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# DISCRETE SEQUENTIAL SEARCH FOR ONE OF MANY OBJECTS

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Suppose $N$ objects are hidden multinomially in $m$ boxes, where $m$ is known and $N$ is random. The boxes are to be searched sequentially. Associated with a search of box $k$ is a cost $c_k > 0$ and a conditional probability $\alpha_k$ of finding a specific object in box $k$, given that it is hidden there. An optimal strategy is one which minimizes the total expected cost required to find at least one object. If $N$ has a positive-Poisson distribution, then an optimal strategy is shown to take a simple form. Conversely, if for all possible $\{c_k\}$ and $\{\alpha_k\}$ an optimal strategy takes this simple form, then $N$ has a positive-Poisson distribution.

**1. Introduction and summary.** Let $N$ objects be hidden in $m \geqq 3$ boxes where $m$ is known and $N$ is unknown. The quantity $N$ has a prior probability distribution $\mathbf{p} = (p_1, p_2, \cdots)$ where $p_n = P(N = n)$ and $\sum_{n=1}^{\infty} p_n = 1$. If $N_k$ denotes the number of objects in box $k$, let the conditional distribution of $\mathbf{N} = (N_1, \cdots, N_m)$ given $N = n$ be multinomial with parameters $n$ and $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_m)$, where $\boldsymbol{\pi}$ is known. Without loss of generality assume that $\boldsymbol{\pi} > 0$, i.e. that $\pi_i > 0$ for all $i$. Associated with box $k$ are constants $c_k$ and $\alpha_k$, where $c_k > 0$ and $0 \leqq \alpha_k \leqq 1$. Each time box $k$ is searched, the cost $c_k$ is paid and, if there are $n_k$ objects there, the probability of finding at least one object is $1 - (1 - \alpha_k)^{n_k}$. (A box may be searched more than once.) The goal is to search the boxes so that the expected total cost expended to find at least one object is minimized.

Many similar models, but treating only the case $p_1 = 1$, have been studied in the literature. The interested reader is referred to Chew (1967), Kadane (1971), Ross (1969) and Sweat (1970). A fairly recent bibliography of the theory of search and reconnaissance appears in Pollock (1971).

The foregoing model can arise in the following manner: There are $N$ objects in the system; each object is located in box $k$ with probability $\pi_k$ independently of the location of other objects; when box $k$ is searched, each object in the box is found with probability $\alpha_k$ independently of the others. A consequence of this model is that if box $k$ is searched and no object is found, then, by Bayes Theorem, the number $N$ of objects in the system has a new distribution $\mathbf{p}^{(k)}$ and the posterior conditional distribution of $\mathbf{N} = (N_1, \cdots, N_m)$, given $N = n$, is still multinomial,

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but with parameters $n$ and $\boldsymbol{\pi}^{(k)} = (\pi_1{}^{(k)}, \cdots, \pi_m{}^{(k)})$ given by

$$\pi_j{}^{(k)} = \frac{(1 - \delta_{jk}\alpha_k)\pi_j}{1 - \alpha_k \pi_k} \tag{1.1}$$

where $\delta$ is the Kronecker delta. Hence, the new state of the system, assuming an object has not been found, can be represented by the ordered pair $(\mathbf{p}^{(k)}, \boldsymbol{\pi}^{(k)})$; the constants $\mathbf{c} = (c_1, \cdots, c_m)$ and $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_m)$ do not change.

A *searching sequence* is a sequence $\mathbf{s} = (s_1, s_2, \cdots)$ which calls for a search of box $s_t$ at *stage* $t$ (assuming no object has yet been found). A searching sequence is *optimal* from an *initial state* $(\mathbf{p}, \boldsymbol{\pi})$ if it minimizes among all searching sequences the total expected cost to find at least one object. A *strategy* is a function which associates with each initial state a searching sequence. A strategy is *optimal* if it associates with each initial state an optimal searching sequence.

We can now formulate the discrete sequential search problem as a dynamic programming problem and show the existence of a stationary optimal strategy.

Formally, a *dynamic programming problem* is an ordered quadruple $(Q, A, \mu, \rho)$ where $Q$ and $A$ are nonempty Borel sets, $\mu$ is a regular conditional probability on $Q$ given $Q \times A$, and $\rho$ is a Baire function on $Q \times A \times Q$. The set $Q$ is the set of *states*, $A$ is the set of *actions* available at each state, $\mu(q' \mid q, a)$ is the probability of moving to state $q'$, given that we are in state $q$ and take action $a$, and $\rho(q, a, q')$ is the *return* received when moving to state $q'$ from state $q$ after taking action $a$. The process is repeated from the new state $q'$. A *random strategy* is a sequence $\mathbf{z} = (z_1, z_2, \cdots)$ where $z_i$ is a conditional distribution over $a$, given the history of the process up to the $i$th stage. The *risk* of a random strategy from a given state $q$ is the negative of the expected total return into the infinite future. An *optimal random strategy* is a random strategy for which no random strategy has smaller risk from any state. A *stationary strategy* is a random strategy for which there exists a function $v: Q \to A$ such that each $z_i$ assigns probability one to $v(q_i)$. That is, a stationary strategy prescribes for each state a given action to be taken regardless of the history, and is nonrandom.

To formulate the searching problem as a dynamic programming problem, take $Q = \{q_0\} \cup \{(\mathbf{p}, \boldsymbol{\pi}) : \sum_{n=1}^{\infty} p_n = 1, p_n \geqq 0 \text{ for all } n, \sum_{k=1}^{n} \pi_k = 1, \pi_k \geqq 0 \text{ for all } k\}$, where $q_0$ is the state in which an object has been found. Take $A = \{1, 2, \cdots, m\}$, where action $k$ is associated with searching box $k$. For $q \neq q_0$, $\rho(q, k, q') = -c_k$ while $\rho(q_0, k, q') = 0$. From state $(\mathbf{p}, \boldsymbol{\pi})$ action $k$ either moves us to state $q_0$ or to the state $(\mathbf{p}^{(k)}, \boldsymbol{\pi}^{(k)})$ where $\boldsymbol{\pi}^{(k)}$ is given by (1.1) and $\mathbf{p}^{(k)}$, by Bayes Theorem, is given by

$$p_n{}^{(k)} = \frac{p_n(1 - \alpha_k \pi_k)^n}{\sum_j p_j(1 - \alpha_k \pi_k)^j} \, . \tag{1.2}$$

The transition probabilities $\mu$ are defined by

$$\mu(q_0 \mid (\mathbf{p}, \boldsymbol{\pi}), k) = 1 - \textstyle\sum_{n=1}^{\infty} (1 - \alpha_k \pi_k)^n p_n \, ,$$

$$\mu((\mathbf{p}^{(k)}, \boldsymbol{\pi}^{(k)}) \mid (\mathbf{p}, \boldsymbol{\pi}), k) = \textstyle\sum_{n=1}^{\infty} (1 - \alpha_k \pi_k)^n p_n \, , \qquad \mu(q_0 \mid q_0, k) = 1 \, .$$

Hence the search problem is a dynamic programming problem with finite action space and nonpositive return function. Therefore, by Theorem 9.1 (b) of Strauch (1966), there exists a stationary optimal strategy. Henceforth, the word *strategy* will mean stationary strategy. Given an initial state $q = (\mathbf{p}, \boldsymbol{\pi})$, a strategy assigns a searching sequence $\mathbf{s} = (s_1, s_2, \cdots)$. The risk of $\mathbf{s}$ from state $q = (\mathbf{p}, \boldsymbol{\pi})$ is denoted by $r_q(s_1, s_2, \cdots)$. If, given a state $q$, there exists an optimal strategy for which $s_1 = k$, we say *in state $q$ it is optimal to search box $k$.*

The special case in which there is only one object in the system ($p_1 = 1$) is of particular interest. In this case, it is well known (see Black (1965), Blackwell (1962), Matula (1964)) that an optimal strategy is determined by the following surprisingly simple rule: At each stage, it is optimal to search a box $\hat{k}$ if $\alpha_k \pi_k / c_k$ is maximized by $k = \hat{k}$, where $\pi_k$ is the current probability assigned to box $k$.

This rule is very easy to compute. At each stage one finds the maximum of $m$ quantities, the $k$th of which is a function of the current parameters associated with the $k$th box. More generally, a distribution $\mathbf{p}$ has a *separable rule* if there exists a function $g_{\mathbf{p}}$ such that for all $\boldsymbol{\alpha}$ $(0 \leqq \boldsymbol{\alpha} \leqq 1)$, all $\boldsymbol{\pi} > 0$, and all $\mathbf{c} > 0$, in state $(\mathbf{p}, \boldsymbol{\pi})$ it is optimal to search box $\hat{k}$ when the function $g_{\mathbf{p}}(\alpha_k, \pi_k, c_k)$ is maximized by $k = \hat{k}$. The function $g_{\mathbf{p}}$ is called the *test function*. Thus, for $\mathbf{p} = (1, 0, 0, \cdots)$ the test function $\alpha_k \pi_k / c_k$ determines a separable rule. Note that having a separable rule is a characteristic of a distribution in the context of a large class of dynamic programming problems, namely, as $\boldsymbol{\alpha}$, $\boldsymbol{\pi}$, and $\mathbf{c}$ vary. (We will think of $m$ as being permanently fixed.)

Since an optimal strategy in general is very difficult to compute, it is of major interest to ask: (a) for what distributions $\mathbf{p}$ do there exist separable rules, and (b) if in state $(\mathbf{p}, \boldsymbol{\pi})$ for which a separable rule exists for $\mathbf{p}$ selecting, say, action $k$, does the new state $(\mathbf{p}^{(k)}, \boldsymbol{\pi}^{(k)})$ have a distribution $\mathbf{p}^{(k)}$ which also has a separable rule.

It is shown in Section 2 that if $\mathbf{p}$ is a positive-Poisson distribution, then $\mathbf{p}$ has a separable rule, and since from (1.2), $\mathbf{p}^{(k)}$ is also a positive-Poisson distribution, it follows that an optimal strategy is determined by the following simple rule: From any state $(\mathbf{p}, \boldsymbol{\pi})$, search a box $\hat{k}$ for which the test function (to be given by (2.7)) is maximized. Conversely, for $m \geqq 3$ it is shown that if $\mathbf{p}$ has a separable rule, then $\mathbf{p}$ must be positive-Poisson.

**2. Distributions having a separable rule.** This section presents and proves the principal result that the only distributions $\mathbf{p}$ which have a separable rule are the positive-Poisson distributions.

A random variable $X$ is said to have a *positive-Poisson distribution* with parameter $\lambda \geqq 0$ (we say $X$ is $pp(\lambda)$) if either $\lambda > 0$ and

$$P(X = n) = \frac{\lambda^n}{(e^\lambda - 1)n!} \qquad \text{for } n = 1, 2, \cdots,$$

or, corresponding to $\lambda = 0$, $P(X = 1) = 1$. If $X$ has a Poisson distribution with parameter $\lambda > 0$, then the conditional distribution of $X$, given $X > 0$, is $pp(\lambda)$.

For any distribution $\mathbf{p}$ denote by $h_{\mathbf{p}}$ the function

(2.1) $$h_{\mathbf{p}}(x) = 1 - \sum_{n=1}^{\infty} (1 - x)^n p_n .$$

It then follows that the probability of finding at least one object in a search of box $k$ in state $(\mathbf{p}, \boldsymbol{\pi})$ is $h_{\mathbf{p}}(\alpha_k \pi_k)$.

LEMMA 1. *Let* $q = (\mathbf{p}, \boldsymbol{\pi})$. *Then*

(2.2) $$r_q(i, j, s_3, s_4, \cdots) \leqq r_q(j, i, s_3, s_4, \cdots)$$

*if and only if*

(2.3) $$\frac{h_{\mathbf{p}}(\alpha_i \pi_i)}{c_i} \geqq \frac{h_{\mathbf{p}}(\alpha_j \pi_j)}{c_j} .$$

PROOF. By direct computation

(2.4) $$r_q(i, j, s_3, s_4, \cdots) = c_i + c_j[1 - h_{\mathbf{p}}(\alpha_i \pi_i)] + b(i, j) r_{q'}(s_3, s_4, \cdots) ,$$

where $b(i, j)$ is the probability of not finding an object on a search of box $i$ followed by box $j$, and $q'$ is the state $(\mathbf{p}^{(i,j)}, \boldsymbol{\pi}^{(i,j)}) = (\mathbf{p}^{(j,i)}, \boldsymbol{\pi}^{(j,i)})$. It is easy to verify that $b(i, j)$ and $r_{q'}(s_3, s_4, \cdots)$ are symmetric as functions of $i$ and $j$. Therefore, (2.2) holds if and only if (2.3) holds.

We shall be interested in maximizing over $j$ the expression $h_{\mathbf{p}}(\alpha_j \pi_j)/c_j$. If this expression is maximized by $j = i$, box $i$ is called a *most inviting box* from state $(\mathbf{p}, \boldsymbol{\pi})$. The next lemma shows that when $\mathbf{p}$ is $pp(\lambda)$, if box $i$ is most inviting from state $(\mathbf{p}, \boldsymbol{\pi})$, then after a search of a box other than box $i$, it remains most inviting.

LEMMA 2. *Suppose* $\mathbf{p}$ *is* $pp(\lambda)$ *and* $h_{\mathbf{p}}(\alpha_j \pi_j)/c_j$ *is maximized when* $j = i$. *If* $k \neq i$ *then* $h_{\mathbf{p}^{(k)}}(\alpha_j \pi_j^{(k)})/c_j$ *is maximized when* $j = i$.

PROOF. If $\mathbf{p}$ is $pp(\lambda)$, then it follows from (1.2) that $\mathbf{p}^{(k)}$ is $pp(\lambda^{(k)})$, where

(2.5) $$\lambda^{(k)} = \lambda(1 - \alpha_k \pi_k) .$$

Multiplying equations (2.5) and (1.1) yields

(2.6) $$\pi_j^{(k)} \lambda^{(k)} = (1 - \delta_{jk} \alpha_k) \pi_j \lambda \leqq \pi_j \lambda .$$

When $\mathbf{p}$ is $pp(\lambda)$ with $\lambda > 0$, $h_{\mathbf{p}}(x) = \beta(\lambda)[1 - e^{-x\lambda}]$ where $\beta(\lambda) = (1 - e^{-\lambda})^{-1}$ and hence

$$\begin{aligned} h_{\mathbf{p}^{(k)}}(\alpha_j \pi_j^{(k)})/c_j &= \beta(\lambda^{(k)})[1 - \exp(-\alpha_j \pi_j^{(k)} \lambda^{(k)})]/c_j \\ &\leqq \beta(\lambda^{(k)})[1 - \exp(-\alpha_j \pi_j \lambda)]/c_j \\ &= \frac{\beta(\lambda^{(k)}) h_{\mathbf{p}}(\alpha_j \pi_j)}{\beta(\lambda) c_j} \\ &\leqq \frac{\beta(\lambda^{(k)}) h_{\mathbf{p}}(\alpha_i \pi_i)}{\beta(\lambda) c_i} \\ &= \beta(\lambda^{(k)})[1 - \exp(-\alpha_i \pi_i \lambda)]/c_i \\ &= \beta(\lambda^{(k)})[1 - \exp(-\alpha_i \pi_i^{(k)} \lambda^{(k)})]/c_i \end{aligned}$$

since inequality (2.6) is an equality for $j = i$. Therefore,

$$h_{\mathbf{p}^{(k)}}(\alpha_j \pi_j^{(k)})/c_j \leqq h_{\mathbf{p}^{(k)}}(\alpha_i \pi_i^{(k)})/c_i .$$

A similar but simpler argument holds when $\lambda = 0$.

THEOREM 1. *Suppose* $\mathbf{p}$ *is* $pp(\lambda)$. *Then in state* $q = (\mathbf{p}, \boldsymbol{\pi})$ *it is optimal to search box* $i$ *if* $h_{\mathbf{p}}(\alpha_j \pi_j)/c_j$, *given by*

$$[1 - \exp(\alpha_j \pi_j \lambda)]/c_j \quad \textit{when} \quad \lambda > 0 , \tag{2.7a}$$

$$\alpha_j \pi_j / c_j \quad \textit{when} \quad \lambda = 0 , \tag{2.7b}$$

*is maximized by* $j = i$, *and thus,* $\mathbf{p}$ *has a separable rule.*

PROOF. Assume (2.7) is maximized by $j = i$. If (2.7) is zero for every $j$, then every searching sequence has infinite risk and the theorem holds. If (2.7) is positive at $j = i$, then $\pi_i > 0$ and hence every searching sequence with finite risk eventually calls for a search of box $i$. Among all searching sequences which are optimal for state $(\mathbf{p}, \boldsymbol{\pi})$, let $\mathbf{s} = (s_1, s_2, \cdots)$ be one which searches box $i$ at the earliest stage. If $s_1 = i$, we are finished. If $s_1 \neq i$ let $t + 2$ be the earliest stage at which $i$ is searched and denote $s_{t+1}$ by $j$ so that $\mathbf{s} = (s_1, \cdots, s_t, j, i, s_{t+3}, \cdots)$ and for $k \leqq t + 1$, $s_k \neq i$. Let $q^* = (\mathbf{p}^*, \boldsymbol{\pi}^*)$ be the state after searching boxes $s_1, \cdots, s_t$, assuming an object has not yet been found. Since (2.7) is maximized by $j = i$, Lemma 2 applied $t$ times proves that $h_{\mathbf{p}^*}(\alpha_j \pi_j^*)/c_j$ is maximized by $j = i$. By Lemma 1, $r_{q^*}(i, j, s_{t+3}, \cdots) \leqq r_{q^*}(j, i, s_{t+3}, \cdots)$ and hence $r_q(s_1, \cdots, s_t, i, j, s_{t+3}, \cdots) \leqq r_q(s_1, \cdots, s_t, j, i, s_{t+3}, \cdots)$, which contradicts the definition of $\mathbf{s}$.

Theorem 1 yields a searching sequence which is optimal when the initial state $\mathbf{p}$ is $pp(\lambda)$, namely at each stage to search a box which maximizes the current value of (2.7). The proof of the converse, namely, that if $\mathbf{p}$ has a separable rule, then $\mathbf{p}$ is $pp(\lambda)$ for some $\lambda \geqq 0$, follows from four lemmas.

LEMMA 3. *If* $m \geqq 3$, $\pi_1 > 0$, $\pi_2 > 0$ *and* $\boldsymbol{\alpha} = (1, 1, \cdots, 1)$, *then for fixed* $(\mathbf{p}, \boldsymbol{\pi})$ *and fixed* $c_1, c_2$, *there exist* $c_3, \cdots, c_m$ *and* $s_3, \cdots, s_m$ *such that either* $(1, 2, s_3, \cdots, s_m)$ *or* $(2, 1, s_3, \cdots, s_m)$ *is an optimal searching sequence from the state* $(\mathbf{p}, \boldsymbol{\pi})$.

The proof of Lemma 3 is elementary and is omitted.

LEMMA 4. *Suppose* $\mathbf{p}$ *has a separable rule. Then for* $\boldsymbol{\alpha} = (1, 1, 1, \cdots)$ *the function* $h_{\mathbf{p}}(\pi_k)/c_k$ *serves as a test function.*

PROOF. Let the separable rule have test function $g_{\mathbf{p}}$. Let $g_{\mathbf{p}}(1, \pi_k, c_k)$ be maximized by $k = i$ and suppose $h_{\mathbf{p}}(\pi_k)/c_k$ is maximized by $k = j$. If $i = j$, we are finished; assume $i \neq j$. By Lemma 3 there exist $\mathbf{c}'$, $s_3, s_4, \cdots, s_m$ such that $c_i' = c_i$, $c_j' = c_j$, and either $(i, j, s_3, \cdots, s_m)$ or $(j, i, s_3, \cdots, s_m)$ is an optimal searching sequence from state $q = (\mathbf{p}, \boldsymbol{\pi})$ with cost vector $\mathbf{c}'$. Since $h_{\mathbf{p}}(\pi_j)/c_j' \geqq h_{\mathbf{p}}(\pi_i)/c_i'$, Lemma 1 implies that $r_q(i, j, s_3, \cdots, s_m) \geqq r_q(j, i, s_3, \cdots, s_m)$ with cost vector $\mathbf{c}'$. Therefore, $(j, i, s_3, \cdots, s_m)$ is an optimal searching sequence from state $(\mathbf{p}, \boldsymbol{\pi})$ with cost vector $\mathbf{c}'$, and hence $g_{\mathbf{p}}(1, \pi_j, c_j') \geqq g_{\mathbf{p}}(1, \pi_i, c_i')$. But $c_j' = c_j$

and $c_i' = c_i$ and hence $g_{\mathbf{p}}(1, \pi_k, c_k)$ is also maximized by $k = j$. Therefore, in state $(\mathbf{p}, \boldsymbol{\pi})$ it is optimal to search box $i$.

LEMMA 5. *If* $\mathbf{p}$ *has a separable rule, then* $h = h_{\mathbf{p}}$ *satisfies*

$$h(x)h(y) + h(x+y)h(1-x-y) - h(x)h(1-x-y) - h(1-y)h(y) = 0 \tag{2.8}$$

*for all* $(x, y)$ *such that* $x \geqq 0$, $y \geqq 0$, $x + y \leqq 1$.

PROOF. Fix $\boldsymbol{\alpha} = (1, 1, \cdots, 1)$; without loss of generality assume that $m = 3$ by setting $\pi_4 = \pi_5 = \cdots = 0$. Consider a $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ such that each $\pi_k > 0$ and set $c_k = h(\pi_k)$. Since the three quantities $h(\pi_k)/c_k$ equal 1, Lemma 1 implies

$$r_q(1, 2, 3) = r_q(2, 1, 3)\,, \tag{2.9a}$$

$$r_q(3, 1, 2) = r_q(1, 3, 2)\,, \tag{2.9b}$$

$$r_q(2, 3, 1) = r_q(3, 2, 1)\,, \tag{2.9c}$$

where $q = (\mathbf{p}, \boldsymbol{\pi})$. By Lemma 4 and the equality of the quantities $h(\pi_k)/c_k$, there exists for $k = 1, 2, 3$ an optimal searching sequence starting with $k$. Hence, by (2.9), at least four of the six risks of (2.9) are equal. In particular, at least one of the following must hold:

$$r_q(2, 1, 3) - r_q(1, 3, 2) = 0\,, \tag{2.10a}$$

$$r_q(1, 3, 2) - r_q(3, 2, 1) = 0\,, \tag{2.10b}$$

$$r_q(3, 2, 1) - r_q(2, 1, 3) = 0\,. \tag{2.10c}$$

By the definition of $h$ and the fact that $c_k = h(\pi_k)$,

$$\begin{aligned} r_q(s_1, s_2, s_3) &= c_{s_1} + [1 - h(\pi_{s_1})]c_{s_2} + [1 - h(\pi_{s_1} + \pi_{s_2})]c_{s_3} \\ &= h(\pi_{s_1}) + [1 - h(\pi_{s_1})]h(\pi_{s_2}) + [1 - h(\pi_{s_1} + \pi_{s_2})]h(\pi_{s_3})\,. \end{aligned}$$

Hence, letting $\boldsymbol{\pi} = (x, y, 1 - x - y)$, we write (2.10a) as (2.8). Similarly (2.10b) and (2.10c) become

$$h(1-x-y)h(x) + h(1-y)h(y) - h(1-x-y)h(y) - h(1-x)h(x) = 0 \tag{2.11}$$

and

$$h(y)h(1-x-y) + h(1-x)h(x) - h(y)h(x) - h(x+y)h(1-x-y) = 0\,, \tag{2.12}$$

respectively.

We have shown that if $x > 0$, $y > 0$, $x + y < 1$, then at least one of (2.8), (2.11), (2.12) holds. For each fixed $x$, at least one of the equations (2.8), (2.11), (2.12) holds for infinitely many $y$ in $[0, 1 - x]$; denote the appropriate equation by $e_x$. The left-hand sides of (2.8), (2.11), (2.12) are analytic functions since $h$ is analytic. Since the zero function is the only analytic function with non-isolated zeroes, for each $x$ in $(0, 1)$ equation $e_x$ holds for every $y$ in $[0, 1 - x]$.

At least one of the equalities (2.8), (2.11), (2.12) is $e_x$ for infinitely many $x$ in (0, 1); denote the appropriate equation by $e$. For each $y$ in (0, 1), equation $e$ holds for infinitely many $x$ in $[0, 1-y)$; therefore, by the analyticity of the left-hand side of equation $e$, equation $e$ holds for all appropriate $(x, y)$.

Inspection of (2.10) shows that (2.8), (2.11), and (2.12) differ only by a labeling of the boxes. Since equation $e$ holds for all $(x, y)$ satisfying $x \geqq 0$, $y \geqq 0$, $x + y \leqq 1$ equation (2.8) also holds for all $(x, y)$ satisfying $x \geqq 0$, $y \geqq 0$, $x + y \leqq 1$.

LEMMA 6. *The only solutions $h$ to* (2.8) *of the form* (2.1) *for some distribution* $\mathbf{p}$ *are*

$$h(x) = 1 - \frac{e^{\lambda(1-x)} - 1}{e^{\lambda} - 1} \tag{2.13}$$

*for some $\lambda > 0$, or*

$$h(x) = x . \tag{2.14}$$

PROOF. Equation (2.8) may be written for $y > 0$ as

$$\frac{h(x+y) - h(x)}{y} \cdot h(1 - x - y) = \frac{h(y)}{y} \cdot [h(1-y) - h(x)] . \tag{2.15}$$

Since (2.15), for fixed $x$ in (0, 1), is an equality between analytic function of $y$ on $(0, 1-x)$ and $h$ must satisfy $h(0) = 0$ and $h(1) = 1$, letting $y \to 0^+$ yields the differential equation

$$h'(x) \cdot h(1-x) = h'(0) \cdot [1 - h(x)] . \tag{2.16}$$

Similarly, (2.8) may be written for $x > 0$ as

$$h(1-x-y) \cdot \frac{h(x+y) - h(y)}{x} + h(y) \cdot \frac{h(1-x-y) - h(1-y)}{x} = \frac{h(x)}{x} \cdot [h(1-x-y) - h(y)] . \tag{2.17}$$

Letting $x \to 0^+$ in (2.17) yields the following differential equation (where $x$ rather than $y$ is used to denote the argument):

$$h'(x)h(1-x) - h(x)h'(1-x) = h'(0) \cdot [h(1-x) - h(x)] . \tag{2.18}$$

If $x > 0$ then $h'(x) > 0$; (2.16) may be written as

$$h(1-x) = h'(0) \cdot \frac{1 - h(x)}{h'(x)} \tag{2.19}$$

for $x$ in some interval. Differentiating both sides of (2.19) yields

$$h'(1-x) = h'(0) \cdot \frac{[h'(x)]^2 + h''(x)[1 - h(x)]}{[h'(x)]^2} . \tag{2.20}$$

Substituting into (2.18) yields

$$h'(0)[h'(x)]^2 - h'(0)h(x)h''(x) = [h'(0)]^2 h'(x) . \tag{2.21}$$

The constant $h'(0)$ is nonzero since if $h'(0)$ were zero, then dividing both sides of (2.16) by $h'(x)$ gives us the false conclusion that $h(1-x)$ is zero in some interval. Divide both sides of (2.21) by $h'(0)$ to obtain

$$[h'(x)]^2 - h(x)h''(x) = h'(0)h'(x) . \tag{2.22}$$

Since $h$ is increasing on $[0, 1]$, $h'$ is a function of $h$. Let $u = h(x)$ and $q(u) = h'(x)$; equation (2.22) becomes

$$q(u) - uq'(u) = a_1 \tag{2.23}$$

where $a_1$ is the constant $h'(0)$; therefore

$$q(u) = a_1 + a_2 u \tag{2.24}$$

for $u$ in some interval and $a_2$ some constant. If $a_2 = 0$, (2.24) yields $h'(x) = a_1$ in some interval; therefore, (2.14) holds on $[0, 1]$. If $a_2 \neq 0$, then (2.24) yields

$$h(x) = \pm\exp(a_4 x + a_5) - a_6 \tag{2.25}$$

for some interval where $a_4$, $a_5$, $a_6$ are constants. The constraint that $h(x)$ be of the form (2.1) for some distribution $\mathbf{p}$ simplifies (2.25) into (2.13). Since (2.13) holds for $x$ in some interval, (2.13) holds on $[0, 1]$.

THEOREM 2. *If $m \geqq 3$ and $\mathbf{p}$ has a separable rule, then $\mathbf{p}$ is $pp(\lambda)$ for some $\lambda \geqq 0$.*

PROOF. Lemmas 5 and 6.

Theorems 1 and 2 completely characterize the class of distributions which have a separable rule. However, for other distributions $\mathbf{p}$ there may exist for specific values of the constants $\boldsymbol{\alpha}$, $\mathbf{c}$ a function $g_{\mathbf{p}}(\alpha_k, \pi_k, c_k)$ such that for all $\boldsymbol{\pi}$ an optimal strategy prescribes searching a box $\hat{k}$ for which $g_{\mathbf{p}}(\alpha_k, \pi_k, c_k)$ is maximized. A trivial example is the case $\boldsymbol{\alpha} = (1, 0, 0, \cdots, 0)$, where we can take $g_{\mathbf{p}}(\alpha_k, \pi_k, c_k) = \alpha_k$ for all $\mathbf{p}$. Another example is given by the following theorem.

THEOREM 3. *If $c_1 = c_2 = \cdots = c_m$, then for all distributions $\mathbf{p}$ an optimal strategy prescribes searching a box $\hat{k}$ for which the current value of $\alpha_k \pi_k$ is maximized.*

PROOF. The probability of finding at least one ball within $d$ searches when using searching sequence $\mathbf{s}$ is

$$1 - \sum_{n=1}^{\infty} \phi^n p_n , \tag{2.26}$$

where

$$\phi = \sum_{j=1}^{m} \pi_j \prod_{t=1}^{d} (1 - \delta_{js_t} \alpha_j) .$$

Chew (1967) showed that if there is just one ball, then the probability of finding it within $d$ searches is maximized by choosing at each stage a box with maximal value of $\alpha_k \pi_k$; therefore, Chew's rule minimizes $\phi$. Since (2.26) is a decreasing function of $\phi$ and the rule of searching a box with maximal value of $\alpha_k \pi_k$ maximizes the probability of finding at least one ball within $d$ searches uniformly in $d$, then the rule minimizes the expected cost when $c_1 = c_2 = \cdots = c_m$.

**3. The case $m = 2$.** It has heretofore been assumed that $m \geqq 3$. In the case

$m = 2$ we conjecture that every distribution $\mathbf{p}$ has a separable rule and that the function $g_{\mathbf{p}}(\alpha_k, \pi_k, c_k) = h_{\mathbf{p}}(\alpha_k \pi_k)/c_k$ serves as a test function, where $h_{\mathbf{p}}$ is given by (2.1). A proof of the conjecture would follow from a proof of the statement that if one box is more inviting from state $(\mathbf{p}, \boldsymbol{\pi})$, then after a search of the other box, the first box remains more inviting. That this statement is not true in general is shown below in Example 1. Lemma 8, however, shows that the statement is true if there are at most two balls. The following lemma holds for an arbitrary number of balls.

LEMMA 7. *If $N$ has distribution $\mathbf{p}$ and $N'$ has distribution $\mathbf{p}^{(k)}$ given by* (1.2), *then $N$ is stochastically greater than or equal to $N'$.*

PROOF. Let $t_n = \sum_{x=1}^{n} p_x$ and $z = 1 - \alpha_k \pi_k$. Then

$$\begin{aligned}
&[P(N' \leqq n) - P(N \leqq n)] \textstyle\sum_{y=1}^{\infty} z^y p_y \\
&\quad = \textstyle\sum_{x=1}^{n} z^x p_x - t_n \sum_{x=1}^{\infty} z^x p_x \\
&\quad = (1 - t_n) \textstyle\sum_{x=1}^{n} z^x p_x - t_n \sum_{x=n+1}^{\infty} z^x p_x \\
&\quad = (1 - t_n)[z t_1 + \textstyle\sum_{x=2}^{n} z^x (t_x - t_{x-1}) - z^{n+1} t_n] \\
&\qquad\quad + t_n [z^{n+1}(1 - t_{n+1}) + \textstyle\sum_{x=n+2}^{\infty} z^x (t_{x-1} - t_x)] \\
&\quad = z(1 - z)[(1 - t_n) \textstyle\sum_{x=0}^{n-1} z^x t_{x+1} + t_n \sum_{x=n}^{\infty} z^x (1 - t_{x+1})] ,
\end{aligned}$$

which is clearly nonnegative, and hence $P(N \leqq n) \leqq P(N' \leqq n)$ for $n = 1, 2, \cdots$.

Lemma 8 is analogous to Lemma 2.

LEMMA 8. *If $m = 2$, $p_1 + p_2 = 1$, and*

$$h_{\mathbf{p}}(\alpha_1 \pi_1)/c_1 \geqq h_{\mathbf{p}}(\alpha_2 \pi_2)/c_2 \tag{3.1}$$

*where $h_{\mathbf{p}}$ is given by* (2.1), *then*

$$h_{\mathbf{p}^{(2)}}(\alpha_1 \pi_1^{(2)})/c_1 \geqq h_{\mathbf{p}^{(2)}}(\alpha_2 \pi_2^{(2)})/c_2 . \tag{3.2}$$

PROOF. It is sufficient to show

$$\frac{h_{\mathbf{p}^{(2)}}(\alpha_1 \pi_1^{(2)})}{h_{\mathbf{p}^{(2)}}(\alpha_2 \pi_2^{(2)})} \geqq \frac{h_{\mathbf{p}}(\alpha_1 \pi_1)}{h_{\mathbf{p}}(\alpha_2 \pi_2)} . \tag{3.3}$$

By Lemma 7,

$$h_{\mathbf{p}^{(2)}}(\alpha_2 \pi_2^{(2)}) \leqq h_{\mathbf{p}^{(2)}}(\alpha_2 \pi_2) \leqq h_{\mathbf{p}}(\alpha_2 \pi_2)$$

since $h_{\mathbf{p}}(\alpha_2 \pi_2)$ is an expectation of an increasing function of the random number of objects. To prove the inequality

$$h_{\mathbf{p}^{(2)}}(\alpha_1 \pi_1^{(2)}) \geqq h_{\mathbf{p}}(\alpha_1 \pi_1) , \tag{3.4}$$

we use (1.1) and (1.2) to reduce inequality (3.4) to

$$\textstyle\sum_{x=1}^{2} (1 - \alpha_1 \pi_1)^x p_x \sum_{y=1}^{2} (1 - \alpha_2 \pi_2)^y p_y - \sum_{n=1}^{2} (1 - \alpha_1 \pi_1 - \alpha_2 \pi_2)^n p_n \geqq 0 ,$$

the left-hand side of which factors into

$$\alpha_1 \pi_1 \alpha_2 \pi_2 [p_2^2 (1 - \alpha_1 \pi_1)(1 - \alpha_2 \pi_2) + p_2 (1 - \alpha_1 \pi_1 - \alpha_2 \pi_2) + p_1] ,$$

which is clearly nonnegative.

THEOREM 4. *If $m = 2$ and $p_1 + p_2 = 1$, an optimal strategy prescribes searching a box $\hat{k}$ for which $h_{\mathbf{p}}(\alpha_k \pi_k)/c_k$ is maximized.*

PROOF. Theorem 4 may be proved by the proof of Theorem 1, except that Lemma 8 plays the role of Lemma 2.

We can neither prove nor find a counterexample to the proposition that for $m = 2$ every distribution $\mathbf{p}$ has a separable rule. The following example shows that Lemma 8 is not true without the condition that $p_1 + p_2 = 1$.

EXAMPLE 1. Let $\mathbf{c} = (1, 5)$, $\boldsymbol{\alpha} = (.99, .10)$, $\boldsymbol{\pi} = (.01, .99)$, and $\mathbf{p}$ be such that $p_1 = p_{500} = .5$. Note that box 1 is clearly more inviting because of its lower cost and the fact that a search of either box will find an object with probability near .5. However, a search of either box without finding an object makes it highly probable that only one object is present and hence box 2 becomes more inviting.

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