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INVARIANT TESTS FOR UNIFORMITY ON COMPACT RIEMANNIAN MANIFOLDS BASED ON SOBOLEV NORMS[1]

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Several invariant tests for uniformity of a distribution on the circle, the sphere and the hemisphere have been proposed by Rayleigh, Watson, Bingham, Ajne, Beran and others. In this paper a class of invariant tests for uniformity on compact Riemannian manifolds containing many of the known ones is presented and studied (the asymptotic theory as well as some local optimality properties for this class of tests are given). The examples include two new tests, one for the sphere and the other for the hemisphere. Let X be a compact Riemannian manifold, μ the normalized volume element (the *uniform distribution* of X) and ν_n the empirical distribution corresponding to a sequence of i.i.d. X-valued random variables. The statistics in which these tests are based are just convergent weighted sums of the squares of the Fourier coefficients of $\nu_n(\omega) - \mu$ with respect to any orthonormal basis of $L_2(X, \mu)$ consisting of eigenfunctions of the Laplacian. An additional condition is imposed on the weights, namely that weights corresponding to coefficients of eigenfunctions in the same eigenspace of the Laplacian be equal (this condition is essential for the invariance of the tests). These statistics are related to *Sobolev norms* and so, the tests are called *Sobolev tests*. In connection with Sobolev statistics, it is interesting to note that the Sobolev norms of index $-s$, $s > (\dim X)/2$, metrize the weak-star topology of $\mathscr{P}(X)$, the space of Borel probability measures on X. A theorem about weak convergence of empirical distributions on compact manifolds, useful in proving some of the asymptotic results for Sobolev statistics, is also included. One of the sections (Section 2) is almost entirely devoted to give a short review of the facts needed in the paper about Riemannian manifolds, the Laplacian and Sobolev spaces.

1. Introduction. The *uniform distribution* on a compact Riemannian manifold is defined as the Borel measure which extends the normalized volume element of the space (for the definition of the volume element see e.g. Helgason (1962) page 291). Since the volume element is invariant under isometries, so is the uniform distribution. Then, it is natural to impose the same kind of invariance on any procedure for deciding whether a given distribution is the uniform one or is not. In this paper we study a class of invariant tests for uniformity on

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compact Riemannian manifolds based on invariant norms that metrize the weak-star topology on their spaces of Borel probability measures.

Let X be a compact Riemannian manifold and μ the uniform distribution on X. The tests considered here consist in rejecting the hypothesis of uniformity of a given distribution on X for large values of statistics of the fomr

$$(1.1)\qquad T_n^{(s)}(\{\alpha_k\})(\omega) = n\|(\sum \alpha_k \sigma_k^{s/2}\pi_k)(\nu_n(\omega) - \mu)\|_{-s}^2$$

where $\nu_n(\omega)$ is the nth empirical distribution, π_k is the orthogonal projection of $L_2(X, \mu)$ onto the kth eigenspace of the Laplace–Beltrami operator (Laplacian) of X acting on the space of Schwartz distributions by duality, σ_k is the sequence of eigenvalues of the Laplacian Δ in increasing order, $\{\alpha_k\}$ is a sequence of real numbers such that $\sup |\alpha_k \sigma_k^{s/2}| < \infty$ (this condition can be weakened) and $\|\cdot\|_{-s}$ is a Sobolev norm of negative index $-s < -(\dim X)/2$ defined through Δ ((2.5)). The statistics of type (1.1) will be called *Sobolev statistics* as they are based on Sobolev norms, and the tests they induce, *Sobolev tests*. Let $\{f_i\}$ be an orthonormal basis of $L_2(X, \mu)$ consisting of eigenfunctions of Δ and let E_k be the eigenspace of Δ of eigenvalue σ_k, $k = 0, 1, \cdots$ ($\sigma_0 = 0$). Then, (1.1) is in fact

$$(1.2)\qquad T_n^{(s)}(\{\alpha_k\})(\omega) = n \sum_{k=1}^\infty \alpha_k^2 \sum_{f_i \in E_k} [\int_X f_i d(\nu_n(\omega) - \mu)]^2 ,$$

i.e., a weighted sum of squares of Fourier coefficients of $\nu_n(\omega) - \mu$ with respect to the orthonormal system $\{f_i\}$ with weights depending only on the eigenspaces (see Section 2). The condition $\sup |\alpha_k \sigma_k^{s/2}| < \infty$ for some $s > (\dim X)/2$ ensures the convergence of the series (1.2) regardless of the underlying distribution of the observations defining ν_n.

Section 2 contains some reviewing on Riemannian manifolds, the Laplacian and Sobolev norms. Nothing is proved except for a lemma on Sobolev spaces which we believe known but know of no reference for it. This lemma (Lemma 2.1) is used in showing that the norm $\|\cdot\|_{-s}$, for $s > (\dim X)/2$, metrizes the weak-star topology of the space $\mathscr{P}(X)$ of Borel probability measures on X. Clearly, this fact provides some ground for studying statistics of type (1.1).

Let ν be any probability measure on X and $\{\nu_n\}_{n=1}^\infty$ the empirical distributions associated with ν. Define processes on $L_2(X, \nu)$ by $Z_n^{(\nu)}(\omega, f) = n^{\frac{1}{2}} \int_X f d(\nu_n(\omega) - \nu)$, $n = 1, \cdots$, and let $Z^{(\nu)}$ be the centered Gaussian process with the same covariance as $Z_n^{(\nu)}$. Let B_s be the unit ball of the Sobolev space $H_s(X)$, $s > (\dim X)/2$; then B_s is a compact subset of $C(X)$. In Section 3 we prove that the probability measures $\mathscr{L}(Z_n^{(\nu)} \mid B_s)$ induced by $Z_n^{(\nu)}$ on $C(B_s)$ converge weakly to the one induced by $Z^{(\nu)}$, $\mathscr{L}(Z^{(\nu)} \mid B_s)$.

Using the result of Section 3 we obtain the limiting distribution of $T_n^{(s)}$ under the null hypothesis (Theorem 4.1). The consistency properties of Sobolev tests are given in Theorem 4.4. The asymptotic theory of these tests is completed with the limiting distribution of Sobolev statistics under any alternative (Theorem 4.7), obtained by a method adapted from Beran (1969). These results, which form Section 4, indicate that the behavior of a Sobolev statistic for large samples distinguishes very clearly between the null hypothesis case (Theorem 4.1, Lemma 4.3) and the case of any alternative against which the test is consistent.

If the manifold is homogeneous, Sobolev statistics can be expanded in terms of only *zonal functions* (Proposition 5.2) and so they become effectively computable in many cases. Another consequence of this fact is that these statistics can be related to those of Beran (1968); such a relation implies that Sobolev tests satisfy some local optimality properties (Theorems 5.3 and 5.4).

Finally, Section 6 is devoted to the important cases of the circle, the sphere and the projective plane (hemisphere). There is no new result here for the circle; it is just seen that several important invariant tests are Sobolev. For the sphere and the projective plane we give two new invariant tests for uniformity computable with $O(n^2)$ operations, n being the sample size (Propositions 6.3 and 6.4). Moreover, a procedure is outlined for constructing invariant tests for uniformity on the sphere and the projective plane computable with only $O(n)$ operations, consistent against prescribed alternatives and with asymptotic approximations to their power available. This procedure is illustrated with two examples: Rayleigh's and Bingham's tests. Some properties for this last test are also deduced.

About notation, perhaps the following should be remarked. If $\{\nu_n\}$ is a sequence of probability measures on a metric space X converging weakly to a probability ν (i.e., converging to ν in the weak-star topology of $C'(X)$), we will write $w^* - \lim \nu_n = \nu$. If ν is a measure on X then $(\ ,\ )_\nu$ and $\|\cdot\|_\nu$ will denote respectively the inner product and the norm of $L_2(X, \nu)$.

2. Riemannian manifolds. The Laplacian. Sobolev norms. A Riemannian manifold is a differentiable manifold X such that the tangent space M_x at each point $x \in X$ has a strictly positive definite inner product g_x defined on it which is smooth in the sense that the mapping $x \to g_x(\mathbf{v}(x), \mathbf{w}(x))$ is C^∞ for every pair of C^∞ vector fields $\mathbf{v}$ and $\mathbf{w}$ (Helgason (1962) page 44 ff.). Given a smooth curve $\gamma(t) \subset X$, $t \in [a, b]$, if $\dot\gamma(t)$ denotes the tangent to $\gamma(t)$, the length of γ is defined to be

$$L(\gamma) = \int_a^b g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))^{\frac{1}{2}}\, dt\,.$$

If moreover X is connected, the function

$$d(x, y) = \inf\{L(\gamma) : \gamma \text{ joints } x \text{ and } y\}$$

is a distance on X, i.e. (X, d) is a metric space, and d is called the Riemannian metric of X (Helgason (1962) pages 50–51). If a diffeomorphism of X onto X preserves the inner product of X, it is called an *isometry* of X. A mapping of X onto X is an isometry in this sense if and only if it is an isometry of the metric space (X, d) onto itself (Helgason (1962) pages 60–61).

The *volume* of any set $A \subset X$ with piecewise smooth boundary can be defined in the following way: if the set A is contained in a coordinate patch with coordinates, say, $x_1, \cdots, x_n$, $n = \dim X$, and if we define $\varphi(x) = (x_1(x), \cdots, x_n(x)) \in \mathbb{R}^n$

and $g_{ij}(x) = g_x(\partial/\partial x_i, \partial/\partial x_j)$, then the volume of A is $\int_{\varphi(A)} |\det(g_{ij}(\varphi^{-1}(p)))|^{\frac{1}{2}}\, dp$ (usual Riemann integration on $\mathbb{R}^n$); if A intersects several coordinate patches (we may assume them to be only a finite number because X is compact), decompose A into a disjoint union of sets with piecewise smooth boundaries, each contained in a coordinate patch, and define the volume of A additively. This definition is independent of the set of coordinate systems used (a change of coordinates has the effect of a change of variables in a multiple integral). Now, the volume can be uniquely extended to a Borel measure on X. This Borel measure, normalized, is the probability measure that we call the *uniform distribution* on X or the *normalized volume element* of X, μ. From its definition it is clear that the uniform distribution μ is *invariant* under isometries. Moreover, if there are enough isometries on X—for instance, if X is homogeneous— it is the only invariant Borel probability.

Three compact Riemannian manifolds are already in use in statistics: the circle, the sphere and the projective plane or hemisphere. These spaces are the setting for the statistics of directional data (Mardia (1972) and the references there). Beran (1968) gave tests for uniformity on compact homogeneous spaces and many of such spaces are Riemannian manifolds too; this work is connected with his (see Section 5). Compact Lie groups which might have some interest in statistics are the tori T^n, T being the circle, and some groups of matrices like the unitary and the orthogonal groups. Among the potentially interesting—from the statistical point of view—homogeneous compact Riemannian manifolds other than the sphere and the hemisphere, perhaps we should mention the Grassmann manifolds $M_k(\mathbb{R}^n)$ (= the set of linear subspaces of dimension k of $\mathbb{R}^n$). Smooth compact surfaces (without boundary) of $\mathbb{R}^n$ are also compact Riemannian manifolds. For these and other examples see Warner (1971).

The Laplacian (Laplace–Beltrami operator) of a Riemannian manifold X is the differential operator defined, in coordinates, by the formula

$$\Delta = -\bar{g}^{-\frac{1}{2}} \sum_j \partial/\partial x_j \sum_i g^{ij}\bar{g}^{\frac{1}{2}}\, \partial/\partial x_i$$

where $\bar{g} = |\det(g_{ij})|$ and $(g^{ij}) = (g_{ij})^{-1}$ (Helgason (1962) page 387). We want to remark the following properties of the Laplacian on a compact Riemannian manifold X: a) Δ is an elliptic second order operator and so its eigenfunctions are in $C^\infty(X)$; b) Δ is self adjoint; c) Δ is invariant under isometries (given an isometry Φ of X, $\Phi(\Delta)$ is defined as $\Phi(\Delta)(f)(x) = \Delta(f \circ \Phi)(\Phi^{-1}(x)))$; d) the eigenspaces of Δ (acting on $C^2(X)$) are finite dimensional; e) if μ is the normalised volume element of X, eigenspaces corresponding to different eigenvalues are orthogonal in $L_2(X, \mu)$ and their orthogonal sum is dense in $L_2(X, \mu)$; and f) the eigenvalues of Δ are all greater than or equal to zero and have no limit point. (References: Helgason (1962) pages 387–388 and Warner (1971) pages 254–256.) These properties show that the eigenfunctions and the eigenspaces of Δ are very natural objects to consider when treating invariance questions.

We now give a brief description of Sobolev spaces. For any real s, the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm given by the bilinear form

$$(u, v)_s = \int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}(1 + |\xi|^2)^s \, d\xi ,$$

where $\hat{u}$ denotes the Fourier transform of u, is called the Sobolev space of order s of $\mathbb{R}^n$, $H_s(\mathbb{R}^n)$. These spaces are in fact spaces of tempered distributions, $H_0(\mathbb{R}^n) = L_2(\mathbb{R}^n)$ and $H_s \subset H_{s'}$ for $s > s'$ (Yoshida (1965) pages 55, 57–59 and 155). One of the main properties of Sobolev spaces is Sobolev's lemma: given $k \in \mathbb{Z}_+ \cup \{0\}$, if $s > k + n/2$ then every distribution in $H_s(\mathbb{R}^n)$ is defined by a function in $C^k(\mathbb{R}^n)$ and the injection of H_s into C^k is continuous (Yosida (1965) pages 174–175). We will see below a complement to this lemma. Let now X be a compact manifold of dimension n; if $\{U_i\}$ is a finite covering of X by coordinate patches and $\{\rho_i\}$ is a partition of unity subordinated to this covering (Helgason (1962) page 8), then the Sobolev spaces of X are the Hilbert spaces of distributions given by

$$\text{(2.1)} \qquad \begin{aligned} H_s(X) &= \{u \in \mathscr{D}'(X) : \rho_i u \in H_s(\mathbb{R}^n) \ \text{ for each } \ i\} \\ (u, v)_s &= \sum (\rho_i u, \rho_i v)_s \, . \end{aligned}$$

This definition is independent of the covering and the partition of unity (Nirenberg (1970) pages 150 and 157–158).

For $\mathbb{R}^n$ Sobolev spaces are defined by summability conditions on the Fourier transform of functions (and distributions). The same is true for compact manifolds if the Fourier coefficients are suitably defined. This is consequence of the following fact: if A is an invertible elliptic operator of order m, then the formula

$$\text{(2.2)} \qquad (u, v)'_{sm} = (A^s u, A^s v)_\mu$$

where $(\ ,\ )_\mu$ is the inner product of $L_2(X, \mu)$ and the real power of the operator A is defined as in Seeley (1968), defines another Hilbert space structure on $H_{sm}(X)$ and there exist constants c_1 and c_2 such that $c_1||u||_{sm} \leqq ||u||'_{sm} \leqq c_2||u||_{sm}$ (Seeley (1968) page 301 and Nirenberg (1970) pages 157–158). Let $\{\sigma_k\}_{k=0}^\infty$ be the set of different eigenvalues of Δ in increasing order, $\{E_k\}_{k=0}^\infty$ the set of eigenspaces (E_k corresponds to σ_k for each k) and $\{f_i\}_{i=0}^\infty$ an orthonormal basis of $L_2(X, \mu)$ consisting of real eigenfunctions of Δ. If we take $A = \Delta + E$ in (2.2), E being the expectation operator with respect to μ, we have

$$\text{(2.3)} \qquad \begin{aligned} (u, v)_s' &= ((\Delta + E)^{s/2}u, (\Delta + E)^{s/2}v)_\mu \\ &= u(1)\overline{v(1)} + \textstyle\sum_{k=1}^\infty \sigma_k^s \sum_{f_i \in E_k} u(f_i)\overline{v(f_i)} , \end{aligned}$$

where $u(f_i)$ is understood in the distributional sense, in particular, if u is a function, $u(f_i) = \int_X u f_i \, d\mu$. Then, for $s > 0$, $f \in L_2(X, \mu)$ is in $H_s(X)$ if and only if

$$\text{(2.4)} \qquad (||f||_s')^2 = (Ef)^2 + \textstyle\sum_{k=1}^\infty \sigma_k^s \sum_{f_i \in E_k} (f, f_i)_\mu^2 < \infty ,$$

and a distribution $T \in \mathscr{D}'(X)$ is in $H_{-s}(X)$ if and only if

$$\text{(2.5)} \qquad (||T||'_{-s})^2 = (T1)^2 + \textstyle\sum_{k=1}^\infty \sigma_k^{-s} \sum_{f_i \in E_k} (Tf_i)^2 < \infty .$$

The norm (2.5) is, for every fixed s, dual to the norm (2.4), i.e. $\|T\|'_{-s} = \sup\{Tf : f \in H_s(X), \|f\|_s' \leqq 1\}$. As a consequence of the invariance property of Δ, the norms (2.4) and (2.5) are invariant under isometries. The Sobolev norms that we are going to use are precisely the norms (2.5). Let us note that the local character of the Sobolev lemma ensures its validity for manifolds. Therefore, if $s > (\dim X)/2$ then $\mathscr{P}(X) \subset H_{-s}(X)$ ($\mathscr{P}(X)$ denotes the set of Borel probability measures on X). On the other hand no discrete probability belongs to $H_{-s}(X)$ for $s \leqq (\dim X)/2$ because the same is true for $\mathbb{R}^n$: $\hat{\delta}_x(\xi) = \exp(i\xi x)$ and this function is square integrable with respect to the measure $(1 + |\xi|^2)^s\, d\xi$ if and only if $s < -n/2$. So, $\mathscr{P}(X)$ is in $H_{-s}(X)$ if and only if $s > (\dim X)/2$.

In order to prove that Sobolev norms of order $-s$, $s > (\dim X)/2$, metrize the weak-star topology of $\mathscr{P}(X)$, we need the following lemma.

LEMMA 2.1. *Let X be a compact Riemannian manifold and d its Riemannian metric. Then, for every $s \in ((\dim X)/2, (\dim X)/2 + 1]$ and $\alpha \in (0, s - (\dim X)/2)$, there exists a constant $c_{s,\alpha} > 0$ such that*

$$|f(x) - f(y)| \leqq c_{s,\alpha}\|f\|_s[d(x, y)]^\alpha \tag{2.6}$$

for every $f \in H_s(X)$, where $H_s(X)$ is taken as a subset of $C(X)$.

PROOF. If $\dim X = n$, by definition (2.1) we only need to prove (2.6) for $\mathbb{R}^n$ (the constant, of course, will not be necessarily the same). In this case, if $f \in H_s(\mathbb{R}^n)$ and $s > n/2$,

$$\begin{aligned}
&|h|^{-\alpha}|f(x + h) - f(x)| \\
&\quad = (2\pi)^{-n/2}\left|\int_{\mathbb{R}^n} |h|^{-\alpha}(\exp(-i\langle h, t\rangle) - 1)\exp(-i\langle x, t\rangle)\hat{f}(t)\, dt\right| \\
&\quad = (2\pi)^{-n/2}\left|\int_{\mathbb{R}^n} |h|^{-\alpha}|t|^{-\alpha}(\exp(-i\langle h, t\rangle) - 1)|t|^\alpha \exp(-i\langle x, t\rangle)\hat{f}(t)\, dt\right| \\
&\quad \leqq 2(2\pi)^{-n/2}\int_{\mathbb{R}^n} |t|^\alpha|\hat{f}(t)|\, dt \\
&\quad = 2(2\pi)^{-n/2}\int_{\mathbb{R}^n} |t|^\alpha(1 + |t|^2)^{-s/2}(1 + |t|^2)^{s/2}|\hat{f}(t)|\, dt \\
&\quad \leqq 2(2\pi)^{-n/2}\int_{\mathbb{R}^n} |t|^{2\alpha}(1 + |t|^2)^{-s}\, dt)^{\frac{1}{2}}\|f\|_s,
\end{aligned}$$

the last integral being convergent for $\alpha < s - n/2$. (The first inequality follows from these two:

$$|\exp(-i\langle h, t\rangle) - 1| \leqq 2 \qquad \text{and} \qquad |\exp(-i\langle h, t\rangle) - 1| \leqq |\langle h, t\rangle| \leqq |h||t|).\ \square$$

The last lemma is clearly true for any norm equivalent to the norm (2.1), in particular for the norm defined by (2.4). We will call *Sobolev norm of order s* any norm on $H_s(X)$ equivalent to the norm (2.1).

THEOREM 2.2. *If X is a compact Riemannian manifold, any Sobolev norm of order $-s$ with $s > (\dim X)/2$ metrizes the weak-star topology of $\mathscr{P}(X)$.*

PROOF. Since the weak-star topology of $\mathscr{P}(X)$ is metrizable (for a proof see Dudley (1966)) we only need to prove that for any sequence $\{\nu_n\} \subset \mathscr{P}(X)$, $w^* - \lim \nu_n = \nu$ if and only if $\lim \|\nu_n - \nu\|_{-s} = 0$. The unit ball B_s of $H_s(X)$ for any Sobolev norm is equicontinuous and equibounded by Lemma 2.1;

therefore, Theorem 7 in Dudley (1966) proves that if $w^* - \lim \nu_n = \nu$, then $\lim \|\nu_n - \nu\|_{-s} = 0$. Since $H_s(X)$ contains a dense subset of $C(X)$ (for example, $C^\infty(X)$), a standard approximation argument proves that if $\lim \int_X h\, d\nu_n = \int_X h\, d\nu$ for every function $h \in H_s(X)$, then the same is true for every continuous function $f \in C(X)$, and this gives the converse. □

At this point we drop the primes in the notation for the norms defined by (2.4) and (2.5), and hereafter only these Sobolev norms will be used. Then, the statistic $T_n^{(s)}(\{\alpha_k\})$ defined by (1.1) becomes

$$T_n^{(s)}(\{\alpha_k\}) = n \sum_{k=1}^\infty \alpha_k^2 \sum_{f_i \in E_k} [\int_X f_i\, d(\nu_n - \mu)]^2 ,$$

and if $\{X_i\}_{i=1}^\infty$ is the sequence of independent identically distributed random variables defining ν_n, $n = 1, 2, \cdots$ (i.e. $\nu_n = n^{-1} \sum_{i=1}^n \delta_{X_i}$), then

$$T_n^{(s)}(\{\alpha_k\}) = n^{-1} \sum_{k=1}^\infty \alpha_k^2 \sum_{f_i \in E_k} [f_i(X_1) + \cdots + f_i(X_n)]^2 . \tag{2.7}$$

By the Sobolev lemma $T_n^{(s)}(\{\alpha_k\})$ exists whenever $\sup |\alpha_k \sigma_k^{s/2}| < \infty$ and $s > (\dim X)/2$. If ν is the distribution of X_1, a sufficient condition for the existence of the statistic (1.1), in view of (2.7), is $\sum_{k=1}^\infty \alpha_k^2 \sum_{f_i \in E_k} \int_X f_i^2\, d\nu < \infty$; as we will see in the next section (Lemma 3.1), this last condition is weaker than the former one.

3. A limit theorem for empirical distributions. Let ν be a Borel probability measure on the compact Riemannian manifold X and let $\nu_n(\omega)$, $n = 1, 2, \cdots$ be, as in the previous section, the empirical distributions obtained from a sequence of independent ν-distributed random variables $\{X_i\}_{i=1}^\infty$ defined on some probability space $(\Omega, \mathscr{B}, P)$ and with values in X. Following Strassen and Dudley (1969) define processes $Z_n^{(\nu)}$, $n = 1, 2, \cdots$ and $Z^{(\nu)}$ on $L_2(X, \nu)$ as follows:

$$Z_n^{(\nu)}(\omega)(f) = n^{\frac{1}{2}} \int_X f\, d(\nu_n(\omega) - \nu) , \tag{3.1}$$

and $Z^{(\nu)}$ as the centered Gaussian process on $L_2(X, \nu)$ with covariance $EZ^{(\nu)}(f)Z^{(\nu)}(g) = \int_X (f - \int_X f\, d\nu)(g - \int_X g\, d\nu)\, d\nu$. Then, by the central limit theorem, the finite dimensional distributions of $Z_n^{(\nu)}$ converge in law to the corresponding ones of $Z^{(\nu)}$. But this is not enough for finding the limiting distribution of $T_n^{(s)}(\{\alpha_k\})$ under the null hypothesis $\nu = \mu$ unless the sequence $\{\alpha_k\}$ has all but a finite number of terms equal to zero.

Let B_s be the closed unit ball of $H_s(X)$ for any $s > (\dim X)/2$ and let $d(f, g) = \sup_{x \in X} |f(x) - g(x)| = \|f - g\|_\infty$. Then, (B_s, d) is a compact metric space by Lemma 2.1 and the Arzelà–Ascoli theorem.

Again following Strassen and Dudley we will say that the central limit theorem for empirical distributions holds on $C(B_s, d)$ (for short, $C(B_s)$) whenever the following two conditions are satisfied:

a) $Z^{(\nu)} | B_s$ is sample continuous for every probability measure ν on X, thus defining a probability measure $\mathscr{L}(Z^{(\nu)} | B_s)$ on the space $C(B_s)$;

b) for every probability measure ν on X,

$$\mathscr{L}(Z_n^{(\nu)} \mid B_s) \to_{w*} \mathscr{L}(Z^{(\nu)} \mid B_s) \quad \text{in} \quad C'(C(B_s))$$

as $n \to \infty$ ($Z_n^{(\nu)}(\omega) \mid B_s$ clearly has continuous trajectories; also, $Z_n^{(\nu)}(\omega)$ is linear for every $\omega \in \Omega$). The object of this section is to prove that the central limit theorem for empirical distributions holds on $C(B_s)$ for every $s > (\dim X)/2$ (note that the question has no sense for smaller values of s).

The following is the main lemma.

LEMMA 3.1. *For every $s > (\dim X)/2$ and every Borel probability measure ν on X, the inclusion map $I: H_s(X) \to L_2(X, \nu)$ is Hilbert–Schmidt.*

PROOF. For every $s > (\dim X)/2$ the function $x \to \|\delta_x\|_{-s}$ (where δ_x denotes, as usual, point mass at x) is continuous: applying Lemma 2.1,

$$|\|\delta_x\|_{-s} - \|\delta_{x'}\|_{-s}| \leqq \|\delta_x - \delta_{x'}\|_{-s} = \sup_{\|f\|_s \leqq 1} |f(x) - f(x')| \leqq c_{s,\alpha}[d(x, x')]^\alpha$$

for any of the numbers α allowed by Lemma 2.1. In particular, since X is compact, there exists a finite positive constant K such that

$$\|\delta_x\|_{-s}^2 \leqq K \tag{3.2}$$

for every $x \in X$.

If we set $\tilde{f}_0 = 1$ and $\tilde{f}_i = \sigma_k^{-s/2} f_i$ for every $f_i \in E_k$, $k = 1, 2, \cdots$, then $\{\tilde{f}_i\}_{i=0}^\infty$ is an orthonormal basis of $H_s(X)$. Therefore, the identity I is Hilbert–Schmidt if and only if

$$1 + \textstyle\sum_{k=1}^\infty \sigma_k^{-s} \sum_{f_i \in E_k} \int_X f_i^2 \, d\nu < \infty , \tag{3.3}$$

But by (3.2),

$$\begin{aligned} 1 + \textstyle\sum_{k=1}^\infty \sigma_k^{-s} \sum_{f_i \in E_k} \int_X f_i^2 \, d\nu &= \textstyle\int_X (1 + \sum_{k=1}^\infty \sigma_k^{-s} \sum_{f_i \in E_k} f_i^2(x)) \, d\nu(x) \\ &= \textstyle\int_X \|\delta_x\|_{-s}^2 \, d\nu(x) \leqq K \end{aligned}$$

for every $\nu \in \mathscr{P}(X)$. Hence (3.3) holds. □

We now prove continuity of $Z^{(\nu)} \mid B_s$ by applying the above lemma.

LEMMA 3.2. *Fore every Borel probability measure ν on X and every $s > (\dim X)/2$ the process $Z^{(\nu)} \mid B_s$ is sample continuous.*

PROOF. If T is a compact metric space, x_t, $t \in T$, is a centered L_2-continuous Gaussian process and $\{h_n\}_{n=1}^\infty$ is an orthonormal basis of the linear span of $\{x_t : t \in T\}$ in $L_2(\Omega, P)$, then x_t is sample continuous if and only if the series $\sum (x_t, h_n) L(h_n)$ converges uniformly in t with probability one, L being the isonormal process of $L_2(\Omega, P)$ (Dudley (1973) page 69). The if part of this statement, which is what we need, is trivial. If $\{g_i\}_{i=0}^\infty$ is an orthonormal basis of $L_2(X, \nu)$ and $\{G_i\}_{i=0}^\infty$ is a sequences of independent $N(0, 1)$ random variables, the above condition reduces to the convergence of the series $\sum_{i=0}^\infty (f, g_i)_\nu G_i$ uniformly in $f \in B_s$ with probability one. A sufficient condition for this to hold, in view of Kolmogorov's inequality, is that there exist a sequence $\{\tau_i^2\}$ of positive real numbers such that

$\sum \tau_i^2 < \infty$ and $(f, g_i)_\nu^2 \leqq \tau_i^2$ for every i and every $f \in B_s$. Let $\{\tilde{f}_k\}_{k=0}^\infty$ be as in Lemma 3.1. Then we can take $\tau_i^2 = \sum_k (\tilde{f}_k, g_i)_\nu^2$ because on one hand

$$\sum \tau_i^2 = \sum_i \sum_k (\tilde{f}_k, g_i)_\nu^2 = \sum_k \sum_i (\tilde{f}_k, g_i)_\nu^2 = \sum_k (\tilde{f}_k, \tilde{f}_k)_\nu^2 < \infty$$

by Lemma 3.1, and on the other, if $f = \sum_k a_k \tilde{f}_k$ with $\sum a_k^2 \leqq 1$, i.e., if $f \in B_s$, then

$$(f, g_i)_\nu^2 = (\sum_k a_k \tilde{f}_k, g_i)_\nu^2 = [\sum_k a_k (\tilde{f}_k, g_i)_\nu]^2 \leqq (\sum_k a_k^2)(\sum_k (\tilde{f}_k, g_i)_\nu^2) \leqq \tau_i^2 . \ \square$$

Next we prove uniform tightness of the sequence of laws $\{\mathscr{L}(Z_n^{(\nu)} \mid B_s)\}_{n=1}^\infty$. We will apply Theorem 2.3 of De Acosta (1970) which asserts that a set of (Borel) probability measures $\{\mu_\alpha\}_{\alpha \in I}$ on a Banach space Y is uniformly tight if and only if

a) it is flatly concentrated, and

b) there exists a w^*-total subset A of Y' such that for every $Z \in A$ the set of probability measures on R, $\{\mu_\alpha \circ Z^{-1}\}_{\alpha \in I}$ is uniformly tight. (A set of probability measures $\{\mu_\alpha\}$ on Y is flatly concentrated if for every $\varepsilon > 0$ and $\delta > 0$ there exists a finite dimensional subspace E of Y such that $\mu_\alpha(E^\varepsilon) \geqq 1 - \delta$, where $E^\varepsilon = \{y \in Y : \inf_{y' \in E} \|y - y'\| \leqq \varepsilon\}$.)

LEMMA 3.3. *The sequence* $\{\mathscr{L}(Z_n^{(\nu)} \mid B_s)\}_{n=1}^\infty$ *of probability measures on* $C(B_s)$ *is uniformly tight for every probability measure* ν *on* X *and for every* $s > (\dim X)/2$.

PROOF. Let us define, for every natural n,

$$F_n = \{A \in C(B_s) : A \text{ is linear (on } H_s),\ Af_0 = 0 \text{ and } Af_k = 0,\ k = n+1, \cdots\}$$

(therefore, $\dim F_n = n$) and, for every $\varepsilon > 0$ and natural n,

$$M_{n,\varepsilon} = \{A \in C(B_s) : A \text{ is linear (on } H_s) \text{ and } (Af_0)^2 + \sum_{k=n+1}^\infty (A\tilde{f}_k)^2 \leqq \varepsilon^2\} .$$

Then $M_{n,\varepsilon}$ is contained in F_n^ε: for every $A \in M_{n,\varepsilon}$ define $\bar{A}$ by $\bar{A}f = \sum_{i=1}^n (f, f_i)_\mu Af_i$; then $\bar{A} \in F_n$ and so, if d denotes distance in $C(B_s)$, we obtain

$$\begin{aligned} d(A, F_n) \leqq d(A, \bar{A}) &= \sup_{f \in B_s} |(f, f_0)_\mu Af_0 + \textstyle\sum_{i=n+1}^\infty (f, f_i)_\mu Af_i| \\ &= \sup_{f \in B_s} |(f, f_0)_\mu Af_0 + \textstyle\sum_k \sigma_k^{s/2} \sum_{f_i \in E_k, i \geqq n+1} (f, f_i)_\mu A\tilde{f}_i| \\ &\leqq \sup_{f \in B_s} [(f, f_0)_\mu^2 + \textstyle\sum_k \sigma_k^s \sum_{f_i \in E_k, i \geqq n+1} (f, f_i)_\mu^2]^{\frac{1}{2}} \\ &\qquad \times [(Af_0)^2 + \textstyle\sum_{i=n+1}^\infty (A\tilde{f}_i)^2]^{\frac{1}{2}} \leqq \varepsilon , \end{aligned}$$

i.e. $M_{n,\varepsilon} \subset F_n^\varepsilon$.

Using this inclusion and Lemma 3.1 we prove that for every $\varepsilon > 0$, $\lim_{n \to \infty} P\{Z_r^{(\nu)} \in F_n^\varepsilon\} = 1$ uniformly in r (which clearly implies that the sequence $\{\mathscr{L}(Z_r^{(\nu)} \mid B_s)\}_{r=1}^\infty$ is flatly concentrated). We have

$$\begin{aligned} P\{Z_r^{(\nu)} \in F_n^\varepsilon\} \geqq P\{Z_r^{(\nu)} \in M_{n,\varepsilon}\} &= 1 - P\{\textstyle\sum_{k=n+1}^\infty (Z_r^{(\nu)}(\tilde{f}_k))^2 \geqq \varepsilon^2\} \\ &\geqq 1 - \varepsilon^{-2} \textstyle\int_\Omega \sum_{k=n+1}^\infty (Z_r^{(\nu)}(\tilde{f}_k))^2 \, dP \\ &= 1 - \varepsilon^{-2} \textstyle\sum_k \sigma_k^{-s} \sum_{f_i \in E_k, i \geqq n+1} \int_X (f_i - \int_X f_i \, d\nu)^2 \, d\nu \\ &\geqq 1 - \varepsilon^{-2} \textstyle\sum_k \sigma_k^{-s} \sum_{f_i \in E_k, i \geqq n+1} \int_X f_i^2 \, d\nu . \end{aligned}$$

But the last quantity, independent of r, converges to 1 for every $s > (\dim X)/2$ by (3.3).

Hence, by Theorem 2.3 in De Acosta (1970), the lemma will be proved if we find a w^*-total subset W of $(C(B_s))'$ such that the sequence $\{\mathscr{L}(Z_n^{(\nu)} \mid B_s) \circ w^{-1}\}_{n=1}^\infty$ is uniformly tight for every w in W. Since B_s is compact metric we can take W to be $\{\delta_f : f \in H_s(X)\}$ and, $Z_n^{(\nu)}(\omega)$ being linear, we only need to show that $\{\mathscr{L}(Z_n^{(\nu)} \mid B_s) \circ \delta_f^{-1}\}_{n=1}^\infty = \{\mathscr{L}(Z_n^{(\nu)}(f)\}_{n=1}^\infty$ is uniformly tight for every f in H_s. But this is true by the central limit theorem. □

Finally,

THEOREM 3.4. *Let X be a compact Riemannian manifold. Then the central limit theorem for empirical distributions holds on* $C(B_s)$ *for every* $s > (\dim X)/2$.

PROOF. Clear from Lemmas 3.2 and 3.3 by Prokhorov's theorem. □

4. Asymptotic theory of Sobolev tests. In this section, X will still be a general compact Riemannian manifold. We begin with the limiting distribution of $T_n^{(s)}$ under the null hypothesis.

THEOREM 4.1. *Let* $\{X_i\}_{i=1}^\infty$ *be a sequence of independent random variables with values in X and with distribution* μ*, the uniform distribution on X. Then for every* $s > (\dim X)/2$, *we have*

$$w^* - \lim_n \mathscr{L}\{T_n^{(s)}(\{\alpha_k\})\} = \mathscr{L}\{\textstyle\sum_{k=1}^\infty \alpha_k^2 H_k\}\,, \tag{4.1}$$

where $\{H_k\}_{k=1}^\infty$ *is a sequence of independent random variables such that, for each k,* H_k *is Chi-square with* $\dim E_k$ *degrees of freedom.*

PROOF. No generality is lost in assuming $\sup_k |\alpha_k \sigma_k^{s/2}| \leqq 1$. Define the following seminorm on $C(B_s)$:

$$h(A) = \sup_{f \in B_s} |A \circ \textstyle\sum_k \alpha_k \sigma_k^{s/2} \pi_k(f)|\,.$$

Since $(\sum_k \alpha_k \sigma_k^{s/2}\pi_k)(B_s) \subseteqq B_s$ as $\sup_k |\alpha_k \sigma_k^{s/2}| \leqq 1$, we have $0 \leqq h(A) \leqq \|A\|_\infty$ and, h being subadditive, this proves the continuity of h. Hence, by Theorem 3.4,

$$w^* - \lim_{n\to\infty} \mathscr{L}\{h^2(Z_n^{(\mu)})\} = \mathscr{L}\{h^2(Z^{(\mu)})\}\,.$$

And this proves the theorem because

$$h^2(Z_n^{(\mu)}(\omega)) = \|Z_n^{(\mu)}(\omega) \circ (\textstyle\sum_k \alpha_k \sigma_k^{s/2}\pi_k)\|_{-s}^2 = T_n^{(s)}(\{\alpha_k\})(\omega)$$

and

$$\begin{aligned} h^2(Z^{(\mu)}(\omega)) &= \|Z^{(\mu)}(\omega) \circ (\textstyle\sum_k \alpha_k \sigma_k^{s/2}\pi_k)\|_{-s}^2 \\ &= \textstyle\sum_{k=1}^\infty \alpha_k^2 \sum_{f_i \in E_k} (Z^{(\mu)}(f_i)(\omega))^2 \\ &= \textstyle\sum_k \alpha_k^2 H_k\,. \end{aligned}$$

□

Theorem 4.1 is still valid under the weaker condition

$$\textstyle\sum_{k=1}^\infty (\dim E_k)\alpha_k^2 < \infty\,, \tag{4.2}$$

but the above proof does not apply to this case. However one can use the central limit theorem for Hilbert space. And there are even simple direct proofs (Watson (1961) and Giné (1973)).

It is easy to see that the distribution function of the right term of (4.1) is absolutely continuous; in particular the convergence of the distribution functions in (4.1) is uniform.

The asymptotic behavior of the tail of the limiting distribution in (4.1) has been studied by Zolotarev (1961) although not with complete generality (just for $\dim E_1 \geqq 3$).

As for the speed of convergence, if the number of nonvanishing α_k's is infinite, there are not many results; for the circle and for tori the method of Sazonov (1969) can be applied and yields bounds of the order of $n^{-\frac{1}{3}}$ and worse. If the number of α_k's different from zero is finite, Theorem 4.1 is just a consequence of the multidimensional central limit theorem and in this case one may even find asymptotic expansions for the distribution of $T_n^{(s)}$ (e.g. by applying Bhachattarya's work (1971)).

Next we examine the consistency properties of the tests based on rejecting the hypothesis of uniformity of a distribution on X for large values of $T_n^{(s)}$.

LEMMA 4.2. *Let $\{X_i\}_{i=1}^\infty$ be a sequence of independent, identically distributed random variables with values in X and let ν be their common distribution. Then, for every $s >$ $(\dim X)/2$,*

$$\text{a.s.-}\lim_{n\to\infty} n^{-1} T_n^{(s)}(\{\alpha_k\}) = \|\textstyle\sum_k \alpha_k \sigma_k^{s/2} \pi_k(\nu - \mu)\|_{-s}^2 . \tag{4.3}$$

PROOF. Let $\nu_n(\omega)$ be the empirical distribution corresponding to $\{X_i\}_{i=1}^\infty$. Then, as it is well known, $w^* - \lim \nu_n(\omega) = \nu$ for almost every ω in Ω and so, by Theorem 2.2, a.s.-$\lim \|\nu_n - \nu\|_{-s} = 0$. Hence,

$$\begin{aligned}\lim_{n\to\infty} & [n^{-1} T_n^{(s)}(\{\alpha_k\})(\omega)]^{\frac{1}{2}} \\ &= \lim_{n\to\infty} \|\textstyle\sum_k \alpha_k \sigma_k^{s/2} \pi_k(\nu_n(\omega) - \mu)\|_{-s} \\ &= \lim_{n\to\infty} \|(\textstyle\sum_k \alpha_k \sigma_k^{s/2} \pi_k)(\nu_n - \nu) + (\textstyle\sum_k \alpha_k \sigma_k^{s/2} \pi_k)(\nu - \mu)\|_{-s} \\ &= \|(\textstyle\sum_k \alpha_k \sigma_k^{s/2} \pi_k)(\nu - \mu)\|_{-s}\end{aligned}$$

almost surely. □

This lemma says nothing about consistency of the test if $(\sum_k \alpha_k \sigma_k^{s/2} \pi_k)(\nu - \mu) = 0$. In this situation the following holds:

LEMMA 4.3. *If $\pi_k(\nu) = 0$ whenever $\alpha_k \neq 0$, $k = 1, 2, \cdots$, then*

$$w^* - \lim_{n\to\infty} \mathscr{L}\{T_n^{(s)}(\{\alpha_k\})\} = \mathscr{L}\{\textstyle\sum_k \alpha_k^2 \sum_{f_i \in E_k} (Z^{(\nu)}(f_i))^2\} , \tag{4.4}$$

where $Z^{(\nu)}$ is defined as in Section 3.

PROOF. By definition, $T_n^{(s)}(\{\alpha_k\}) = \sum_k \alpha_k^2 \sum_{f_i \in E_k} [Z_n^{(\nu)}(f_i) + n^{\frac{1}{2}} \int_X f_i \, d\nu]^2$. But since $\int_X f_i \, d\nu = 0$ for every $f_i \in E_k$ with $\alpha_k \neq 0$, this expression becomes $T_n^{(s)}(\{\alpha_k\}) = \sum_k \alpha_k^2 \sum_{f_i \in E_k} (Z_n^{(\nu)} f_i)^2$ and we can apply the result of Section 2 in exactly the same way as we did in Theorem 4.1 □

From the last two lemmas we obtain:

THEOREM 4.4. *The tests based on rejecting uniformity for large values of* $T_n^{(s)}(\{\alpha_k\})$ *are consistent against an alternative* ν *if and only if* ν *is not orthogonal to every* E_k *such that* $\alpha_k \neq 0$.

PROOF. If the condition holds then, by Lemma 4.2, for every $\alpha > 0$ we have $\lim_{n\to\infty} P\{T_n^{(s)}(\{\alpha_k\}) > \alpha\} = 1$ and the test is consistent. On the contrary, if the condition does not hold then, by Lemma 4.3, this limit is smaller than one for every $\alpha > 0$ and so the test is not consistent. □

Like Theorem 4.1, Lemmas 4.2 and 4.3 are valid in more generality: the condition

$$\sum_{k=1}^{\infty} \alpha_k^2 \sum_{f_i \in E_k} \int_X f_i^2 \, d\nu < \infty \tag{4.5}$$

is enough. Hence, Theorem 4.4 also holds under this condition.

We end this section giving the asymptotic distribution of $T_n^{(s)}$ under alternatives. For this, as well as for other parts of this section, the work of Beran (1969) on invariant tests for distributions on the circle has been of great help.

The following lemma gives an alternative expression for $T_n^{(s)}$ which allows us to use Beran's method (1969).

LEMMA 4.5. *The series*

$$g(x, y) = \sum_{k=1}^{\infty} \alpha_k \sum_{f_i \in E_k} f_i(x) f_i(y) \tag{4.6}$$

converges in $L_2(X, \mu)$ *for every fixed* x *in* X. *Moreover the following identity holds*:

$$T_n^{(s)}(\{\alpha_k\}) = n^{-1} \int_X [\sum_{j=1}^{n} g(x, X_j)]^2 \, d\mu(x) . \tag{4.7}$$

PROOF. Since, for $s > (\dim X)/2$, δ_x is in $H_{-s}(X)$ for every x, we have (note $\sup |\alpha_k \sigma_k^{s/2}| < \infty$):

$$\sum_{k=1}^{\infty} \alpha_k^2 \sum_{f_i \in E_k} f_i^2(x) = \|(\sum_{k=1}^{\infty} \alpha_k \sigma_k^{s/2} \pi_k)(\delta_x)\|_{-s}^2 < \infty ,$$

so that $g(\cdot, y) \in L_2(X, \mu)$. Now,

$$\begin{aligned} T_n^{(s)}(\{\alpha_k\}) &= n\|(\sum_k \alpha_k \sigma_k^{s/2} \pi_k)(\nu_n - \mu)\|_{-s}^2 \\ &= n^{-1} \sum_{k=1}^{\infty} \alpha_k^2 \sum_{f_i \in E_k} (\sum_{j=1}^{n} f_i(X_j))^2 \\ &= n^{-1} \|\sum_{k=1}^{\infty} \alpha_k \sum_{f_i \in E_k} (\sum_{j=1}^{n} f_i(X_j)) f_i\|_{\mu}^2 \\ &= n^{-1} \int_X [\sum_{j=1}^{n} g(x, X_j)]^2 \, d\mu(x) . \end{aligned}$$

□

In the proof of Theorem 1 of Beran (1969) the problem of finding the asymptotic distribution of a sequence of random variables of the type (4.7), X being the circle, is reduced to the central limit theorem by an argument which is a direct generalisation of the following simple observation: let $\{x_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $Ex_1 \neq 0$ and $Ex_1^2 = \sigma^2 < \infty$; then,

$$n^{-\frac{3}{2}}[(\sum_{i=1}^{n} x_i)^2 - (nEx_1)^2] = n^{-\frac{1}{2}}(\sum_{i=1}^{n} x_i - nEx_1)(n^{-1} \sum_{i=1}^{n} x_i + Ex_1) ,$$

and this implies

$$w^* - \lim \mathscr{L}\{n^{-\frac{3}{2}}[(\sum_{i=1}^{n} x_i)^2 - (nEx_1)^2]\} = N(0, 2\sigma|Ex_1|) .$$

Similarly,

$$w^* - \lim \mathscr{L}\{n^{-k+\frac{1}{2}}[(\sum_{i=1}^n x_i)^k - (nEx_1)^k]\} = N(0, \sigma k|Ex_1|^{k-1}) .$$

And even for nonintegral powers the analogous expression holds provided the random variables are positive and have a continuous distribution.

The following generalisation applies to statistics of the form (4.7).

PROPOSITION 4.6. *Let* $(U, \mathscr{F}, \mu)$ *and* $(V, \mathscr{G}, \nu)$ *be probability spaces,* $\{V_i\}_{i=1}^\infty$ *a sequence of* i.i.d. *random variables with values in* V *and common distribution* ν, *and* $g: U \times V \to \mathbb{R}$ *a measurable function. Suppose* $g \in L_k(U \times V, \mu \times \nu)$ *for some integer* $k > 1$. *Define*

$$r(u) = \int_V g(u, v)\, d\nu(v) \qquad \text{and} \qquad \sigma^2 = \mathrm{Var}\,[k \int_U r^{k-1}(u) g(u, V_1)\, d\mu(u)] .$$

Then, $r \in L_k(U, \mu)$, σ^2 *is finite and*

$$w^* - \lim \mathscr{L}\{n^{-k+\frac{1}{2}}[\int_U (\sum_{i=1}^n g(u, V_i))^k\, d\mu(u) - n^k \int_U r^k(u)\, d\mu(u)]\} = N(0, \sigma) .$$

The proof of this proposition is easy but somewhat cumbersome. The details may be found in Giné (1973).

Again, there is no difficulty in proving that, if g is positive and bounded, an analogue to Proposition 4.6 holds for real powers of g.

The following theorem gives the asymptotic distribution of $T_n^{(s)}$ under alternatives.

THEOREM 4.7. *Let* ν *be a probability measure on* X *such that* $\pi_k(\nu) \neq 0$ *for some* $k \geq 1$ *with* $\alpha_k \neq 0$, *let* g *be as in* (4.6), *and define*

$$r(x) = \int_X g(x, y)\, d\nu(y) .$$

Then,

$$w^* - \lim_{n\to\infty} \mathscr{L}\{n^{-\frac{1}{2}}[T_n^{(s)}(\{\alpha_k\}) - ET_n^{(s)}(\{\alpha_k\})]\} = N(0, \sigma) , \tag{4.8}$$

where

$$\sigma^2 = 4[\int_X (\int_X r(x) g(x, y)\, d\mu(x))^2\, d\nu(y) - (\int_X r^2\, d\mu)^2] .$$

PROOF. By Lemma 4.5 and Proposition 4.6 we only need to prove:

(a) $g \in L_2(X \times X, \mu \times \nu)$, and
(b) $\lim n^{-\frac{1}{2}}(ET_n^{(s)}(\{\alpha_k\}) - n \int_X r^2(y)\, d\mu(y)) = 0$.

We have, for (a):

$$\begin{aligned}\int_X g^2\, d\mu\, d\nu &= \int_X (\sum_k \alpha_k^2 \sum_{f_i \in E_k} f_i^2(x))\, d\nu(x) = \sum_k \alpha_k^2 \sum_{f_i \in E_k} \int_X f_i^2\, d\nu \\ &\leq \sup |\alpha_k^2 \sigma_k^s| \sum_k \sigma_k^{-s} \sum_{f_i \in E_k} \int_X f_i^2\, d\nu < \infty\end{aligned}$$

by Lemma 3.1, i.e. $g \in L_2(X \times X, \mu \times \nu)$; as for (b),

$$\begin{aligned}ET_n^{(s)}\{\alpha_k\} &= \int_X \int_X g^2(x, y)\, d\mu(x)\, d\nu(y) + (n-1) \int_X [\int_X g(x, y)\, d\nu(y)]^2\, d\mu(x) \\ &= \text{constant} + (n-1) \int_X r^2(x)\, d\mu(x) ,\end{aligned}$$

and from this, (b) follows. □

Again, this theorem is valid for $\{\alpha_i\}$ satisfying only condition (4.5).

Theorem 4.7 may be used for giving asymptotic approximations to the power of the test for uniformity based on $T_n^{(s)}$ against any alternative (simple). Some numerical results for Ajne's test on the circle can be found in Beran (1969).

5. Sobolev tests on homogeneous compact manifolds. A Riemannian manifold is called homogeneous if its group of isometeries, G, acts transitively on it. For every $x \in X$, let K_x denote the isotropy group of x, i.e. the subgroup of isometries leaving x fixed. With this notation we have:

DEFINITION 5.1. A differentiable function f on X is a zonal function with respect to a point $x_0 \in X$ if it is invariant under K_{x_0} i.e. if it is constant on the orbits of K_{x_0}.

For instance, on the sphere, the zonal functions with respect to a pole are the functions constant on the parallels.

X is called two-point homogeneous if, for every x, K_x acts transitively on the spheres centered at x or what is the same, if for any set of four points x_1, x_2, y_1, y_2 with $d(x_1, y_1) = d(x_2, y_2)$, there exists $\Phi \in G$ such that $\Phi(x_i) = y_i$, $i = 1, 2$. Zonal functions on two-point homogeneous spaces have the following characterization: a function is zonal with respect to x if and only if it depends only on the distance to x (Berger et al. (1971), III-C.1.7.).

Let X be a homogeneous compact Riemannian manifold with only a finite number of connected components—in what follows X is assumed to be connected; then, its group of isometies G is a compact Lie group, the isotropy group K_x of every $x \in X$ is a closed subgroup of G (Kobayashi and Nomizu (1963) page 239) and the homogeneous manifold of left cosets G/K_x is mapped diffeomorphically onto X by the mapping $[\Phi] \to \Phi(x)$, where $[\Phi] = \Phi K_x$ (Warner (1971) page 123). The normalised Haar measure of G gives a measure on $G/K_x \cong X$ in a natural way; this induced measure is precisely the uniform distribution μ on X. These facts on compact homogeneous manifolds will be used in proving some of the results below.

One of the tools we use in this section is a direct generalisation for the eigenfunctions of Δ of the equation of the cosine of a difference and also of the addition formula for spherical harmonics. Let us describe it (for a proof, see Giné (1975) or (1973)). If X is compact homogeneous, $x_0 \in X$ and E_k is an eigenspace of Δ, then there exists a uniquely determined real function $f_{x_0}^{(k)} \in E_k$ zonal with respect to x_0, orthogonal to every function in E_k vanishing at x_0, normalised (w.r.t. the $L_2(X, \mu)$ norm) and positive at x_0. And if $\{f_i\}_{i=1}^{\dim E_k}$ is any orthonormal ($L_2(X, \mu)$ sense) basis for E_k, then

$$\sum_{i=1}^{\dim E_k} f_i(x)\overline{f_i(y)} = (\dim E_k)^{\frac{1}{2}} f_{x_0}^{(k)}(\Phi_{x,x_0}(y)) , \tag{5.1}$$

where Φ_{x,x_0} is any isometry of X mapping x into x_0. Moreover, $\sum |f_i(x)|^2 \equiv \dim E_k$ and $f_x^{(k)}(x) = (\dim E_k)^{\frac{1}{2}}$. If X is two-point homogeneous, then the right hand side of (5.1) can be replaced by $(\dim E_k)^{\frac{1}{2}} h_k(d(x, y))$, h_k being the real function defined as

(5.2) $$h_k(d(x, y)) = f_x^{(k)}(y) = f_y^{(k)}(x) .$$

As a direct consequence of (5.1) and (5.2) we have the following proposition (which needs no proof).

PROPOSITION 5.2. *Let X be compact homogeneous and, for every x and y, set*

(5.3) $$h(x, y) = \sum_{k=1}^{\infty} \alpha_k^2 (\dim E_k)^{\frac{1}{2}} f_{x_0}^{(k)}(\Phi_{x,x_0}(y))$$

and

(5.4) $$g(x, y) = \sum_{k=1}^{\infty} \alpha_k (\dim E_k)^{\frac{1}{2}} f_{x_0}^{(k)}(\Phi_{x,x_0}(y)) .$$

Then, (4.7) *is valid with the function g given by* (5.4). *The following identity is also true*:

(5.5) $$T_n^{(s)}(\{\alpha_k\}) = n^{-1} \sum_{r,l=1}^{n} h(X_r, X_l) .$$

If X is two-point homogeneous, then

(5.3)′ $$h(x, y) = \sum_{k=1}^{\infty} \alpha_k^2 (\dim E_k)^{\frac{1}{2}} h_k(d(x, y))$$

and

(5.4)′ $$g(x, y) = \sum_{k=1}^{\infty} \alpha_k (\dim E_k)^{\frac{1}{2}} h_k(d(x, y)) .$$

The simplification of $T_n^{(s)}$ is significant at least for two-point homogeneous spaces because it proves that in these spaces Sobolev statistics only depend on the distance between observations and also because the function h_k seems to be easier to obtain than the functions f_i: zonal harmonics in two-point homogeneous compact manifolds are solutions of just ordinary second order differential equations. If the function h can be effectively determined, one may be able to compute $T_n^{(s)}$ from (5.4) with at most $O(n^2)$ operations. If only a finite number of the α's are nonzero and the relevant f_i's are known, then one can compute $T_n^{(s)}$ with only $O(n)$ operations just using (2.7); however, (5.4) could be useful even in this case (e.g. see Section 6).

We now turn to another consequence of Proposition 5.2. Beran (1968) considers statistics of the form

(5.6) $$T_n(f) = n^{-1} \int_G [\sum_{i=1}^{n} f(\Phi(X_i)) - n]^2 \, d\mu(\Phi) ,$$

where μ is the Haar normalized measure of G and f is a bounded probability density with respect to μ. He proves (Theorem 5) that the test consisting in rejecting uniformity for large values of $T_n(f)$ is most powerful invariant except for terms of order $O(\alpha^3)$ against the family of alternatives $\{f_\alpha \circ \Phi : f_\alpha = \alpha(f - 1) + 1, \alpha \in [-1, 1], \Phi \in G\}$ i.e., against the collection of probability measures $f_\alpha(\Phi(x)) \, d\mu(x)$, $\alpha \in [-1, 1]$, $\Phi \in G$. More precisely, there exist constants c_1, c_2 and c_3 such that the following restricted expansion is valid near $\alpha = 0$:

$$\int_G \prod_{i=1}^{n} f_\alpha(\Phi(X_i)) \, d\mu(\Phi) = c_1 + \alpha^2(c_2 + c_3 T_n(f)(X_1, \cdots, X_n)) + O(\alpha^3) .$$

In our case $c_1 = 1$, $c_2 = -n \int_X (f(x) - 1)^2 \, d\mu(x)$ and $c_3 = n$. Let us remark that considerations on invariance and the Neyman–Pearson lemma imply that the most powerful invariant test for uniformity against $f\,d\mu$ is to reject the null hypothesis for large values of

$$\int_G \prod_{i=1}^n f(\Phi(X_i)) \, d\mu(\Phi) \, .$$

Thus the test based on $T_n(f)$ may be considered as most powerful invariant against alternatives of the form $f_\alpha \circ \Phi \, d\mu$ for α small. The next two theorems assert that Sobolev tests on homogeneous manifolds have this property (at least if $s > \dim X$ or if the coefficients α_k satisfy $\sum_k \alpha_k \dim E_k < \infty$) and that on two-point homogeneous manifolds they are all the tests satisfying this property for alternatives which are zonal with respect to some point.

THEOREM 5.3. *Let X be a homogeneous compact (connected) Riemannian manifold and let $\{\alpha_k\}$ be a sequence of real numbers such that the function g defined by* (5.4) *is bounded. Then the test for uniformity based on $T_n^{(s)}(\{\alpha_k\})$ is most powerful invariant except for terms of order $O(\alpha^3)$ against the family of densities*

$$\{f_{\alpha,x} : f_{\alpha,x}(y) = \alpha g(x, y)/\|g(x, y\|_\infty + 1, \alpha \in [-1, 1], x \in X\} \, . \tag{5.7}$$

PROOF. Let us set $f_x = f_{1,x}$ for simplicity. Since f_x is zonal w.r.t. x, for every $\Phi \in G$ we have $f_x \circ \Phi = f_{\Phi^{-1}(x)}$ and so the set of densities (5.7) is invariant under isometries. By Theorem 5 of Beran (1968) we only need to prove that $T_n^{(s)}(\{\alpha_k\}) = cT_n(f_x)$ for some constant c and $x \in X$. According to (5.6) and (5.7), we have

$$T_n(f_x) = (n\|g\|_\infty^2)^{-1} \int_G [\textstyle\sum_{i=1}^n g(x, \Phi(X_i))]^2 \, d\mu(\Phi) \, .$$

Now,

$$g(x, \Phi(y)) = \textstyle\sum_k \alpha_k(\dim E_k) f_x^{(k)}(\Phi(y)) = \sum_k \alpha_k(\dim E_k) f_{\Phi^{-1}(x)}^{(k)}(y) = g(\Phi^{-1}(x), y) \, .$$

Therefore, using this fact, the properties of the Haar measure and that $g(\Phi(x), y)$, as a function of Φ, is constant on left K_x cosets, we obtain:

$$\begin{aligned} T_n(f_x) &= (n\|g\|_\infty^2)^{-1} \int_G [\textstyle\sum_{i=1}^n g\Phi^{-1}(x), X_i)]^2 \, d\mu(\Phi) \\ &= (n\|g\|_\infty^2)^{-1} \int_G [\textstyle\sum_{i=1}^n g(\Phi(x), X_i)]^2 \, d\mu(\Phi) \\ &= (n\|g\|_\infty^2)^{-1} \int_X [\textstyle\sum_{i=1}^n g(y, X_i)]^2 \, d\mu(y) = \|g\|_\infty^{-2} T_n^{(s)}(\{\alpha_k\}) \, . \end{aligned}$$ □

Under an additional hypothesis satisfied by compact two-point homogeneous manifolds, a converse to Theorem 5.3 is true.

THEOREM 5.4. *Let X be a homogeneous compact (connected) Riemannian manifold with the property that the subspace of E_k consisting of zonal functions with respect to a point $x \in X$ has dimension one for every $k = 1, 2, \cdots$. Then, if f_x is a bounded density zonal with respect to x, the statistic $T_n(f_x)$ is of the Sobolev type. In fact the function g defining $T_n(f_x)$ throughout* (4.7) *is $g(y, z) = f_y(z) - 1$, with f_y given by $f_y = f_x \circ \Phi_{y,x}$.*

PROOF. By the same computation carried out in the proof of Theorem 5.3,

$$T_n(f_x) = n^{-1} \int_X [\textstyle\sum_{i=1}^n g(y, X_i)]^2 \, d\mu(y) \, .$$

Now, f_x being zonal with respect to x, the hypothesis ensures that g has an expansion of the form (5.4). In fact,

$$g(x, y) = \sum_{k=1}^{\infty} (f_x, f_x^{(k)})_\mu f_x^{(k)}(y) .$$

Then, $T_n(f_x) = T_n^{(s)}(\{\alpha_k\})$ with $\alpha_k = (f_x, f_x^{(k)})_\mu/(\dim E_k)^{\frac{1}{2}}$ (therefore, the sequence $\{\alpha_k\}$ satisfies condition (4.2)). □

Berger et al. (1971), III, C.1.8, show that the hypothesis of Theorem 5.4 is satisfied by compact two-point homogeneous Riemannian manifolds.

Although $T_n(f_x)$ is not a strict Sobolev statistic (not necessarily $\sup |\alpha_k \sigma_k^{s/2}| < \infty$), the results of Section 4 apply to it. In particular, several results of Beran (1969) have been extended to spaces other than the circle.

6. Examples: the cases of the circle, the sphere and the projective plane. Let X be the circle $x^2 + y^2 = 1$ of $\mathbb{R}^2$. Then $d\mu = d\theta/2\pi$ (θ denotes arc length starting at (1, 0)), $\Delta = -d^2/d\theta^2$, $\sigma_k = k^2$, E_k is the linear span of $\cos k\theta$ and $\sin k\theta$ (E_0 is $\mathbb{C}$ or $\mathbb{R}$: since Δ is self adjoint we may just consider the real $L_2(X, \mu)$) and, for $k \geqq 1$, an orthonormal basis for E_k is $2^{\frac{1}{2}} \cos k\theta$ and $2^{\frac{1}{2}} \sin k\theta$. The function h_k defined by (5.2) is $h_k(\theta) = 2^{\frac{1}{2}} \cos k\theta$. Hence, by Proposition 5.2,

$$T_n^{(s)}(\{\alpha_k\}) = 2n^{-1} \sum_{k=1}^{\infty} \alpha_k^2 \sum_{i,j=1}^{n} \cos k(X_i - X_j) . \tag{6.1}$$

Perhaps the most interesting statistic of this type is Watson's one (Watson (1961)):

$$U_n^2 = n \int_0^1 [\nu_n(0, x) - x - \int_0^1 (\nu_n(0, x) - x)\, dx]^2\, dx .$$

(Here the circle is taken of unit length). This is an invariant modification of the Cramér–von Mises statistic. Some trivial computation involving Parseval's formula shows

$$4\pi^2 U_n^2 = T_n^{(1)}(\{k^{-1}\}) . \tag{6.2}$$

The limiting distribution of $T_n^{(1)}(\{k^{-1}\})$ under the null hypothesis is $\mathscr{L}(\sum_{k=1}^{\infty} k^{-2} H_k)$ where $\{H_k\}_{k=1}^{\infty}$ is a sequence of independent identically distributed random variables with a chi-square distribution of two degrees of freedom (Theorem 4.1). Using a modification of the method of Zolotarëv (1961) consisting in inverting the characteristic function by means of an integration by residues one can show (Watson (1961)) that the distribution function of $\sum_{k=1}^{\infty} k^{-2} H_k$, say $F(x)$, has the expression

$$F(x) = [1 + 2 \sum_{k=1} (-1)^k \exp(-k^2 x/2)]\chi_{(0,\infty)}(x) . \tag{6.3}$$

This series is alternating and perfectly suitable for computation.

Watson's statistic has been very well studied. Besides the information one can deduce as a consequence of the previous theory in this paper, we must mention the work of M. Stephens (1963) and (1964) about rapidity of convergence of the law of U_n^2 to its asymptotic distribution and, mainly, about the exact distribution of this statistic for small samples as well as about its lower tail for any sample size.

The oldest invariant test for uniformity on the circle is Rayleigh's test which consists in rejecting uniformity for large values of the statistic R_n^2/n, where n is sample size and R_n is the resultant of the sample, the circle being considered as the set of unit vectors of $\mathbb{R}^2$. Clearly we have

$$R_n^2/n = n^{-1}[(\sum_{i=1}^n \cos X_i)^2 + (\sum_{i=1}^n \sin X_i)^2] = n^{-1} \sum_{i,j=1}^n \cos (X_i - X_j)$$
$$= T_n^{(1)}(\{1, 0, \cdots\})/2 .$$

Besides its properties as a Sobolev test, this test is most powerful invariant for uniformity against the von Mises distribution $c \cdot \exp(k \cos (\theta - \theta_0))$ (Ajne (1968)). Even intuition favors this statistic as the natural candidate for testing uniformity against unimodal alternatives.

Another interesting Sobolev test on the circle is Ajne's test (Ajne (1968)), based on the statistic

$$A_n = n^{-1} \int_0^1 [N(\theta) - n/2]^2 \, d\theta , \tag{6.4}$$

where $N(\theta)$ is the number of points of the sample located on the half circle centered at θ. Some easy computation shows that

$$A_n = 4\pi^{-1} T_n^{(1)}(\{\alpha_k\}) \quad \text{with} \quad \alpha_{2k} = 0 \quad \text{and} \quad \alpha_{2k+1} = (-1)^k (2k + 1)^{-1} . \tag{6.5}$$

Ajne (1968) proved that this test is locally most powerful invariant against distributions with density constant on a hemisphere and on its complement. The asymptotic distribution of this statistic under the null hypothesis was first found by Watson (1967a) who also gave it in the form (5.5). For a study of the power of Ajne's test against the natural alternatives, see Beran (1969). J. S. Rao (1972) and E. Rothman (1972) have interesting generalisations of this test still falling into the category of Sobolev tests.

We now examine some examples for the sphere and for the projective plane (obtained from the sphere by identifying antipodal points). If $X = S^2$ is the unit sphere $x^2 + y^2 + z^2 = 1$ of $\mathbb{R}^3$, then $d\mu = (4\pi)^{-1} \sin \theta \, d\theta \, d\phi$ (where (θ, ϕ) are the usual spherical coordinates) is the uniform distribution on X; the Laplacian is given by $-\Delta = \partial^2/\partial\theta + \cot \theta \, \partial/\partial\theta + (\sin^2 \theta)^{-1} \partial^2/\partial\phi^2$, its eigenvalues are $\sigma_k = k(k + 1)$, $k = 1, 2, \cdots$ and $\sigma_0 = 1$, and their multiplicities, $\dim E_k = 2k + 1$, $k = 0, 1, \cdots$. An orthonormal basis for the eigenspace E_k of eigenvalue $\sigma_k = k(k + 1)$ is $\{f_k^{(m)}\}_{m=-k}^k$ with

$$f_k^{(0)}(\theta, \phi) = (2k + 1)^{\frac{1}{2}} P_k(\cos \theta) ,$$
$$f_k^{(m)}(\theta, \phi) = [(2k + 1)(k - m)!/(k + m)!]^{\frac{1}{2}} P_k^{(m)}(\cos \theta) \cos m\phi ,$$

and

$$f_k^{(-m)}(\theta, \phi) = [(2k + 1)(k - m)!/(k + m)!]^{\frac{1}{2}} P_k^{(m)}(\cos \theta) \sin m\phi ,$$
$$m = 1, 2, \cdots, k ,$$

where P_k and $P_k^{(m)}$ are the Legendre and associated Legendre functions:

$$P_k(x) = (k! \, 2^k)^{-1} (d^k/dx^k)(x^2 - 1)^k ,$$

and

$$P_k^{(m)}(x) = (-1)^m(1 - x^2)^{m/2}(d^m/dx^m)(P_k(x)) .$$

(Gradschtein and Ryzhik (1965).)

Since $f_k^{(0)}(x) = (2k + 1)^{\frac{1}{2}}P_k(\cos \widehat{x_0 x})$ ($\widehat{ab}$ is the unoriented angle between a and b as vectors in R^3) and S^2 is two-point homogeneous, we have, in the notation of Section 5,

$$f_x^{(k)}(y) = (2k + 1)^{\frac{1}{2}}P_k(\cos \widehat{xy})$$

and

$$h_k(r) = (2k + 1)^{\frac{1}{2}}P_k(\cos r) , \quad 0 \leqq r \leqq \pi , \tag{6.6}$$

for $k = 1, 2, \cdots$. Formula (5.1), for the particular basis defined above, becomes

$$\sum_{m=-k}^{k} f_k^{(m)}(x)f_k^{(m)}(y) = (2k + 1)P_k(\cos \widehat{xy}) . \tag{6.7}$$

Equation (6.7) is known as the addition formula for spherical harmonics.

Proposition 5.2 together with (6.6) gives concrete expressions for $T_n^{(s)}$ on the sphere.

If in S^2 we identify diametrically opposite points, we obtain the projective plane P^2. In P^2 we can use the same coordinates (θ, ϕ) as in S^2 (or the Cartesian coordinates (x, y, z)) and the functions on P^2 are the ones on S^2 with the appropriate symmetry property, i.e. such that $f(\theta, \phi) = f(\pi - \theta, \pi + \phi)$ (or $f(x, y, z) = f(-x, -y, -z)$). Using the definitions of P_k and $P_k^{(m)}$ we easily obtain

LEMMA 6.1. *If $f \in E_k$, E_k being the eigenspace of Δ (on S^2) of eigenvalue $k(k + 1)$, then*

$$f(\theta, \phi) = (-1)^k f(\pi - \theta, \pi + \phi) .$$

COROLLARY 6.2. *The eigenspaces of Δ on P^2 are $\{E_{2k}\}_{k=0}^{\infty}$, where E_{2k} is the eigenspace of Δ on S^2 of eigenvalue $2k(2k + 1)$. The eigenvalue of E_{2k} is also $2k(2k + 1)$, $k = 1, 2, \cdots$.*

Corollary 6.2 makes it unnecessary to treat P^2 very specifically: if $T_n^{(s)}(\{\alpha_k\})$ is a Sobolev statistic for the sphere satisfying $\alpha_{2k+1} = 0$, $k = 0, 1, \cdots$, then it belongs to the projective plane too, i.e. it can be used for testing uniformity on P^2.

Formula (5.5) is very suitable for computation. Using it we now give a statistic such that the corresponding test is consistent against any alternative and such that for sample size n it can be effectively computed with $O(n^2)$ operations.

PROPOSITION 6.3. *Let $X_1, \cdots, X_n$ be n independent observations from some distribution on S^2; then the statistic F_n defined by*

$$F_n(X_1, \cdots, X_n) = 3n/2 - (4/n\pi)[\textstyle\sum_{i<j\leqq n} \widehat{X_i X_j} + \sum_{i<j\leqq n} \sin \widehat{X_i X_j}] \tag{6.8}$$

is a Sobolev statistic of type $T_n^{(\frac{3}{2})}(\{\alpha_k\})$ with $\alpha_k \neq 0$ for every $k \in N$ and $\sup_k |\alpha_k \sigma_k^{\frac{3}{4}}| < \infty$. The invariant test for uniformity consisting in rejecting it for large values of F_n

is consistent against any alternative. The asymptotic distribution of F_n under uniformity is

$$\mathscr{L}\{\sum_{k=1}^{\infty}[2^{-2k}(2k-1)^{-2}((2k-1)!!/k!)^2 H_{2k-1} + 2^{-2k-1}(2k-1)^{-1}(k-1)^{-1}((2k-1)!!/k!)^2 H_{2k}]\},$$

where $\{H_r\}_{r=1}^{\infty}$ is a sequence of independent random variables with $\mathscr{L}(H_r) = \chi^2_{2r+1}$, and $(2k-1)!! = (2k-1)(2k-3)\cdots 1$.

PROOF. An alternative form for (6.8) is

$$F_n = n^{-1}\sum_{i,j=1}^{n}[(1 - 2\widehat{X_iX_j}/\pi) + (\tfrac{1}{2} - 2(\sin\widehat{X_iX_j})/\pi)] \tag{6.8$'$}$$

and so, if

$$h(r) = (1 - 2r/\pi) + (\tfrac{1}{2} - 2(\sin r)/\pi)$$

then

$$F_n = n^{-1}\sum_{i,j} h(\widehat{X_iX_j}).$$

Now, by (5.5) we only need to see that h admits an expansion of type (5.3)$'$ with $\sup_k |\alpha_k\sigma^{\frac{3}{2}}| < \infty$, $\alpha_k \neq 0$ for every $k \geqq 1$ and $s = \frac{3}{2}$. But

$$1 - 2r/\pi = \sum_{k=1}^{\infty} 2^{-2k}(2k-1)^2(4k-1)[(2k-1)!!/k!]^2 P_{2k-1}(\cos r) \tag{6.9}$$

and

$$\tfrac{1}{2} - 2(\sin r)/\pi = \sum_{k=1}^{\infty} 2^{-2k-1}(2k-1)^{-1}(k+1)^{-1}(4k+1) \times [(2k-1)!!/k!]^2 P_{2k}(\cos r) \tag{6.10}$$

for $r \in [0, \pi]$ (Gradschtein and Ryzhik (1965)) and $(2k-1)!!/k! \cong (\pi k)^{-\frac{1}{2}}2^k$ as $k \to \infty$. Comparison of (6.9) and (6.10) with (5.3)$'$ and trivial computation show that $F_n = T_n^{(\frac{3}{2})}(\{\alpha_k\})$ with $\alpha_k \neq 0$ for all $k \geqq 1$ and $\sup|\alpha_k\sigma_k^{\frac{3}{2}}| < \infty$. The remaining statements are now a consequence of Theorems 4.1 and 4.4. □

The first term of (6.8)$'$, $n^{-1}\sum_{i,j}(1 - 2\widehat{X_iX_j}/\pi)$, is Beran's form of Ajne's statistic for the sphere (defined similarly to (6.4); see Beran (1968) page 193). It appeared originally with a misprint and it has been cited at least twice in the literature (Watson (1967b) page 378 and Mardia (1972) page 282) still with misprints.

The second term of (6.8)$'$ is interesting because it gives a test for uniformity on the projective plane consistent against any alternative (and a test on the sphere consistent against any alternative with density symmetric with respect to the center). More concretely:

PROPOSITION 6.4. *Let $X_1, \cdots, X_n$ be n independent observations of a distribution on P^2; then the statistic defined by*

$$G_n = 2^{-1}n - 4(n\pi)^{-1}\sum_{i<j}\sin\widehat{X_iX_j} \tag{6.11}$$

is a Sobolev statistic on P^2 of type $T_n^{(\frac{3}{2})}(\{\alpha_k\})$ with $\sup_k|\alpha_k\sigma_k^{\frac{3}{2}}| < \infty$ and $\alpha_k \neq 0$ for all $k \geqq 1$. The invariant test for uniformity based on G_n is consistent against any *alternative and the asymptotic distribution of G_n under the null hypothesis is*

$$\mathscr{L}\{\sum_{k=1}^{\infty} 2^{-2k-1}(2k-1)^{-1}(k-1)^{-1}((2k-1)!!/k!)^2 H_{2k}\},$$

where $\{H_r\}$ is as in Proposition 6.3.

PROOF. As Proposition 6.3 but taking Corollary 6.2 into account. □

On the sphere, G_n does not give a test consistent against any alternative, of course, but the set of alternatives against which it is consistent contains, by Theorem 4.4, the densities which are symmetric with respect to the origin (and not a.s. constant).

F_n and G_n have the inconvenience of requiring $O(n^2)$ computations for samples of size n, and so does Ajne's statistic on the sphere. However, if the number of coefficients α_k different from zero is finite, then $T_n^{(s)}(\{\alpha_k\})$ can be computed with $O(n)$ operations using (2.7), as remarked before. Still, we can use (5.5) to obtain efficient expressions for $T_n^{(s)}$ even in this case. Call $\mathbf{x}_i$ the unit vector corresponding to the observation X_i and set $P_k(x) = \sum_{r=0}^{k} \gamma_r^{(k)} x^r$, $k = 1, 2, \cdots$. Then by (6.6),

$$\begin{aligned}\sum_{i,j} h_k(\widehat{X_i X_j}) &= (2k+1)^{\frac{1}{2}} \sum_{i,j} P_k(\mathbf{x}_i \cdot \mathbf{x}_j)\\ &= (2k+1)^{\frac{1}{2}} \sum_{r=0}^{k} \gamma_r^{(k)} \sum_{i,j} (\mathbf{x}_i \cdot \mathbf{x}_j)^r\\ &= (2k+1)^{\frac{1}{2}}[\gamma_0^{(k)} n^2\\ &\quad + \sum_{r=1}^{k} \gamma_r^{(k)} \sum_{i_1,\cdots,i_r=1}^{3} (\sum_{i=1}^{n} x_i^{(i_1)} \cdots x_i^{(i_r)})^2],\end{aligned} \tag{6.12}$$

where $\mathbf{x}_i \cdot \mathbf{x}_j$ is the Euclidean inner product in R^3 and $x_i^{(j)}$ is the jth coordinate of the vector $\mathbf{x}_i$ with respect to an orthogonal system. Let us take $k = 1$. Then $P_1(x) = x$ and so (6.12) gives

$$\sum_{i,j} h_1(\widehat{X_i X_j}) = 3^{\frac{1}{2}} \sum_{j=1}^{3} (\sum_{i=1}^{n} x_i^{(j)})^2 = 3^{\frac{1}{2}} R_n^2,$$

R_n denoting the length of the resultant of the sample, $R_n = |\sum_{i=1}^{n} \mathbf{x}_i|$. Therefore,

$$T_n^{(s)}(\{1, 0, \cdots\}) = 3R_n^2/n. \tag{6.13}$$

For a modern reference to this statistic, see Mardia (1972). Let us take now $k = 2$. Then, $P_2(x) = (3x^2 - 1)/2$, and so, by (6.12), we have

$$\begin{aligned}\sum_{i,j} h_2(\widehat{X_i X_j}) &= 5^{\frac{1}{2}}[-n^2/2 + 3 \sum_{j,k=1}^{3} (\sum_{i=1}^{n} x_i^{(j)} x_i^{(k)})^2/2]\\ &= 5^{\frac{1}{2}}[-n^2/2 + 3(\text{trace } T^2)/2],\end{aligned} \tag{6.14}$$

with $T = \sum_{i=1}^{n} T(\mathbf{x}_i)$ and

$$T(\mathbf{x}) = \begin{pmatrix} x^2 & yx & zx \\ xy & y^2 & zy \\ xz & yz & z^2 \end{pmatrix} \tag{6.15}$$

for any vector $\mathbf{x} = (x, y, z)$. Therefore,

$$T_n^{(s)}(\{0, 1, 0, \cdots\}) = 5[-n/2 + 3(\text{trace } T^2)/2n]. \tag{6.16}$$

The statistic (6.16) was first introduced by C. Bingham (1964) in his Ph. D.

thesis, but it has remained unpublished, to our knowledge, until 1972 (Mardia). We obtained this statistic independently, just in the way described above, with the intention of constructing easily computable tests for uniformity on P^2. Bingham obtained (6.16) as an asymptotic form of the invariant likelihood ratio statistic for testing uniformity against $c \cdot \exp(\sum_{i=1}^{3} k \cos^2 \widehat{x_i x})\, d\mu(x)$, x_1, x_2, x_3 orthogonal. Denote by B_n this statistic, i.e.

$$B_n = T_n^{(s)}(\{0, 1, 0, \cdots\}) . \tag{6.16$'$}$$

Then, the theory of the previous two sections, applied to B_n gives the following proposition.

PROPOSITION 6.5. *The statistic B_n satisfies*:

a) *its asymptotic distribution under the hypothesis of uniformity is* $\chi_5^2/5$ (Bingham (1964));

b) *the test based on B_n is consistent against any alternative ν such that* $\int_{S^2} f_2^{(m)}(x)\, d\nu(x) \neq 0$ *for some* $m = -2, \cdots, 2$;

c) *the test based on B_n is most powerful invariant except for terms of order* $O(\alpha^3)$ *against the family*

$$\{f_{\alpha,x} : f_{\alpha,x}(y) = 1 + \alpha P_2(\cos \widehat{xy}),\ \alpha \in [-1, 1],\ x \in S^2\} ;$$

d) *the limiting distribution of* $n^{-\frac{1}{2}}[B_n - n\alpha^2/5]$ *under the alternative* $f_{\alpha,x}$ *is* $N(0, 2\alpha[1 + 2\alpha/7 - \alpha^2/5]^{\frac{1}{2}}/5^{\frac{1}{2}})$.

PROOF. a) is consequence of Theorem 4.1, b) comes from Theorem 4.4, c) from Theorem 5.3 and d) from Theorem 4.7. The computations involved in proving c) (mainly based in the orthogonality properties of spherical harmonics and in the addition formula) are very straightforward and we omit them. For other alternatives, they may become rather complicated. □

The consistency property b) can be written in a more convenient form:

COROLLARY 6.6. *The test based on B_n is consistent against a probability measure ν if and only if*

$$\int_{S^2} (T(\mathbf{x}) - I/3)\, d\nu(\mathbf{x}) \neq 0 , \tag{6.17}$$

where $T(\mathbf{x})$ is as defined in (6.15) *and I is the identity* 3×3 *matrix.*

PROOF. Consequence of the identity

$$T(\mathbf{x}) - I/3 = 15^{-\frac{1}{2}} \begin{pmatrix} -3^{-\frac{1}{2}} f_2^{(0)} + f_2^{(2)} & f_2^{(-2)} & -f_2^{(1)} \\ f_2^{(-2)} & -3^{-\frac{1}{2}} f_2^{(0)} - f_2^{(2)} & -f_2^{(-1)} \\ -f_2^{(1)} & -f_2^{(-1)} & 2 \cdot 3^{-\frac{1}{2}} f_2^{(0)} \end{pmatrix},$$

which can be easily verified. □

Among the distributions satisfying (6.15) there are the ones with density $h(\theta, \phi)$ such that: a) $h(\cdot, \phi)$ is symmetric about $\theta/2$ and monotonic between 0 and $\pi/2$ for every ϕ, and b) $h(\cdot, \phi)$ is not a constant for any ϕ in a subset of

$[0, 2\pi]$ of positive Lebesgue measure. In fact these densities are not orthogonal to $P_2(\cos\theta)$.

The last two tests in this section are the best examples of Sobolev tests on the sphere. Using (6.12), one can construct other tests with the limiting distributions under the null hypothesis and under alternatives as described in Section 4, with prescribed consistency properties and computable with relatively few operations.

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