

ON A THEOREM OF MORIMOTO CONCERNING SUFFICIENCY  
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We prove in a discrete setting that if for all test functions,  $t$ , there is a  $\mathbf{B}$  measurable test function,  $s$ , such that  $E_p(t) = E_p(s)$  for all  $p \in P$  then some subfield of  $\mathbf{B}$  is sufficient for  $P$ .

The purpose of this note is to call attention to the fact that the conclusion of Theorem 5 of Morimoto (1972) can be strengthened. In the following we use the notation and definitions of Morimoto (1972). In particular,  $P$  is a family of discrete distributions on a set  $X$  and all subsets of  $X$  are measurable. We assume as in Morimoto (1972) that  $p(A) = 0$  for all  $p \in P$  implies  $A = \emptyset$ .

The strengthened version of Morimoto's Theorem 5 is as follows:

**THEOREM.** *Let  $\mathbf{B}$  be a  $\sigma$ -field such that for any test function  $t(x)$  there is a  $\mathbf{B}$  measurable test function  $s(x)$  with  $E(t(x)|p) = E(s(x)|p)$  for all  $p \in P$ . Then  $\mathbf{B}$  contains a sufficient subfield, i.e.,  $\mathbf{B} > \mathbf{B}(\mathbf{M})$ .*

(Morimoto's conclusion is that  $\mathbf{T}(\mathbf{B}) > \mathbf{M}$  which implies that  $\mathbf{B}$  is pairwise sufficient, but not that  $\mathbf{B}$  is sufficient.)

**PROOF.** Write  $P = \{p_\omega : \omega \in \Omega\}$  where  $\Omega$  is a well-ordered set. The collection of subsets  $(\mathbf{M})$  defined in Morimoto (1972) may be rewritten using a transfinite induction as

$$(1) \quad \mathbf{M} = \{T_{\omega_i} : i = 1, 2, \dots, I_\omega \leq \infty\}$$

where

$$(2) \quad p_\omega(x) > 0 \quad \text{for all } x \in \bigcup_{i=1}^{I_\omega} T_{\omega_i} \quad \text{and} \quad P_{\omega'}(\bigcup_{\omega \leq \omega'} \bigcup_{i=1}^{I_\omega} T_{\omega_i}) = 1,$$

and the sets  $T_{\omega_i}$  are mutually disjoint. (The statement  $I_\omega \leq \infty$  above is intended to mean that the index set  $\{i\}$  is countable, but possibly infinite.)

Now,  $V \in \mathbf{B}(\mathbf{M})$  if and only if  $V$  may be written

$$V = \bigcup_{\omega \in \Omega_0} \bigcup_{i \in I(\omega)} T_{\omega_i} = \bigcup_{i=1}^{\infty} \bigcup_{\omega \in \Omega_i} T_{\omega_i}$$

where  $\Omega_j \subset \Omega$ ,  $j = 0, 1, \dots$ . In order to prove that  $\mathbf{B} > \mathbf{B}(\mathbf{M})$  it therefore suffices to prove that any set of the form

$$Q = \bigcup_{\omega \in \Omega'} T_\omega$$

satisfies  $Q \in \mathbf{B}$ , where  $\Omega' \subset \Omega$  and  $T_\omega = T_{\omega, i(\omega)}$ .

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As a consequence of the characterization described in (7) of Morimoto (1972) and of our definition (2) each set  $T_\omega$  may be written in the form

$$(3) \quad T_\omega = \{x : p_\omega(x) > 0, p_\xi(x) = 0 \quad \forall \xi < \omega, \\ \text{and } p_\omega(x) = k_i p_{\omega_i}(x); i = 1, 2, \dots, I_\omega \leq \infty\}$$

where  $k_i > 0$  and  $\omega_i \geq \omega$ . (Note that the indices  $\omega_i$  depend on  $\omega$ , although this is not indicated by the notation.) After defining  $0/0 = 0$  we may rewrite (3) as

$$(4) \quad T_\omega = \{x : p_\omega(x) / \sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) = 1 \quad \forall \{\alpha_i\} \ni \alpha_i \geq 0, \sum_{i=1}^{I_\omega'} \alpha_i = 1 \\ \text{and } p_\xi(x) = 0 \quad \forall \xi < \omega\}.$$

We now prove

LEMMA. *There exists a vector  $(\alpha_1, \dots)$  in the simplex defined in (4) which satisfies*

$$(5) \quad x \in T_\omega \Leftrightarrow p_\omega(x) / \sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) = 1 \quad \text{and} \quad p_\xi(x) = 0 \quad \forall \xi < \omega.$$

PROOF. Consider the metric space,  $M$ , consisting of points  $(\alpha_1, \dots, \alpha_i, \dots)$ ,  $i = 1, \dots, I_\omega'$ , satisfying  $0 \leq \alpha_i \leq \max(2^{-i}, 2^{-i}k_i^{-1})$ ; and with metric,  $\rho$ , given from the sup ( $L_\infty$ ) norm:  $\rho(\alpha, \beta) = \sup\{|\alpha_i - \beta_i| : i = 1, \dots, I_\omega'\}$ . This is a complete metric space.

If  $x \notin \bigcup_{i=1}^{I_\omega'} T_{\omega_i}$  then either  $p_\xi(x) = 0$  for all  $\xi < \omega$  or  $p_\omega(x) = 0$  so that the r.h.s. of (5) cannot hold true.

For given  $x \in D_\omega = \bigcup_{i=1}^{I_\omega'} T_{\omega_i} - T_\omega$  let  $S_x$  denote the set of points in  $M$  which satisfy  $p_\omega(x) / \sum_{i=1}^{I_\omega'} \alpha_i = \sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x)$ . For  $x \in D_\omega$   $p_\omega(x) > 0$  and there is some index,  $\omega_i^x$ , say, such that  $p_\omega(x) \neq k_i p_{\omega_i^x}(x)$ . It follows that if  $\beta \in S_x$  and  $\beta'$  satisfies  $\beta_{i'} = \beta_i$  for  $i \neq \omega_i^x$  and  $\beta'_{\omega_i^x} \neq \beta_{\omega_i^x}$  then  $\beta' \notin S_x$ . Hence the interior of  $S_x$  is empty.

The fact that  $\alpha_i \leq \max(2^{-i}, 2^{-i}k_i^{-1}p_{\omega_i}^{-1}(x))$  and the dominated convergence theorem lead to the conclusion that  $S_x$  is closed in  $M$ . Since  $D_\omega$  is countable the Baire category theorem may then be invoked to establish the existence of a point  $\alpha' \in M$  such that  $p_\omega(x) / \sum_{i=1}^{I_\omega'} \alpha'_i \neq \sum_{i=1}^{I_\omega'} \alpha'_i k_i p_{\omega_i}(x)$ , for all  $x \in D_\omega$ . The vector  $\alpha'' = \alpha' / \sum_{i=1}^{I_\omega'} \alpha'_i$  is in the simplex described in (4) and satisfies the conclusion of (5).  $\square$

Fix any vector in the simplex which satisfies (5). Then  $T_\omega = R_\omega - S_\omega$  where

$$R_\omega = \{x : p_\omega(x) / \sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) \geq 1 \quad \text{and} \quad p_\xi(x) = 0 \quad \forall \xi < \omega\} \subset \bigcup_{i=1}^{I_\omega'} T_{\omega_i},$$

and

$$S_\omega = \{x : p_\omega(x) / \sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) > 1 \quad \text{and} \quad p_\xi(x) = 0 \quad \forall \xi < \omega\}.$$

Note that if  $p_\omega(x) > 0$  and  $p_\xi(x) > 0$  for some  $\xi < \omega$  then  $p_\omega(x) / l p_\xi(x) < 1$  for  $l$  sufficiently large. Since  $\{x : p_\omega(x) > 0\}$  is countable we may thus rewrite  $R_\omega$  as

$$R_\omega = \{x : p_\omega(x) / [\sum_{i=1}^{I_\omega'} \alpha_i k_i p_{\omega_i}(x) + \sum_{j=1}^{J_\omega} l_j p_{\xi_j}(x)] \geq 1\}$$

where the  $\xi_j$  satisfy  $\xi_j < \omega$  and the  $l_j$  are suitable positive constants and  $0 \leq J_\omega \leq \infty$ .  $S_\omega$  has a similar expression. Hence

$$(6) \quad Q = \bigcup_{\omega \in \Omega'} (R_\omega - S_\omega).$$

Since  $S_\omega \subset R_\omega$  for all  $\omega \in \Omega'$  and since for  $\omega \neq \omega' (\bigcup_{i=1}^{I_\omega} T_{\omega i}) \cap (\bigcup_{i=1}^{I_{\omega'}} T_{\omega' i}) = \emptyset$  we have  $R_{\omega'} \cap R_\omega = \emptyset$  for all  $\omega, \omega' \in \Omega'$  with  $\omega' \neq \omega$ . The expression (6) may thus be rewritten as

$$(7) \quad Q = \bigcup_{\omega \in \Omega'} R_\omega - \bigcup_{\omega \in \Omega'} S_\omega.$$

To prove the theorem it therefore suffices to show that any set of the form  $\bigcup_{\omega \in \Omega'} R_\omega$  or  $\bigcup_{\omega \in \Omega'} S_\omega$  is an element of  $\mathbf{B}$ .

Let  $W' = \bigcup_{\omega \in \Omega'} A(p_\omega) = \{x : p_\omega(x) > 0 \text{ for some } \omega \in \Omega'\}$ .

Let  $t = \chi_{\bigcup_{\omega \in \Omega'} R_\omega}$  and let  $s$  be the  $\mathbf{B}$  measurable function generated by the hypothesis of the theorem such that  $E(s|p) = E(t|p)$  for all  $p \in P$ .  $t' = t$  is the essentially unique test function relative to the measure given by  $p_\omega + \sum \alpha_i k_i p_{\omega_i} + \sum l_j p_{\xi_j}$  maximizing  $E(t'|p_\omega)$  subject to the side conditions

$$E(t'|p_{\omega_i}) \leq E(t|p_{\omega_i}) \quad \text{and} \quad E(t'|p_{\xi_j}) \leq E(t|p_{\xi_j});$$

$$i = 1, \dots, I'_\omega, j = 1, \dots, J_\omega.$$

Hence  $s(x) = t(x)$  for all  $x \in W'$ .

Since  $t(x) = 0$  for  $x \notin W'$  we thus have  $s(x) \geq t(x)$  for all  $x \in X$ . Suppose  $s(x) > t(x)$  for some  $x \in X$ . Then, for some  $p_\omega, p_\omega(x) > 0$  and  $E(s|p_\omega) > E(t|p_\omega)$ , a contradiction.

It follows that  $s(x) = t(x)$  for all  $x$  so that  $\bigcup_{\omega \in \Omega'} R_\omega \in \mathbf{B}$ . Similarly  $\bigcup_{\omega \in \Omega'} S_\omega \in \mathbf{B}$  (use the fact that  $\mathbf{B}$  is a  $\sigma$ -field). As described above this proves that  $\mathbf{B} > \mathbf{B}(\mathbf{M})$ , which is the desired conclusion.

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