n=1}^{\infty} \bigcup_{j=1}^n D(F_{nj}) \cup D(F) \subset D, \quad C(F_j) = \mathbb{R} - D(F_j).$$

Given such countable set  $D$ , let  $H$  denote a (discrete) cumulative distribution function on  $\mathbb{R}$  such that

$$h(x) \equiv H(x) - H(x-0) > 0 \quad \forall x \in D, \quad \sum_{x \in D} h(x) = 1.$$

With this notation we define a strictly monotone function  $T: \mathbb{R} \rightarrow \mathbb{R}$  by

$$T(x) = x + H(x)$$

and random variables  $X_j^* = X_j^*(X_j, U_j)$  by

$$(5.1) \quad X_j^* = T(X_j) - U_j h(X_j), \quad j = 1, \dots, n.$$

These definitions give a suitable connection between the ranks, the randomized ranks, the given random variables, and the starred random variables, needed in the proofs of the main theorems.

**LEMMA 5.1.** *Under  $\mathcal{D}$  and under  $\mathcal{D}_0$ :*

(a)  $X_1^*, \dots, X_n^*$  are independent random variables each of them having a continuous distribution.

(b) If  $F_j^*$  denotes the cumulative distribution function of  $X_j^*$  then

$$(5.2) \quad \begin{aligned} F_j^*(y) &= F_j(T^{-1}(y)) && \text{if } y \in T(\mathbb{R}), \\ &= F_j(x-0) + \frac{F_j(x) - F_j(x-0)}{T(x) - T(x-0)} (y - T(x-0)) && \text{if } y \in [T(x-0), T(x)], \quad x \in D. \end{aligned}$$

(c)  $X_i < X_j \Rightarrow X_i^* < X_j^*$ ;

$$X_j = \inf \{x \in \mathbb{R} : T(x) \geq X_j^*\}, \quad X^{(j)} = \inf \{x \in \mathbb{R} : T(x) \geq X^{*(j)}\}.$$

(d)  $R_i(X_1^*, \dots, X_n^*) = R_i^*$  a.e.  $\forall i = 1, \dots, n$ .

(e)  $X_1, \dots, X_n$  i.i.d. implies the independence of  $R^*$  and  $X^{(j)}$  and consequently the independence of  $R^*$  and  $\tau = \tau(X^{(j)})$ .

(f)  $R_i + 1 - \sum_{j=1}^n I_{\{0\}}(R_i - R_j) \leq R_i^* \leq R_i \quad \forall i = 1, \dots, n$ .

(g) Continuity of  $F_j$  for all  $j = 1, \dots, n$  implies  $R^* = R$  a.e.

**PROOF.** (a) Independence is immediate, continuity is gotten from the strict monotonicity of  $T$  by the following conditional argument

$$\begin{aligned} 0 &\leq P(X_j^* = x) = \int P(T(y) - U_j h(y) = x) dF_j(y) \\ &= \int I_{\{0\}}(h(y)) I_{\{x\}}(T(y)) dF_j(y) = \int_{\mathbb{R}-D} I_{\{x\}}(T(y)) dF_j(y) \\ &\leq \int_{C(F_j)} I_{\{T^{-1}(x)\}}(y) dF_j(y) = 0. \end{aligned}$$

(b) For all  $x, y \in \mathbb{R}$  put  $g(x, y) = \int_0^1 I_{(-\infty, y]}(T(x) - uh(x)) du$ . Then by the

independence of  $X_j$  and  $U_j$  we have

$$F_j^*(y) = P(T(X_j) - U_j h(X_j) \leq y) = \int g(x, y) dF_j(x)$$

and furthermore

$$\begin{aligned} g(x, y) &= 1 && \text{if } T(x) \leq y, \\ &= 0 && \text{if } T(x) > y + h(x), \\ &= 1 - (T(x) - y)/h(x) && \text{if } y < T(x) \leq y + h(x), \\ &= 1 && \text{if } T(x) \leq y, \\ &= 0 && \text{if } T(x - 0) > y, \\ &= (y - T(x - 0))/(T(x) - T(x - 0)) && \text{if } T(x - 0) \leq y < T(x). \end{aligned}$$

Thus we get from the strict monotonicity of  $T$

$$\begin{aligned} F_j^*(y) &= P(T(X_j) \leq y) \\ &+ \sum_{x \in D} I_{[T(x-0), T(x)]}(y) \frac{F_j(x) - F_j(x - 0)}{T(x) - T(x - 0)} (y - T(x - 0)) \\ &= F_j(T^{-1}(y)) && \text{if } y \in T(\mathbb{R}) \\ &= F_j(x - 0) + \frac{F_j(x) - F_j(x - 0)}{T(x) - T(x - 0)} (y - T(x - 0)) \\ &&& \text{if } y \in [T(x - 0), T(x)], \quad x \in D. \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad X_i < X_j &\Rightarrow X_i^* = T(X_i) - U_i h(X_i) \leq T(X_i) < X_j + H(X_i) \\ &\leq T(X_j) - h(X_j) \leq T(X_j) - U_j h(X_j) = X_j^*. \end{aligned}$$

The second part is also an immediate consequence of the definition.

(d) On one side we have by definition

$$\begin{aligned} R_i^* &= \sum_{j=1}^n I_{(0, \infty)}(R_i - R_j) + \sum_{j=1}^n I_{(0)}(R_i - R_j) I_{[0, \infty)}(U_j - U_i) \\ &= \sum_{j=1}^n I_{(0, \infty)}(X_i - X_j) + \sum_{j=1}^n I_{(0)}(X_i - X_j) I_{[0, \infty)}(U_j - U_i), \end{aligned}$$

on the other side we get from part (c) a.e. the equality

$$\begin{aligned} R_i(X_1^*, \dots, X_n^*) &= \sum_{j=1}^n I_{[0, \infty)}(X_i^* - X_j^*) \\ &= \sum_{j=1}^n I_{(0, \infty)}(X_i - X_j) + \sum_{j=1}^n I_{[0, \infty)}(X_i^* - X_j^*) I_{(0)}(X_i - X_j) \\ &= \sum_j I_{(0, \infty)}(X_i - X_j) + \sum_j I_{[0, \infty)}((U_j - U_i)h(X_i)) I_{(0)}(X_i - X_j) \\ &= \sum_j I_{(0, \infty)}(X_i - X_j) + \sum_j I_{(0)}(X_i - X_j) I_{[0, \infty)}(U_j - U_i), \end{aligned}$$

since  $P(X_i = X_j, h(X_i) \leq 0) = 0$  if  $i \neq j$  by the independence of  $X_i$  and  $X_j$ .

(e)  $X_1, \dots, X_n$  i.i.d. implies  $X_1^*, \dots, X_n^*$  i.i.d. Therefore the independence of  $R(X_1^*, \dots, X_n^*)$  and  $X^{*(\cdot)}$  (cf. Witting and Nölle (1970)) and part (d) imply the independence of  $R^*$  and  $X^{*(\cdot)}$ . Thus part (c) proves the statement (e).

(f) From the representation of  $R_i^*$  in the proof of part (d) we have

$$\begin{aligned} \text{(5.3)} \quad R_i^* &= R_i - \sum_{j=1}^n I_{(0)}(X_i - X_j) I_{(-\infty, 0)}(U_j - U_i) \\ &\geq R_i + 1 - \sum_{j=1}^n I_{(0)}(R_i - R_j). \end{aligned}$$

(g) This is an immediate consequence of (5.3).  $\square$

LEMMA 5.2. For each  $n \in \mathbb{N}$  let  $X_n$  and  $Y_n$  be  $n$ -dimensional random vectors defined on some measure space  $(\Omega, \mathfrak{A})$ . Let  $P$  and  $Q$  be probability measures on  $(\Omega, \mathfrak{A})$  and let  $X_n$  and  $Y_n$  be independent with respect to  $P$  and  $Q$  for each  $n \in \mathbb{N}$ . Finally assume  $\mathfrak{L}(Y_n | P) = \mathfrak{L}(Y_n | Q) \forall n \in \mathbb{N}$ . Then, for each sequence  $\{f_n\}$  of Borel-measurable functions  $f_n$  from  $\mathbb{R}_{2n}$  to some  $\mathbb{R}_{k_n}$ , the following implication is true:

$$\{\mathfrak{L}(X_n | Q)\} \text{ contiguous to } \{\mathfrak{L}(X_n | P)\} \text{ implies contiguity of } \\ \{\mathfrak{L}(f_n(X_n, Y_n) | Q)\} \text{ to } \{\mathfrak{L}(f_n(X_n, Y_n) | P)\}.$$

PROOF. For each Borel-measurable subset  $B_n \subset \mathbb{R}_{k_n}$  we get:

$$P^{f_n(X_n, Y_n)}(B_n) = P(f_n(X_n, Y_n) \in B_n) = P((X_n, Y_n) \in f_n^{-1}(B_n)) \\ = E_P I_{f_n^{-1}(B_n)}(X_n, Y_n) = \int (E_P I_{f_n^{-1}(B_n)}(x, Y_n)) P^{X_n}(dx),$$

and a similar equation for  $Q$  instead of  $P$ .

Because of  $\mathfrak{L}(Y_n | P) = \mathfrak{L}(Y_n | Q)$  this implies the following chain of implications which conclude the proof:

$$P^{f_n(X_n, Y_n)}(B_n) \rightarrow 0 \implies P^{X_n}(\{x : E_P I_{f_n^{-1}(B_n)}(x, Y_n) > \varepsilon\}) \rightarrow 0 \quad \forall \varepsilon > 0 \\ \implies Q^{X_n}(\{x : E_P I_{f_n^{-1}(B_n)}(x, Y_n) > \varepsilon\}) \rightarrow 0 \quad \forall \varepsilon > 0 \\ \implies Q^{X_n}(\{x : E_Q I_{f_n^{-1}(B_n)}(x, Y_n) > \varepsilon\}) \rightarrow 0 \quad \forall \varepsilon > 0 \\ \implies \int (E_Q I_{f_n^{-1}(B_n)}(x, Y_n)) Q^{X_n}(dx) \rightarrow 0 \\ \implies Q^{f_n(X_n, Y_n)}(B_n) \rightarrow 0. \quad \square$$

LEMMA 5.3. Let  $F^*$  and  $F_j^*$  denote the distribution functions of  $X_j^*$  under  $F$  and  $F_j$ , respectively. Then, with  $\tilde{b}_n$  according to (2.4):

- (a)  $\{\mathbf{X}_1^n F_j\}$  contiguous to  $\{F^{(n)}\}$  implies contiguity of  $\{\mathbf{X}_1^n F_j^*\}$  to  $\{F^{*(n)}\}$ .
- (b)  $\int (\tilde{b}_{nF} \circ F) dF_j = \int (\tilde{b}_{nF} \circ F^*) dF_j^*$ .
- (c)  $\{\mathbf{X}_1^n F_j\}$  contiguous to  $\{F^{(n)}\}$  implies

$$(5.4) \quad nP_F(A_n) \rightarrow 0 \implies \sum_{j=1}^n P_{F_j}(A_n) \rightarrow 0,$$

and

$$(5.5) \quad \sum_{j=1}^n c_j \int ((\tilde{b}_n - \tilde{b}_{nF}) \circ F^*) dF_j^* \rightarrow 0.$$

PROOF. (a) This is an immediate consequence of Lemma 5.2 and the definition of the starred random variables.

(b) From (5.2) and (2.1) we get  $\forall x \in \mathbb{R}, \forall u \in (0, 1) : (\tilde{b}_{nF} \circ F^*)(T(x) - uh(x)) = (\tilde{b}_{nF} \circ F)(x)$ . This implies  $(\tilde{b}_{nF} \circ F^*)(X_j^*) = (\tilde{b}_{nF} \circ F)(X_j)$  and thus the equality.

$$(c) \quad nP_F(A_n) \rightarrow 0 \implies P_F^{(n)}(\mathbf{X}_1^n A_n^c) = (1 - P_F(A_n))^n \rightarrow e^0 = 1 \\ \implies \prod_1^n P_{F_j}(A_n^c) = (\mathbf{X}_1^n P_{F_j})(\mathbf{X}_1^n A_n^c) \rightarrow 1 \quad (\text{by contiguity}) \\ \implies 0 \leq \sum_1^n P_{F_j}(A_n) \leq -\log \prod_1^n (1 - P_{F_j}(A_n)) \rightarrow 0.$$

This proves (5.4). For the proof of (5.5) we get from (5.1), (5.2), and (2.1),

the following equality:

$$\begin{aligned}
 \int (\bar{b}_n \circ F^*) dF_j^* &= \int \int (\bar{b}_n \circ F^*)(T(x) - uh(x)) dF_j(x) d\lambda(u) \\
 &= \int_{\mathbb{R}^D} (\bar{b}_n \circ F^*)(T(x)) dF_j(x) \\
 &\quad + \sum_{x \in D} (F_j(x) - F_j(x - 0)) \int \bar{b}_n(F^*(T(x) - uh(x))) d\lambda(u) \\
 &= \int_{C(F_j)C(F)} (\bar{b}_n \circ F) dF_j \\
 &\quad + \sum_{x \in D(F_j)C(F)} (F_j(x) - F_j(x - 0)) \bar{b}_n(F(x)) \\
 &\quad + \sum_{x \in D(F_j)D(F)} (F_j(x) - F_j(x - 0)) \\
 &\quad \times \int \bar{b}_n(F(x - 0) + (F(x) - F(x - 0))(1 - u)) d\lambda(u) \\
 &= \int_{C(F)} (\bar{b}_n \circ F) dF_j + \int_{D(F_j)D(F)} (\bar{b}_{nF} \circ F) dF_j.
 \end{aligned}$$

Similarly we get

$$\int (\bar{b}_{nF} \circ F^*) dF_j^* = \int_{C(F)} (\bar{b}_{nF} \circ F) dF_j + \int_{D(F_j)D(F)} (\bar{b}_{nF} \circ F) dF_j.$$

Therefore it suffices to show

$$(5.6) \quad \sum_j c_j \int_{C(F)} ((\bar{b}_n - \bar{b}_{nF}) \circ F) dF_j \rightarrow 0.$$

If we define

$$(5.7) \quad S(F) \equiv \{x \in \mathbb{R} : F(x - \varepsilon) < F(x) < F(x + \varepsilon) \quad \forall \varepsilon > 0\}$$

we have

$$(5.8) \quad F(C(F)S(F)) \subset (0, 1) - \bigcup_{x \in D(F)} [F(x - 0), F(x)],$$

and

$$(5.9) \quad \lambda((0, 1)F(C(F) - S(F))) = 0.$$

Thus by (2.1)

$$\int_{C(F)S(F)} ((\bar{b}_n - \bar{b}_{nF}) \circ F) dF_j = 0,$$

and by (5.4)

$$(5.10) \quad \sum_{j=1}^n P_{F_j}(C(F) - S(F)) \rightarrow 0,$$

since

$$\begin{aligned}
 P_{F_j}(C(F) - S(F)) &= P(F^{-1}(U_1) \in C(F) - S(F)) \\
 &\leq P(U_1 \in F(\mathbb{R}), (F \circ F^{-1})(U_1) \in F(C(F) - S(F))) \\
 &= P(U_1 \in F(C(F) - S(F))) = 0 \quad \text{by (5.9)}.
 \end{aligned}$$

Therefore we get from (2.4)

$$\begin{aligned}
 &|\sum_j c_j \int_{C(F)} ((\bar{b}_n - \bar{b}_{nF}) \circ F) dF_j| \\
 &= |\sum_j c_j \int_{C(F)-S(F)} ((\bar{b}_n - \bar{b}_{nF}) \circ F) dF_j| \\
 &\leq 2 \max_j |c_j| \sup_x |\bar{b}_n(x)| \sum_j P_{F_j}(C(F) - S(F)) \rightarrow 0,
 \end{aligned}$$

which concludes proof.  $\square$

LEMMA 5.4. (a) Put  $F^{-1}(u) = \inf \{x \in \mathbb{R} : F(x) \geq u\}$ ,  $0 < u < 1$ , and  $\tilde{F}(x) = \lim_{t \downarrow 0} F(x - t) = F(x - 0)$ ,  $x \in \mathbb{R}$ . Then

$$\begin{aligned} (F \circ F^{-1})(u) &= u && \text{if } u \in F(\mathbb{R}) - F(D(F)), \\ &= F(x) && \text{if } u \in (F(x - 0), F(x)], \quad x \in D(F), \\ &= F(x) && \text{if } u = F(x - 0) \notin F(\mathbb{R}), \quad x \in D(F), \\ (\tilde{F} \circ F^{-1})(u) &= u && \text{if } u \in F(\mathbb{R}) - F(D(F)), \\ &= F(x - 0) && \text{if } u \in (F(x - 0), F(x)], \quad x \in D(F), \\ &= F(x - 0) && \text{if } u = F(x - 0) \notin F(\mathbb{R}), \quad x \in D(F). \end{aligned}$$

(b) Given  $u \in C(F \circ F^{-1}) \cap C(\tilde{F} \circ F^{-1})$  and  $j_n \in \{1, \dots, n\}$  such that  $j_n/n \rightarrow u$ . Then

$$\begin{aligned} F(X_n^{(j_n)}) &\rightarrow (F \circ F^{-1})(u) \quad \text{in } F^{(n)}\text{-probability,} \\ \tilde{F}(X_n^{(j_n)}) &\rightarrow (\tilde{F} \circ F^{-1})(u) \quad \text{in } F^{(n)}\text{-probability.} \end{aligned}$$

(c) For each  $u \in (0, 1)$  define random variables  $s_{nu}$  and  $t_{nu}$  according to

$$s_{nu} = T_{nk-1}, \quad t_{nu} = T_{nk}, \quad \text{if } T_{nk-1} \leq nu < T_{nk}, \quad k = 1, \dots, g_n.$$

Then, for each  $u \in C(F \circ F^{-1}) \cap C(\tilde{F} \circ F^{-1}) \cap (F(\mathbb{R}) - F(D(F)))$ ,

$$\begin{aligned} n^{-1}s_{nu} &\rightarrow u \quad \text{in } F^{(n)}\text{-probability,} \\ n^{-1}t_{nu} &\rightarrow u \quad \text{in } F^{(n)}\text{-probability,} \end{aligned}$$

and for each  $u \in (F(x - 0), F(x))$ ,  $x \in D(F)$ ,

$$\begin{aligned} n^{-1}s_{nu} &\rightarrow F(x - 0) \quad \text{in } F^{(n)}\text{-probability,} \\ n^{-1}t_{nu} &\rightarrow F(x) \quad \text{in } F^{(n)}\text{-probability.} \end{aligned}$$

(d) Define  $\bar{b}_n(\cdot, \tau_n) : (0, 1) \rightarrow \mathbb{R}$  according to

$$\bar{b}_n(u, \tau_n) = n\tau_n^{-1} \int_{T_{nk-1}/n}^{T_{nk}/n} b \, d\lambda \quad \text{if } T_{nk-1} \leq nu < T_{nk}, \quad k = 1, \dots, g_n.$$

Then, for  $\lambda$ -almost all  $u \in (0, 1)$ ,

$$\bar{b}_n(u, \tau_n) \rightarrow b_F(u) \quad \text{in } F^{(n)}\text{-probability.}$$

PROOF. (a) Obvious.

(b) Immediate from  $\mathfrak{L}(U_n^{(j_n)}) \rightarrow \mathfrak{L}(u)$  and  $\mathfrak{L}(F^{-1}(U_n^{(j_n)})) = \mathfrak{L}(X_n^{(j_n)} | F^{(n)})$ .

(c) From the definition of  $s_{nu}$  and  $t_{nu}$  we get

$$s_{nu} < [nu + 1] \leq t_{nu}, \quad X_n^{(t_{nu})} = X_n^{([nu+1])}$$

and thus

$$t_{nu} = \sum_{i=1}^n I_{(-\infty, X_n^{([nu+1])}]}(X_i), \quad s_{nu} = \sum_{i=1}^n I_{(-\infty, X_n^{([nu+1])})}(X_i).$$

Therefore by Glivenko-Cantelli

$$\begin{aligned} t_{nu}/n - F(X_n^{([nu+1])}) &\rightarrow 0 \quad \text{in } F^{(n)}\text{-probability,} \\ s_{nu}/n - \tilde{F}(X_n^{([nu+1])}) &\rightarrow 0 \quad \text{in } F^{(n)}\text{-probability.} \end{aligned}$$

Application of part (a) and part (b) concludes the proof of part (c).

(d) Let  $L(b)$  denote the Lebesgue set of  $b$  (e.g. Rudin (1970), page 158) and put

$$A = L(b) \cap C(F \circ F^{-1}) \cap C(\tilde{F} \circ F^{-1}) \cap (F(\mathbb{R}) - F(D(F))).$$

From part (c) and (2.1) we get for each  $u \in A$

$$\bar{b}_n(u, \tau_n) = (t_{nu} - s_{nu})^{-1} n \int_{s_{nu}/n}^{t_{nu}/n} b \, d\lambda \rightarrow b(u) = b_F(u) \quad \text{in } F^{(n)}\text{-probability}$$

and for each  $u \in (F(x - 0), F(x))$ ,  $x \in D(F)$ ,

$$\bar{b}_n(u, \tau_n) \rightarrow (F(x) - F(x - 0))^{-1} \int_{F(x-0)}^{F(x)} b \, d\lambda = b_F(u) \quad \text{in } F^{(n)}\text{-probability.}$$

Because of  $\lambda(L(b)) = 1$  and because of the monotonicity of  $F \circ F^{-1}$  and  $\tilde{F} \circ F^{-1}$  we have

$$\lambda(A + \sum_{x \in D(F)} (F(x - 0), F(x))) = 1,$$

which concludes the proof.  $\square$

**PROOF OF THEOREM 2.1.** Given  $F$  and  $F_j$  define  $F^*$ ,  $F_j^*$ , and  $X_j^*$  according to (5.1) and (5.2). Putting

$$T_{b_F}^* = \sum_1^n c_i b \circ F^* \circ X_i^*$$

we have under the assumptions of part (a):

$$ET_{b_F}^* = \sum c_i \int b \, d\lambda = 0, \quad \text{Var } T_{b_F}^* = \sum c_i^2 \int b^2 \, d\lambda = 1.$$

Thus  $\mathfrak{L}(T_{b_F}^*) \rightarrow N(0, 1)$  by Lindeberg–Feller theorem, since  $\max_i c_i^2 \rightarrow 0$  by assumption. Furthermore we get from Hájek and Šidák (1967), Theorem V.1.4.a and formula (10) in the proof of Theorem V.1.5.a, the statement

$$(5.11) \quad E(S_b^* - T_{b_F}^*)^2 \rightarrow 0.$$

For the proof of part (b) this and the contiguity assumption imply (cf. Lemma 5.3. (a))

$$(5.12) \quad S_b^* - T_{b_F}^* \rightarrow 0 \quad \text{in } \times_1^n F_j^*\text{-probability.}$$

Therefore and because of Lemma 5.3. (b), (c), it suffices to prove

$$\mathfrak{L}(T_{b_F}^* - \sum_j c_j \int (\bar{b}_n \circ F^*) \, dF_j^*) \rightarrow N(0, 1).$$

Because of Lemma 5.3. (a), (1.3), (1.1), and (2.4) this is an immediate consequence of the Theorem and Remark 1 of Behnen and Neuhaus (1975).

(c) From part (a) we have that  $\{\varphi_b^*\}$  is asymptotically level  $\alpha$  for  $H_0$ . Also  $K_{b_F}$  is well defined since (2.4) implies  $1 + dc_j(\bar{b}_{nF} \circ F)$  to be an  $F$ -density for  $d$  small enough (or  $n$  large enough):

$$\begin{aligned} \int (\bar{b}_{nF} \circ F) \, dF &= \int (\bar{b}_{nF} \circ F \circ F^{-1}) \, d\lambda \\ &= \int_{F(\mathbb{R})} \bar{b}_{nF} \, d\lambda + \sum_{x \in D(F)} \bar{b}_{nF}(F(x))(F(x) - F(x - 0)) \\ &= \int_{F(\mathbb{R})} \bar{b}_n \, d\lambda + \sum_{x \in D(F)} \int_{(F(x-0), F(x))} \bar{b}_n \, d\lambda = \int \bar{b}_n \, d\lambda = 0. \end{aligned}$$

First we get by  $\sum_j c_j^2 \bar{b}_{nF}^2(F^*(X_j^*)) \rightarrow \int b_F^2 \, d\lambda = \int b^2 \, d\lambda = 1$  in  $F^{*(n)}$ -probability

and by Taylor-expansion

$\log \prod_j (1 + dc_j \check{b}_{nF} \circ F^* \circ X_j^*) + d^2/2 - dT_{\check{b}_{nF}}^* \rightarrow 0$  in  $F^{*(n)}$ -probability, since  $\int (\check{b}_{nF} - b)^2 d\lambda \rightarrow 0$ . Secondly we have by (5.2) and (2.1) (cf. proof of Lemma 5.3. (b))

$$\check{b}_{nF} \circ F^* \circ X_j^* = \check{b}_{nF} \circ F \circ X_j.$$

This implies (cf. (5.11))

$$\log \prod_j (1 + dc_j \check{b}_{nF} \circ F \circ X_j) + d^2/2 - dS_b^* \rightarrow 0 \text{ in } F^{*(n)}\text{-probability.}$$

Thus, part (a) and Hájek and Šidák (1967), Corollary VI.1.2, imply contiguity of  $K_{b_F}$  to  $\{F^{(n)}\}$ . Therefore we get from part (a) and part (b) asymptotic power-equivalence of  $\{\varphi_b^*\}$  and Neyman–Pearson test for  $\{F^{(n)}\}$  against  $\mathcal{G}$  for each  $\vartheta \in K_{b_F}$ , which concludes the proof.  $\square$

**PROOF OF THEOREM 2.2.** Let  $\{F^{(n)}\} \in H_0$  be given. Then we get from Theorem 2.1 under the assumption of part (a):

$$\mathfrak{L}(S_{b_F}^*) \rightarrow N(0, \int b_F^2 d\lambda);$$

and under the assumptions of part (b):

$$\mathfrak{L}(S_{b_F}^* - \delta_n(b_F, \{\mathbf{X}_1^n F_j\})) \rightarrow N(0, \int b_F^2 d\lambda);$$

and finally under the assumptions of part (c):

$$\{I_{\{S_{b_F}^* > k_n(\int b_F^2 d\lambda)^{1/2}\}}\}$$

is asymptotically most powerful for  $\{F^{(n)}\}$  against  $K_{b_F}$  at level  $\alpha$ , since the monotonicity of  $b$  implies  $\int b_F^2 d\lambda > 0$  for all  $F$  not being one-point measures. Since we may assume  $\check{b}_{nF} = \check{b}_{nF}$ , which implies for  $\delta_n(\cdot, \cdot)$  according to (2.3)

$$\delta_n(b_F, \{\mathbf{X}_1^n F_j\}) = \delta_n(b, \{\mathbf{X}_1^n F_j\}),$$

and since we have from Hájek (1969), page 130, under  $H_0$

$$\text{Var}(\bar{S}_b | \tau) = \frac{1}{n-1} \sum_{i=1}^n (\bar{b}_i)^2,$$

it suffices (by contiguity) to prove (for each  $\{F^{(n)}\} \in H_0$ )

$$(5.13) \quad \bar{S}_b - S_{b_F}^* \rightarrow 0 \text{ in } F^{(n)}\text{-probability,}$$

and

$$(5.14) \quad \frac{1}{n-1} \sum_{i=1}^n (\bar{b}_i)^2 \rightarrow \int b_F^2 d\lambda \text{ in } F^{(n)}\text{-probability.}$$

For the proof we state

$$\begin{aligned} E(\bar{S}_b - S_{b_F}^*)^2 &= \sum_i c_i^2 E(b_{R_i}(\tau) - b_{FR_i^*})^2 \\ &\quad + \sum_{i \neq j} c_i c_j E(b_{R_i}(\tau) - b_{FR_i^*})(b_{R_j}(\tau) - b_{FR_j^*}). \end{aligned}$$



Then we get by Lemma 5.1 and (5.3) on one hand

$$\begin{aligned} E((b_{R_i}(\tau) - b_{FR_i^*})^2 | \tau) &= \frac{1}{n} \sum_i (b_i(\tau) - b_{Fi})^2 \\ &= \int (b_{[nu+1]}(\tau) - b_{F[nu+1]})^2 d\lambda(u), \end{aligned}$$

and on the other hand if  $i \neq j$

$$\begin{aligned} E((b_{R_i}(\tau) - b_{FR_i^*})(b_{R_j}(\tau) - b_{FR_j^*}) | \tau) \\ &= \frac{1}{n(n-1)} \sum_{i \neq j} (b_i(\tau) - b_{Fi})(b_j(\tau) - b_{Fj}) \\ &= -\frac{1}{n-1} \int (b_{[nu+1]}(\tau) - b_{F[nu+1]})^2 d\lambda(u). \end{aligned}$$

Because of  $\sum_{i \neq j} c_i c_j = (\sum_i c_i)^2 - \sum_i c_i^2 = -1$  we have

$$\begin{aligned} E(\bar{S}_b - S_{\bar{b}_F}^*) &= \frac{n}{n-1} E \int (b_{[nu+1]}(\tau) - b_{F[nu+1]})^2 d\lambda(u) \\ &\leq 2 \frac{n}{n-1} (E \int (b_{[nu+1]}(\tau) - b_F(u))^2 d\lambda(u) \\ &\quad + \int (b_{F[nu+1]} - b_F(u))^2 d\lambda(u)). \end{aligned}$$

But  $\int (b_F(u) - b_{F[nu+1]})^2 d\lambda \rightarrow 0$  by assumption. Because of  $1/(n-1) \sum_i (\bar{b}_i)^2 = n/(n-1) \int b_{[nu+1]}^2(\tau) d\lambda(u)$  it is therefore sufficient for the proof of (5.13) and (5.14) to prove

$$(5.15) \quad E \int (b_{[nu+1]}(\tau) - b_F(u))^2 d\lambda(u) \rightarrow 0.$$

For this proof let  $\bar{b}_n(\cdot, \tau)$  denote the (random) function defined in Lemma 5.4 (d). Then we have

$$\begin{aligned} E \int (b_{[nu+1]}(\tau) - b_F(u))^2 d\lambda(u) \\ \leq 2E \int (b_{[nu+1]}(\tau) - \bar{b}_n(u, \tau))^2 d\lambda(u) + 2E \int (\bar{b}_n(u, \tau) - b_F(u))^2 d\lambda(u), \end{aligned}$$

and

$$\begin{aligned} E \int (b_{[nu+1]}(\tau) - \bar{b}_n(u, \tau))^2 d\lambda(u) \\ &= E \sum_{j=1}^g \int_{T_{j-1}^{j/n}}^{T_j^{j/n}} (\dots)^2 d\lambda \\ &= E \sum_{j=1}^g \left(\frac{n}{\tau_j}\right)^2 \left(\int_{T_{j-1}^{j/n}}^{T_j^{j/n}} (b_{[nv+1]} - b(v)) d\lambda(v)\right)^2 \int_{T_{j-1}^{j/n}}^{T_j^{j/n}} d\lambda(u) \\ &\leq E \sum_{j=1}^g \int_{T_{j-1}^{j/n}}^{T_j^{j/n}} (b_{[nv+1]} - b(v))^2 d\lambda(v) \\ &= \int (b_{[nv+1]} - b(v))^2 d\lambda(v) \rightarrow 0 \quad \text{by assumption (1.2)}. \end{aligned}$$

For the other term we get from Lemma 5.4 (d)

$$\bar{b}_n(U_1, \tau) \rightarrow b_F(U_1) \quad \text{in } \lambda \times F^{(n)}\text{-probability,}$$

and thus

$$(\lambda \times P_F^{(n)})(A_{nk}) \rightarrow 0 \quad \forall k \in \mathbb{N},$$

if we put

$$A_{nk} = \{(u, x) \in (0, 1) \times \mathbb{R}_n : (\bar{b}_n(u, \tau(x)) - b_F(u))^2 > k\}.$$

Therefore we get by the dominated convergence theorem and from the definition of  $\bar{b}_n(\cdot, \tau)$  for each  $k \in \mathbb{N}$ :

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} E \int (\bar{b}_n(u, \tau) - b_F(u))^2 d\lambda(u) \\
 &= \limsup_{n \rightarrow \infty} E(\bar{b}_n(U_1, \tau) - b_F(U_1))^2 \\
 &= \limsup_{n \rightarrow \infty} E(\bar{b}_n(U_1, \tau) - b_F(U_1))^2 I_{A_{nk}}(U_1, X) \\
 &\leq 2 \limsup_{n \rightarrow \infty} E \bar{b}_n^2(U_1, \tau) I_{A_{nk}}(U_1, X) \\
 &\leq 2 \limsup_n E \left( \sum_{j=1}^g I_{[T_{j-1}, T_j)}(nU_1) \left( \frac{n}{\tau_j} \right) \left( \int_{T_{j-1}^{j/n}} b^2 d\lambda \right) I_{A_{nk}}(U_1, X) \right) \\
 &\leq 2 \limsup_n E \left( \sum_{j=1}^g I_{[T_{j-1}, T_j)}(nU_1) k I_{A_{nk}}(U_1, X) \right) \\
 &\quad + 2 \limsup_n E \left( \sum_{j=1}^g I_{[T_{j-1}, T_j)}(nU_1) \left( \frac{n}{\tau_j} \right) \int_{T_{j-1}^{j/n}} b^2 I_{(k, \infty)}(b^2) d\lambda \right) \\
 &\leq 2k \limsup_n (\lambda \times P_F^{(n)})(A_{nk}) + 2 \limsup_n E \left( \sum_{j=1}^g \int_{T_{j-1}^{j/n}} b^2 I_{(k, \infty)}(b^2) d\lambda \right) \\
 &= 2 \int b^2 I_{(k, \infty)}(b^2) d\lambda.
 \end{aligned}$$

The last expression converges to zero as  $k$  tends to infinity, which concludes the proof.  $\square$

PROOF OF COROLLARY 2.3. This is obvious from Theorem 2.1 and Theorem 2.2, since for  $\delta_n(b, \mathcal{D}(d))$  according to (2.3) we get (similar to the proof of Lemma 5.3. (b))

$$\begin{aligned}
 \delta_n(b, \mathcal{D}(d)) &= \sum_i c_i \int (\bar{b}_{nF} \circ F)(1 + dc_i(\bar{h}_{nF} \circ F)) dF \\
 &= d \int \bar{b}_{nF} \bar{h}_{nF} d\lambda \rightarrow d \int b_F h_F d\lambda. \quad \square
 \end{aligned}$$

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