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A METHOD OF CONSTRAINED RANDOMIZATION FOR $p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ FACTORIALS

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The statistical analysis which is carried out in conducting a $p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ factorial experiment in blocked designs requires that the treatment combinations be randomly arranged for each treatment run. When the nature of the process under investigation restricts the number of factor levels which can be changed from treatment combination to treatment combination, the usual technique of full randomization cannot be carried out.

This paper presents methods of constrained randomization for $p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ factorial experiments in blocked designs when the requirement on adjacent treatment combinations is that the number, Δ, of factor levels which can be changed is less than t, where $t=\sum_{i=1}^k n_i(p_i-1)$. If $\Delta = t$ this is ordinary full randomization.

The method of constrained randomization contained in this paper requires the construction of an operational sequence in which $\Delta = 1$ for the first and last treatment combinations in the sequence as well as for all adjacent treatment combinations. The existence and construction of such operational sequences present interesting graph theory problems whose formulations and solutions are found in this paper. This method of constrained randomization provides a basis for a statistical analysis utilizing the randomization model, which results in unbiased estimates of treatment effects and an unbiased estimate of experimental error.

1. Introduction. The experimental design considered in this paper is one in which treatments are applied to experimental units in a sequential manner, with results of one treatment being observed before application of the following treatment, and not all treatment combinations can be placed adjacent in the consecutive order. Methods of constrained randomization for 2^n factorials in blocks and for 2^{n-p} fractional replicates have been developed by Tiahrt and Weeks [6]. The general frame and definition of the problem may be found in this reference. However, these methods do not extend to p^n factorials for $p>2$ or, more generally, to a $p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ factorial. Methods of constrained randomization for arbitrary values of the p_i's and n_i's in the $p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ factorial are given in this paper.

Terminology and notations which will be used in further describing this generalized problem are now given.

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NOTATION. The phrases "treatment combination" and "treatment combinations" will be denoted by tc and tcs, respectively. The phrase "$p_1^{n_1} \cdots p_k^{n_k}$ factorial experiment" will be denoted by FE(k).

DEFINITION. Two tcs have *order of adjacency* Δ if the total number of changes in the factor levels in the two adjacent tcs is equal to Δ.

The order of adjacency, Δ, is given by the sum $\Delta = \sum_{s=1}^{k} \sum_{j=1}^{n_s} |x_{ij}^{(s)} - x_{i'j}^{(s)}|$, where tc i is denoted by $(x_{i1}^{(1)}, x_{i2}^{(1)}, \cdots, x_{in_1}^{(1)}, x_{i1}^{(2)}, x_{i2}^{(2)}, \cdots, x_{in_2}^{(2)}, \cdots, x_{i1}^{(k)}, x_{i2}^{(k)}, \cdots, x_{in_k}^{(k)})$ and where $x_{ij}^{(s)} \in \{0, 1, \cdots, p_s - 1\}$ for $s = 1, 2, \cdots, k$.

Restricting the value of Δ to $\Delta < t = \sum_{j=1}^{k} n_j(p_j - 1)$ in a FE(k) induces a compatibility condition on the sequence which prevents full randomization in arranging the sequence of treatment combinations required to conduct the experiment.

DEFINITION 2. An *operational sequence* is any sequence of the tcs which satisfies the compatibility condition imposed on the design by the experimenter and/or the experimental process.

The problem thus becomes one of constructing an experimental design which will provide a method of experimentation under the restriction of the compatibility condition and a means of statistically analyzing the data. This paper investigates the problem for arbitrary values of the p_i's, n_i's and Δ.

DEFINITION 3. An operational sequence is said to be a *cycle* if each tc appears in the sequence and if the first and last tcs in the sequence, when placed in adjacent positions, have order of adjacency Δ. Unless otherwise stated, a cycle for a FE(k) will contain $\prod_{i=1}^{k} p_i^{n_i}$ terms. (e.g., each tc will appear once in the operational sequence.)

A factorial is said to be *cyclic* w.r.t. $\Delta \leqq k$ if there exists a cycle for the factorial which satisfies the compatibility condition $\Delta \leqq k$.

Methods of constructing operational sequences will be based on the results of the next section, which investigates the existence or nonexistence of a cycle for a given factorial and compatibility condition. Those theorems which establish the existence of a cycle also reveal a method of constructing the cycle and hence an operational sequence for the corresponding factorial.

2. Existence and Construction of Cycles.

Without loss of generality, the first term in each cycle will be taken as $(0, 0, \cdots, 0)$, the low level of each factor.

THEOREM 1. *If p is odd, the p^n factorial is not cyclic* w.r.t. $\Delta = 1$.

PROOF OF THEOREM 1. Let $p = 2k + 1$ where $k \in I^+$. If a cycle exists for this factorial, it must contain $(2k + 1)^n$ tcs and the last tc, $(c_1, c_2, \cdots, c_n)$, must satisfy the condition $\sum_{i=1}^{k} |c_i| = 1$. Moving through the cycle from one point to the next is equivalent to the adding ± 1 to one of the n components. Since there are an odd number of points in the cycle, this must be done an even number of times. However, this would mean that the last point, $(c_1, c_2, \cdots, c_n)$

is such that $\sum_{i=1}^{n} c_i$ is an even number. This contradicts the requirement so the construction of such a cycle is impossible.

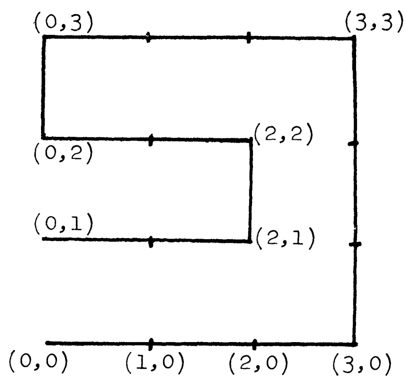
It should be noted for future reference that the proof was based solely on the fact that a cycle cannot contain an odd number of terms if $\Delta = 1$.

The following notations will be used in proving subsequent theorems. The symbolism $(a, b)s(a, b + k)$ will denote the sequence $(a, b), (a, b + 1), (a, b + 2), \cdots, (a\, b + k)$ and $(a, b + k)s(a, b)$ will denote the sequence $(a, b + k), (a, b + k - 1), (a, b + k - 2), \cdots, (a, b)$. The notations $(a + k, b)s(a, b)$ and $(a, b)s(a + k, b)$ are defined in a similar manner. Small Greek letters will be used to denote row vectors. If $\alpha = (a_1, a_2, \cdots, a_n)$ and $\beta = (b_1, b_2, \cdots, b_m)$, (α, β) is defined by $(\alpha, \beta) = (a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m)$.

THEOREM 2. *If p is even, the p^n factorial is cyclic* w.r.t. $\Delta < t$ *where* $t > 1$.

PROOF OF THEOREM 2. Let $p = 2k$ where $k \in I^+$. It suffices to show that the theorem holds for $\Delta = 1$. This will be done by mathematical induction.

(a) If $n = 2$, each tc is denoted by (a, b) where $a, b \in \{0, 1, \cdots, 2k - 1\}$. A cycle for this factorial is the sequence $(0, 0)s(2k - 1, 0), (2k - 1, 1)s(2k - 1, 2k - 1), (2k - 2, 2k - 1)s(0, 2k - 1), (0, 2k - 2)s(2k - 2, 2k - 2), (2k - 2, 2k - 3)s(0, 2k - 3), (0, 2k - 4)s(2k - 2, 2k - 4), \cdots, (0, 2)s(2k - 2, 2), (2k - 2, 1)s(0, 1)$. Notice for the last tc to be $(0, 1)$, p must be even as it is here. The geometrical interpretation of a cycle for a 4^2 factorial is as shown.



(b) Suppose the theorem is true for $n = r$. Then each point in the cycle is represented in r-dimensional space by an r-tuple of the form $(a_1, a_2, \cdots, a_r)$ where $a_i \in \{0, 1, \cdots, 2k - 1\}$. This same point has a representation in $(r + 1)$-dimensional space as $(a_1, a_2, \cdots, a_r, 0)$. Since $\Delta = 1$, a pair of adjacent points in the cycle will have a representation in $(r + 1)$-dimensional space as $(a_1, a_2, \cdots, a_q, \cdots, a_r, 0), (a_1, a_2, \cdots, a_q \pm 1, \cdots, a_r, 0)$. Since there are an even number of points in the cycle, there are $(2k)^r/2$ disjoint pairs of adjacent points where the first pair consists of the first two points in the cycle and the remaining pairs consist of the remaining points taken two at a time in consecutive order. To construct a cycle in $(r+1)$-dimensional space, replace each pair $(a_1, a_2, \cdots, a_q, \cdots, a_r, 0)$, $(a_1, a_2, \cdots, a_q, \pm 1, \cdots, a_r, 0)$ by the following sequence: $(a_1, a_2, \cdots, a_q, \cdots, a_r, 0)$, $(a_1, a_2, \cdots, a_q, \cdots, a_r, 1)$, $\cdots$, $(a_1, a_2, \cdots, a_q, \cdots, a_r, 2k-1)$, $(a_1, a_2, \cdots, a_q \pm 1, \cdots, a_r, 2k-1)$, $(a_1, a_2, \cdots, a_q \pm 1, \cdots, a_r, 2k-2)$, $\cdots$, $(a_1, a_2, \cdots, a_q \pm 1, \cdots, a_r, 0)$. Clearly this construction yields a cycle for the $(2k)^{r+1}$ factorial which satisfies the condition that $\Delta = 1$.

For future reference, notice that the last point in the cycle is $(0, 1, 0, 0, \cdots, 0)$.

Although it was shown in Theorem 1 that a p^n factorial is not cyclic w.r.t. $\Delta = 1$ if p is odd, a cycle for this factorial can be constructed if the condition that each tc can appear only once in the operational sequence is removed. With the freedom to repeat points wherever convenient, it is easy to see how any number of cycles could be constructed.

However, due to the time and economic factors involved, the experimenter may wish to have available a cycle in which repetitions are kept to a minimum.

The next theorem shows that such a cycle can be constructed by repeating one tc once. Information which is crucial to the proof of the theorem is contained in the following lemma.

LEMMA 1. *In a $(2k+1)^n$ factorial with $\Delta = 1$, a noncyclic operational sequence can be constructed in such a way that the first point in the sequence is $(0, 0, \cdots, 0)$, the last point is $(2k, 2k, \cdots, 2k)$, and each of the other $(2k+1)^n - 2$ points appears exactly once.*

PROOF OF LEMMA 1. The proof is by mathematical induction in a manner similar to that used for Theorem 2.

A theorem concerning a cycle with one repeated tc now follows.

THEOREM 3. *If p is odd and $\Delta = 1$, a cycle for the p^n factorial can be constructed by repeating one point once.*

PROOF OF THEOREM 3. Let $p = 2k + 1$ where $k \in I^+$. To illustrate how the cycle will be constructed, consider the case where $k = 1$ and $n = 3$. The cycle for this factorial, starting at $(1, 1, 0)$ and ending at $(0, 1, 0)$, is as follows: $(1,1,0)$, $(1,1,1)$, $(1,1,2)$, $(0,1,2)$, $(0,2,2)$, $(1,2,2)$, $(2,2,2)$, $(2,1,2)$, $(2,0,2)$, $(1,0,2)$, $(0,0,2)$, $(0,0,1)$, $(1,0,1)$, $(2,0,1)$, $(2,1,1)$, $(2,2,1)$, $(1,2,1)$, $(0,2,1)$, $(0,1,1)$, $(0,1,0)$, $(0,2,0)$, $(1,2,0)$, $(2,2,0)$, $(2,1,0)$, $(2,0,0)$, $(1,0,0)$, $(0,0,0)$, $(0,1,0)$. In this sequence, each tc has a representation as (a_1, a_2, a_3) and the sequences in the 2-dimensional subspaces $a_3 = 0$, $a_3 = 1$, and $a_3 = 2$ can be thought of as cycles in which the $(1, 1, a_3)$ point has been omitted. The repeated point is $(0, 1, 0)$.

To prove the theorem in the general case, first consider a $(2k+1)^2$ factorial. If the point $(1, 1)$ is omitted, a cycle for this factorial is $(0, 0)s(2k, 0)$, $(2k, 1)s(2k, 2k)$, $(2k-1, 2k)s(0, 2k)$, $(0, 2k-1)s(2k-1, 2k-1)$, $(2k-1, 2k-2)s(2k-1, 1)$, $(2k-2, 1)s(2k-2, 2k-2)$, $\cdots$, $(2, 2)s(0, 2)$, $(0, 1)$.

Notice that this sequence has been constructed in such a way that any point in the form $(0, a)$ where a is positive and even, is followed by the point $(0, a - 1)$. Hence the last point is $(0, 1)$. If the sequence was reversed, the cycle would omit $(1, 1)$, and start at $(0, 1)$.

In completing the proof, the following notation will be used. If β is an $(n-2)$-dimensional constant vector, $(a, b, \beta)c(d, f, \beta)$ will represent a cycle in the 2-dimensional subspace of an n-dimensional space which starts at (a, b, β) ends at (d, f, β), and omits the point $(1, 1, \beta)$.

Now let $\alpha_1, \alpha_2, \cdots, \alpha_q$, where $q = (2k+1)^{n-2}$, be a sequence in $(n-2)$-dimensional space which satisfies Lemma 1 for a $(2k+1)^{n-2}$ factorial.

The sequence which satisfies the conditions of Theorem 3 is $(1, 1, \alpha_1)$, $(1, 1, \alpha_2)$, $(1, 1, \alpha_3)$, $\cdots$, $(1, 1, \alpha_q)$, $(0, 1, \alpha_q)c(0, 0, \alpha_q)$, $(0, 0, \alpha_{q-1})c(0, 1, \alpha_{q-1})$, $(0, 1, \alpha_{q-2})c(0, 0, \alpha_{q-2})$, $\cdots$, $(0, 1, \alpha_1)c(0, 0, \alpha_1)$, $(0, 1, \alpha_1)$.

Notice that there are an odd number of α's and that the sequences $(a, b, \alpha_t)c(d, t, \alpha_t)$ have been constructed in such a way that if t is odd, $b = 1$ and $a = d = f = 0$. This justifies having the sequence $(0, 1, \alpha_1)c(0, 0, \alpha_1)$ in the last 2-dimensional subspace. The repeated point, $(0, 1, \alpha_1)$, is compatible with the first point, $(1, 1, \alpha_1)$.

Although a p^n factorial is not cyclic w.r.t. $\Delta = 1$ if p is odd and no point is repeated, this is not the case for other values of Δ. The following corollary deals with such cases.

COROLLARY 1. *If p is odd, the p^n factorial is cyclic* w.r.t. $\Delta < t$ *where* $t > 2$.

PROOF OF COROLLARY 1. It suffices to show that the corollary holds for $\Delta \leqq 2$. If the point $(0, 1, \alpha_1)$ is removed as the last point in the sequence constructed in Theorem 3, the resultant sequence is clearly cyclic w.r.t. $\Delta \leqq 2$.

It should be noted that the resultant sequence is a noncyclic operational sequence having order of adjacency $\Delta = 1$, $(1, 1, 0, 0, \cdots, 0)$ as its first term, and $(0, 0, \cdots, 0)$ as its last term. By reversing the terms in this sequence, an operational sequence is generated which has $(0, 0, \cdots, 0)$ as its first term and $(1, 1, 0, 0, \cdots, 0)$ as its last term.

Although these last remarks were not germane to the proof of the corollary, they will be needed in extending these results to a FE(k).

THEOREM 4. *If $\sum_{i=1}^{k} p_i$ is odd, the* FE(k) *is not cyclic* w.r.t. $\Delta = 1$.

PROOF OF THEOREM 4. First note that $p_i^{n_i}$ is odd if and only if p_i is odd. Hence $\prod_{i=1}^{k} p_i^{n_i}$ is odd if and only if $\prod_{i=1}^{k} p_i$ is odd. Thus, if a cycle for this factorial exists, it must contain an odd number of terms. However, it was shown in Theorem 1, independent of the form of the factorial whose factor levels yield an odd number of tcs, that a cycle consisting of an odd number of terms cannot be constructed if $\Delta = 1$.

For convenience, the following notation will be used in the proofs of the theorems which follow. If α is a constant vector and $\beta_1, \beta_2, \cdots, \beta_n$ is a sequence having order of adjacency $\Delta = 1$, $(\alpha, \beta_1)C(\alpha, \beta_n)$ will denote the sequence $(\alpha, \beta_1), (\alpha, \beta_2), \cdots, (\alpha, \beta_n)$. The order of adjacency for this sequence is also $\Delta = 1$. $(\beta_1, \alpha)C(\beta_n, \alpha)$ will denote the sequence $(\beta_1, \alpha), (\beta_2, \alpha), \cdots, (\beta_n, \alpha)$.

THEOREM 5. *If p_i is even for $i = 1, 2, \cdots, k$, the* FE(k) *is cyclic* w.r.t. $\Delta = 1$.

PROOF OF THEOREM 5. The proof is by mathematical induction.

(a) Let $k = 2$. It was shown in Theorem 2 that if p_1 and p_2 are even, the factorials $p_1^{n_1}$ and $p_2^{n_2}$ are each cyclic w.r.t. $\Delta = 1$. Let $\alpha_1, \alpha_2, \cdots, \alpha_q$, where $q = p_1^{n_1}$ and $\alpha_q = (0, 1, 0, 0, \cdots, 0)$ by a cycle for the $p_1^{n_1}$ factorial. Let $\beta_1, \beta_2, \cdots, \beta_m$, where $m = p_2^{n_2}$ be a cycle for the $p_2^{n_2}$ factorial. Then a cycle for the $p_1^{n_1}p_2^{n_2}$ factorial is given by the sequence $(\alpha_1, \beta_1)C(\alpha_1, \beta_m), (\alpha_2, \beta_m)C(\alpha_2, \beta_1), (\alpha_3, \beta_1)C(\alpha_3, \beta_m), (\alpha_4, \beta_m)C(\alpha_4, \beta_1), \cdots, (\alpha_q, \beta_m)C(\alpha_q, \beta_1)$.

Notice that the sequence was constructed in such a way that if a is even in the subsequence $(\alpha_a, \beta_b)C(\alpha_a, \beta_f)$, $b = m$ and $f = 1$. Hence the last point must be $(0, 1, 0, 0, \cdots, 0)$ and is compatible with the first point $(0, 0, \cdots, 0)$, and the sequence is a cycle, as claimed.

(b) Assume the theorem holds for $k = t$. Then the FE(t) is cyclic w.r.t. $\Delta = 1$. Now consider the FE($t + 1$). Replacing the cycle for the $p_1^{n_1}$ factorial in part (a) by the cycle for the FE(t), and the cycle for the $p_2^{n_2}$ factorial in part (a) by the cycle for the $p_{t+1}^{n_{t+1}}$ factorial which, when reversed, starts at $(0, 1, 0, 0, \cdots, 0)$, and ends at $(0, 0, \cdots, 0)$, the construction described in part (a) yields a cycle for the FE($t + 1$). That the last point is $(0, 1, 0, 0, \cdots, 0)$ again follows from the fact that there are an even number of terms in the cycle for the FE(t).

THEOREM 6. *If p_1 is even and p_2 is odd, the* FE(t) *is cyclic* w.r.t. $\Delta = 1$.

PROOF OF THEOREM 6. It was shown in Theorem 3 that a noncyclic operational sequence having order of adjacency $\Delta = 1$ exists for a $p_2^{n_2}$ factorial. Let this operational sequence be denoted by $\beta_1, \beta_2, \cdots, \beta_m$, where $m = p_2^{n_2}$ and $\beta_m = (1, 1, 0, 0, \cdots, 0)$. Let a cycle for the $p_1^{n_1}$ factorial again be denoted by the α's in Theorem 5. Then a cycle for the FE(2) is $(\alpha_1, \beta_1)C(\alpha_1, \beta_m), (\alpha_2, \beta_m)C(\alpha_2, \beta_1), (\alpha_3, \beta_1)C(\alpha_3, \beta_m), \cdots, (\alpha_q, \beta_m)C(\alpha_q, \beta_1)$.

Here again, the last term in the sequence is $(0, 1, 0, 0, \cdots, 0)$. Since the last term is compatible with the first, the sequence is a cycle, as claimed.

It was established in Theorem 4 that if $p_1 p_2$ is odd the FE(2) is not cyclic w.r.t. $\Delta = 1$. However, as in the case for the p^n factorial where p is odd, it would be desirable to have a cycle for this factorial which repeats a minimum number of points. The next theorem deals with this problem.

THEOREM 7. *If p_1 and p_2 are both odd and $\Delta = 1$, a cycle for the* FE(2) *can be constructed by repeating one point once.*

PROOF OF THEOREM 7. Let $\beta_1, \beta_2, \cdots, \beta_m$ be the operational sequence of β's described in Theorem 6. The order of adjacency for this sequence is $\Delta = 1$ and the second from the last term is $\beta_{m-1} = (0, 1, 0, 0, \cdots, 0)$. Let $\alpha_1, \alpha_2, \cdots, \alpha_q$, where $q = p_1^{n_1}$ be a similar operational sequence for the $p_1^{n_1}$ factorial. A cycle for the FE(2) is $(\alpha_1, \beta_1)C(\alpha_1, \beta_{m-1})$, $(\alpha_2, \beta_{m-1})C(\alpha_2, \beta_1)$, $(\alpha_3, \beta_1)C(\alpha_3, \beta_{m-1}) \cdots$ $(\alpha_q, \beta_1)C(\alpha_q, \beta_{m-1})$, $(\alpha_q, \beta_m)C(\alpha_1, \beta_m)$, (α_1, β_{m-1}). The repeated point is the last point $(\alpha_1, \beta_{m-1}) = (0, 0, \cdots, 0; 0, 1, 0, 0, \cdots, 0)$ where the semicolon separates the components of α from the components of β.

It should be observed that the second from the last term is $(0, 0, \cdots, 0; 1, 1, 0, 0, \cdots, 0)$. This will be needed in the next theorem, which is a generalization of Theorem 7.

THEOREM 8. *If p_i is odd for $i = 1, 2, \cdots, k$ and $\Delta = 1$, a cycle for* FE(k) *can be constructed by one repetition of the point* $(0, 0, \cdots, 0; 0, 1, 0, 0, \cdots, 0)$, *where the semicolon separates the last n_k components from the preceding ones. The point preceding the last (or repeated) point can be taken as* $(0, 0, \cdots, 0; 1, 1, 0, 0, \cdots, 0)$.

PROOF OF THEOREM 8. The proof is by mathematical induction.

(a) Theorem 7 establishes the case for $k = 2$.

(b) Suppose the theorem holds for $k = r$. Then there exists a cycle for the $p_2^{n_2}p_3^{n_3} \cdots p_r^{n_r}p_{r+1}^{n_{r+1}}$ factorial which repeats one point. Denote this cycle by $\beta_1, \beta_2, \cdots, \beta_t, \beta_{t+1}$ where $t = \prod_{i=2}^{r+1} p_i^{n_i}$, $\beta_{t+1} = (0, 0, \cdots, 0; 0, 1, 0, 0, \cdots)$ and $\beta_t = (0, 0, \cdots, 0; 1, 1, 0, 0, \cdots, 0)$. Let $\alpha_1, \alpha_2, \cdots, \alpha_q$ be the operational sequence used in Theorem 7 for the $p_1^{n_1}$ factorial. Replacing the points β_{m-1} and β_m in Theorem 7 by β_{t-1} and β_t respectively, the construction of a cycle for the FE$(r + 1)$ is identical to the construction shown in Theorem 7 except that the last point is replaced by (α_1, β_{t+1}).

To complete the discussion of the problem of constructing cycles for a FE(k), consider the case where $\prod_{i=1}^{k} p_i$ is even and some of the p_i's are odd. Since the statistical design is essentially unchanged by permuting the $p_i^{n_i}$'s we can assume that those p_i's which are even occur first in the notation $p_1^{n_1}p_2^{n_2} \cdots p_k^{n_k}$.

THEOREM 9. *If p_i is even for $i = 1, 2, \cdots, r$ and p_j is odd for $j = r + 1, r + 2, \cdots, k$, the* FE$(k)$ *is cyclic* w.r.t. $\Delta = 1$.

PROOF OF THEOREM 9. Let $\alpha_1, \alpha_2, \cdots, \alpha_h$ where $h = \prod_{i=1}^{r} p_i^{n_i}$ by a cycle for the FE(r). That such a cycle exists for $\Delta = 1$ was established in Theorem 5. Let $\beta_1, \beta_2, \cdots, \beta_q$ where $q = \prod_{i=r+1}^{k} p_i^{n_i}$ be the sequence constructed in Theorem 8 omitting the last term, so that there is no repeated point. Replacing the α's and β's in Theorem 6 with the α's and β's just described, the construction of the desired cycle is identical to the construction shown in Theorem 6.

This completes the problem of constructing cycles for a FE(k) with $\Delta = 1$. The following corollary treats the case of constructing cycles for this factorial if no points are to be repeated and the restriction that $\Delta = 1$ is relaxed.

COROLLARY 2. *Any* FE(k) *is cyclic* w.r.t. $\Delta < t$ *where* $t > 2$.

PROOF OF COROLLARY 2. It suffices to show that the corollary holds for $\Delta = 2$. If $\prod_{i=1}^{k} p_i$ is even, Theorem 9 establishes the desired result. If $\prod_{i=1}^{k} p_i$ is odd, the result follows from considering the sequence constructed in Theorem 8 with the last term (the repeated point) omitted.

This completes the proof of the corollary.

In order to construct a cycle for a FE(k) with $\Delta = 1$ and $\prod_{i=1}^{k} p_i$ odd, it was necessary to repeat at least one point. An operational sequence was defined without any requirement of cyclicness, and consequently no repetitions are required in order to construct an operational sequence.

THEOREM 10. *An operational sequence having no repeated terms exists for any* FE(k) *having compatibility condition* $\Delta = 1$.

PROOF OF THEOREM 10. If $\prod_{i=1}^{k} p_i$ is even, Theorem 9 establishes the desired result. If $\prod_{i=1}^{k}$ is odd, the operational sequence constructed in Theorem 8 with the last (or repeated) term omitted is an operational sequence having $\Delta = 1$ and no repeated points.

3. Constrained randomization for a FE(k) **in a randomized nonconsecutive block design.** This section presents methods of constrained randomization for a FE(k) in randomized complete block designs where each block is independent of the others.

Constrained randomization for a FE(k) in nonconsecutive replicates is performed by the following procedure.

(1) A cycle for the factorial under consideration is constructed.

(2) For each replication, randomly assign n_1 factor names to the first n_i components $x_1, x_2, \cdots, x_{n_1}$ of each treatment combination in the cycle, randomly assign n_2 factor names to the next n_2 components $x_{n_1+1}, x_{n_1+2}, \cdots, x_{n_1+n_2}$ of each tc in the cycle, etc.

(3) For each replication, pick an integer at random from the set $\{1, 2, \cdots, P\}$ where P is the number of tcs in the cycle obtained in step (1).

Let K be the integer so chosen. From the sequence $\alpha_1, \alpha_2, \cdots, \alpha_P$ obtained in step (2), construct the sequence $\alpha_K, \alpha_{K+1}, \cdots, \alpha_P, \alpha_1, \alpha_2, \cdots, \alpha_{K-1}$.

The resulting cycle is the random sequence of tcs which was desired.

EXAMPLE 1. As an example of this method, consider the following randomization for the replication of a $2^2 2^3$ factorial with $\Delta = 1$.

(1) Since $2 \cdot 3 = 6$ is even, a cycle containing 36 tcs, none of which are repeated, is needed. Such a cycle, which can be constructed using the method discussed in the proof of Theorems 2, 3 and 6, is shown. (These, and the subsequent cycles, should be read consecutively from the bottom of one column to the top of the next one.)

0000	1011	1100	0111
0010	1001	1110	0101
0020	1002	1120	0102
0021	1012	1121	0112
0022	1022	1122	0122
0012	1021	1112	0121
0002	1020	1102	0120
0001	1010	1101	0110
0011	1000	1111	0100

(2) Suppose the random assignment in step (2) replaces the tc (x_1, x_2, x_3, x_4) by the tc (x_1, x_2, x_4, x_3). The resultant sequence of tcs is

0000	1011	1100	0111
0001	1010	1101	0110
0002	1020	1102	0120
0012	1021	1112	0121
0022	1022	1122	0122
0021	1012	1121	0112
0020	1002	1120	0102
0010	1001	1110	0101
0011	1000	1111	0100

(3) Suppose the random number chosen in step (3) is 31. Then the new sequence of tcs moves the 31st term of the intermediate sequence into the first position. This new sequence is

0121	0012	1021	1112
0122	0022	1022	1122
0112	0021	1012	1121
0102	0020	1002	1120
0101	0010	1001	1110
0100	0011	1000	1111
0000	1011	1100	0111
0001	1010	1101	0110
0002	1020	1102	0120

Note that the last sequence in the example is again a cycle with $\Delta = 1$.

The theorem which follows will show that the randomization procedure will always preserve this property.

THEOREM 11. *The sequence of* tcs *resulting from constrained randomization of a cycle having order of adjacency* $\Delta = k$ $(\leqq k)$ *is a cycle having the same order of adjacency.*

PROOF OF THEOREM 11. Let the cycle constructed in step (1) be denoted by $\alpha_1, \alpha_2, \cdots, \alpha_P$. Step (2) is a rearrangement of the components of each tc and, therefore, leaves invariant the order of adjacency of any adjacent pair of tcs, α_i, and α_{i^+}, say, where $i^+ = (i+1) \bmod (P)$. This is a consequence of the fact that every tc in the sequence received identically the same assignment of factor names. Thus, x_{ir} and x_{i^+r}, the rth components of α_i and α_{i^+}, become the r'th components $x_{ir'}$ and $x_{i^+r'}$ of α_i and α_{i^+} after step (2). Hence $|x_{ir} - x_{i^+r}| = |x_{ir'} - x_{i^+r'}|$ and so $\sum_r |x_{ir} - x_{i^+r}| = \sum_{r'} |x_{ir'} - x_{i^+r'}|$. This holds for every pair of adjacent tcs in the sequence and thus the sequence obtained from step (2) is a cycle having the same order of adjacency as the cycle in step (1).

Step (3), as a cyclic permutation of the sequence obtained from step (2), preserves the relative position of the tcs in the cycle. Hence a pair of adjacent tcs are again adjacent after step (3) and so their order of adjacency is unchanged. Thus, the sequence of tcs resulting from the randomization procedure of a cycle having order of adjacency $\Delta \leqq k$ is a cycle having the same order of adjacency. This completes the proof.

A result which is basic to the development of the randomization model is presented in the following theorem.

THEOREM 12. *For a* FE(k), *let* $R = \prod_{i=1}^{k} n_i!$. *Then, over all distinct cycles obtained by the constrained randomization procedure, each* tc *appears R times in each position of the sequence if there are no repeated points. If* $\prod_{i=1} p_i^{n_i}$ *is odd and* $\Delta = 1$, *repeated* tcs *will occur 2R times in each position of the sequence.*

PROOF OF THEOREM 12. Let β and γ be two tcs in the cycle constructed in step (1) of the randomization process. Let β' and γ' be the tcs which result from applying step (2) to β and γ. Since step (2) assigns the same factor names to every tc in the cycle, $\beta' = \gamma'$ if and only if $\beta = \gamma$. Thus, if β and γ are distinct points, β' and γ' are also distinct and the number of distinct points is unchanged after applying step (2) of the randomization procedure.

Assuming each tc appears only once in the sequence, then over all cycles generated by applying step (3) to one of the R cycles obtained from step (2), each tc will appear once in each position. Hence in applying step (3) to all of the R cycles resulting from step (2), each tc will appear R times in each position of the cycle.

In the case where the cycle contains a repeated tc, over all cycles generated in applying step (3) to one of the R cycles obtained from step (2) the repeated tc will appear twice in each position. Applying this to all of the R cycles resulting from step (2), the repeated tc will appear $2R$ times in each position of the cycle.

4. The randomization model. Consider a FE(k) where $\prod_{i=1}^{k} p_i$ is even or $\Delta \leqq k$ $(k > 1)$, or both. The population of inference under the randomization model is the set of experimental units actually used, or a larger population from which experimental units are randomly chosen. Each tc appears in only one experimental unit in each replication since there are no repeated points for this factorial under the method of constrained randomization.

Let y_{ijk} denote the population response (conceptual yield) to tc k on experimental unit j in replication i. Let $N = \prod_{i=1}^{k} p_i^{n_i}$ and suppose there are r replications. Then $k = 1, 2, \cdots, N$; $j = 1, 2, \cdots, N$; and $i = 1, 2, \cdots, r$.

Using methods identical to those presented in [6], the following statistical properties may be obtained.

(1) An unbiased estimate of any factorial effect $x_1 x_2 \cdots x_n$ is given by $\sum_k \pi_k \bar{y}_{\cdot k}$, where $\sum_k \pi_k = 0$, and the value of π_k is determined by the value of k and the type of effect being estimated.

(2) The average variance of the estimate for $x_1 x_2 \cdots x_n$ is

$$(1/r)S^2[\textstyle\sum_k \pi_k^2 - \sum_{k,k\neq k'} \sum_{k'} \pi_k \pi_{k'}/(N-1)] \,.$$

(3) An unbiased estimate of S^2 may be found from the usual analysis of variance and tests for individual factor effects and interactions may also be obtained. When $\prod_{i=1}^{r} p_i$ is odd and $\Delta = 1$, the cycle which is obtained by the randomization process contains one repeated point. The repeated point is not randomly chosen from all the tcs available but is determined by the methods for construction of cycles as developed in the first section. Moreover, after the repeated point has been determined, the relative positions of the two experimental units to which the repeated tc is applied remain unchanged after step (2) in the randomization process.

This lack of randomness in the selection of tcs and experimental units deprives the corresponding model of many of the properties obtained in the previous sections where no tcs were repeated. However, unbiased estimates for factorial effects can still be obtained using a randomization model.

Since after step (2) in the randomization process the repeated tc is known, the expectations which yield the unbiased estimates are conditional expectations, the conditional event being that the repeated tc is the one observed.

Using the notations and definitions presented for the randomization model with no repeated points, one may write $y_{ijk} = \mu + b_i + t_k + (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot})$ where $i = 1, 2, \cdots, r,\ j = 1, 2, \cdots, N+1,\ k = 1, 2, \cdots, N$. As before, $\sum_i b_i = \sum_k t_k = 0$. The random variable δ_{ij}^k is defined by

$$\begin{aligned} \delta_{ij}^k &= 1 \quad \text{if tc } k \text{ is an experimental unit } j \text{ of rep } i \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Denoting the repeated tc by k^*,

$$\sum_{k=1}^{N} \delta_{ij}^k = 1 , \qquad \text{and} \qquad \begin{aligned} \sum_{j=1}^{N+1} \delta_{ij}^k &= 2 \quad \text{if } k = k^* \\ &= 1 \quad \text{if } k \neq k^* . \end{aligned}$$

With y_{ik} the observed response of tc k in replication i, one has

$$\begin{aligned} y_{ik} = \sum_j \delta_{ij}^k y_{ijk} &= 2\mu + 2b_i + 2t_k + e_{ik} \quad &\text{if } k = k^* \\ &= \mu + b_i + t_k + e_{ik} \quad &\text{if } k \neq k^* \end{aligned}$$

where $e_{ik} = \sum_i \delta_{ij}^k(\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot})$. Note that $\sum_k e_{ik} = 0$.

Distributional properties of δ_{ij}^k and y_{ik} follow.

LEMMA 2. *Under the constrained randomization procedure for a* FE(k) *with* $\prod_{i=1}^{k} p_i$ *odd and* $\Delta = 1$,

(1)
$$\begin{aligned} E[\delta_{ij}^k \mid k^* \textit{ is the repeated point } (\text{rp})] &= \frac{2}{N+1} \quad &\textit{if } k = k^* \\ &= \frac{1}{N+1} \quad &\textit{if } k \neq k^* \end{aligned}$$

(2)
$$E[e_{ik} \mid k^* \textit{ is the } \text{rp}] = 0 .$$

PROOF OF LEMMA 2. All the probabilities are conditional probabilities determined by the constrained randomization procedure after step (2), in which the repeated tc, k^*, is observed.

PROOF OF (1). By Theorem 12, each nonrepeated tc appears on each experimental unit j with equal frequency and, therefore, P[tc k is on experimental unit $j \mid k \neq k^*] = 1/N+1$. That the repeated tc appears twice as frequently as each nonrepeated tc was also established in Theorem 12. Hence P[tc k is on experimental unit $j \mid k = k^*] = 2/N+1$.

$$E[\delta_{ij}^k \mid k^* \text{ is the rp}] = 1 \cdot P[\delta_{ij}^k = 1 \mid k^* \text{ is the rp}] = \frac{2}{N+1} \quad \text{if} \quad k = k^*$$
$$= \frac{1}{N+1} \quad \text{if} \quad k \neq k^* .$$

PROOF OF (2). Consider

$$\begin{aligned} E[e_{ik} \mid k^* \text{ is the rp}] &= \textstyle\sum_j E[\delta_{ij}^k \mid k^* \text{ is the rp}](\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot}) \\ &= \frac{2}{N+1} \textstyle\sum_j (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot}) \quad \text{if} \quad k = k^* \\ &= \frac{1}{N+1} \textstyle\sum_j (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot}) \quad \text{if} \quad k \neq k^* \\ &= 0 \quad \text{for all} \quad k . \end{aligned}$$

This completes the proof of the lemma.

The following theorem presents estimates for tcs.

THEOREM 13. *Unbiased estimates for t_k are given by*

(1) $\hat{t}_{k^*} = \frac{N+1}{2N} \bar{y}_{\cdot k^*} - \bar{y}_{\cdot\cdot} \quad$ *if* $\quad k = k^*$

(2) $\hat{t}_k = \bar{y}_{\cdot k} + \frac{1}{2N} \bar{y}_{\cdot k^*} - \bar{y}_{\cdot\cdot} \quad$ *if* $\quad k \neq k^*$.

PROOF OF THEOREM 13. First consider $\bar{y}_{\cdot k}$.

$$\begin{aligned} \bar{y}_{\cdot k} = \frac{1}{r} \textstyle\sum_i y_{ik} &= 2\mu + 2t_{k^*} + \frac{1}{r} \textstyle\sum_i e_{ik^*} \quad \text{if} \quad k = k^* \\ &= \mu + t_k + \frac{1}{r} \textstyle\sum_i e_{ik} \quad \text{if} \quad k \neq k^* . \end{aligned}$$

Since by Lemma 2, $E(e_{ik} \mid k^* \text{ is the rp}) = 0$,

$$\begin{aligned} E[\bar{y}_{\cdot k} \mid k^* \text{ is the rp}] &= 2\mu + 2t_{k^*} \quad \text{if} \quad k = k^* \\ &= \mu + t_k \quad \text{if} \quad k \neq k^* . \end{aligned}$$

Now consider $\bar{y}_{\cdot\cdot}$.

$$\begin{aligned} \bar{y}_{\cdot\cdot} = \frac{1}{N} \textstyle\sum_k \bar{y}_{\cdot k} &= \frac{1}{N}\left[(N+1)\mu + \textstyle\sum_k t_k + t_{k^*} + \frac{1}{r} \textstyle\sum_k \textstyle\sum_i e_{ik}\right] \\ &= \frac{N+1}{N}\mu + \frac{1}{N} t_{k^*} . \end{aligned}$$

PROOF OF (1). For $k = k^*$

$$E[\hat{t}_{k^*}] = E\left[\frac{N+1}{2N}\bar{y}_{\cdot k^*} - \bar{y}_{\cdot\cdot} \,|\, k^* \text{ is the rp}\right]$$

$$= \frac{N+1}{2N}(2\mu + 2t_{k^*}) - \frac{N+1}{N}\mu - \frac{1}{N}t_{k^*} = \left(\frac{N-1}{N} - \frac{1}{N}\right)t_{k^*} = t_{k^*}.$$

PROOF OF (2). If $k \neq k^*$

$$E[\hat{t}_k] = E\left[\bar{y}_{\cdot k} + \frac{1}{2N}\bar{y}_{\cdot k^*} - \bar{y}_{\cdot\cdot}\middle| k^* \text{ is the rp}\right]$$

$$= (\mu + t_k) + \frac{1}{2N}(2\mu + 2t_{k^*}) - \frac{N+1}{N}\mu - \frac{1}{N}t_{k^*} = t_k.$$

This completes the proof of Theorem 13.

The following theorem gives estimates for factorial effects.

THEOREM 14. *Given that k^* is the repeated* tc, *an unbiased estimate of any main effect or interaction $X_1 X_2 \cdots X_n$ where $X_1 X_2 \cdots X_n = \sum_k \pi_k t_k$ and $\sum_k \pi_k = 0$, is $\sum_k \pi_k \bar{y}_{\cdot k} - \frac{1}{2}\pi_{k^*}\bar{y}_{\cdot k^*} = \sum_k \pi_k \hat{t}_k$.*

PROOF OF THEOREM 14. It need only be shown that the contrast of the $\hat{t}_k$'s is $\sum_k \pi_k \bar{y}_{\cdot k} - \frac{1}{2}\pi_{k^*}\bar{y}_{\cdot k^*}$. Thus consider

$$\sum_k \pi_k \hat{t}_k = \pi_{k^*}\left(\frac{N+1}{2N}\bar{y}_{\cdot k^*} - \bar{y}_{\cdot\cdot}\right) + \sum_{k,k\neq k^*}\pi_k\left(\bar{y}_{\cdot k} + \frac{1}{2N}\bar{y}_{\cdot k^*} - \bar{y}_{\cdot\cdot}\right)$$

$$= \frac{1}{2N}(\pi_{k^*} + \sum_{k,k\neq k^*}\pi_k)\bar{y}_{\cdot k^*}$$

$$- (\pi_{k^*} + \sum_{k,k\neq k^*} k_k)\bar{y}_{\cdot\cdot} + \tfrac{1}{2}\pi_{k^*}\bar{y}_{\cdot k^*} + \sum_{k,k\neq k^*}\pi_k\bar{y}_{\cdot k}$$

$$= (\sum_k \pi_k)\left(\frac{1}{2N}\bar{y}_{\cdot k^*} - \bar{y}_{\cdot\cdot}\right) - \tfrac{1}{2}\pi_{k^*}\bar{y}_{\cdot k^*} + \sum_k \pi_k \bar{y}_{\cdot k}$$

$$= \sum_k \pi_k \bar{y}_{\cdot k} - \tfrac{1}{2}\pi_{k^*}\bar{y}_{\cdot k^*}.$$

Due to the randomization constraints imposed on the design, variance estimates for this case have not been obtained.

Extensions. Preliminary investigations indicate that the constrained randomization procedures contained in this paper can be used to design confounding schemes for simple p^n factorials (for instance, a 3^3 factorial) under the restriction that $\Delta \leqq t$ for some values of t.

The possibility of extending these results to confounding schemes for a FE(k) which is restricted by the condition $\Delta \leqq t$ is open to further study.

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