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SPECIAL CASE OF THE DISTRIBUTION OF THE MEDIAN[1]

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Let t be the translation parameter of a process $X(t)$, $-\infty < t < \infty$. The likelihood ratio of the process $X(t)$ at t against $t = 0$ can be written as $\exp[W(t) - \frac{1}{2}|t|]$, $-\infty < t < \infty$, where $W(t)$ is a standard Wiener process. For the absolute error-loss function the best invariant estimator of the translation parameter is the median of the posterior distribution. The distribution of the median for the posterior distribution is obtained, when the prior distribution for t is the Lebesgue measure on the real line.

1. Introduction. Let $X(t)$, $-\infty < t < \infty$, be a stochastic process with t as the non-stationarity point. Let the origin be the true non-stationarity point and let $L(t)$ be the likelihood ratio of the process at t against $t = 0$. $L(t)$ can be treated as proportional to the a posteriori distribution of t, if the Lebesgue measure on the real line is assumed to be the a priori distribution on the non-stationarity point of $X(t)$. It is well known that the median of the a posteriori distribution is the best invariant estimator of the parameter t for an absolute error loss function. We shall treat a special case of $L(t)$, which has the following representation.

$$L(t) = \exp[W(t) - \tfrac{1}{2}|t|], \qquad -\infty < t < \infty \tag{1}$$

where $W(t)$ is a Gaussian process with independent increments and with

$$\begin{aligned} &\text{(i)} \quad W(0) = 0, \\ &\text{(ii)} \quad EW(t) = 0 \quad \text{for all } t. \\ &\text{(iii)} \quad \operatorname{Cov}[W(t_1), W(t_2)] = \delta(t_1, t_2)\min(|t_1|, |t_2|) \\ &\qquad \text{where } \delta(t_1, t_2) = 1 \quad t_1 t_2 > 0 \\ &\qquad \qquad = 0 \quad t_1 t_2 \leqq 0, \end{aligned} \tag{2}$$

i.e., $W(t)$ is a standard Wiener process. It should be noted that the class of stochastic processes satisfying condition (1) is not empty. In fact Rao and Rubin [2] gave a necessary and sufficient condition for a Gaussian process to be the log-likelihood process of a Gaussian process. We shall obtain an expression for the distribution of the median of $L(t)$.

2. Notation and some results. We introduce further notations in terms of

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$L(t)$, where $L(t)$ is defined by (1) and (2). Let for $T > 0$,

$$\begin{aligned} &\text{(i)}\quad X^- = \textstyle\int_{-\infty}^{0} L(t)\,dt\,,\\ \text{(3)}\qquad &\text{(ii)}\quad X^+ = X = \textstyle\int_0^\infty L(t)\,dt\,,\\ &\text{(iii)}\quad Y = \textstyle\int_0^T L(t)\,dt\,, \qquad \text{and}\\ &\text{(iv)}\quad Z = \textstyle\int_T^\infty L(t)\,dt\,. \end{aligned}$$

We can express $P[\text{median} \geqq T]$ as follows.

$$\text{(4)}\qquad P[\text{median} \geqq T] = \int_{-\infty}^{\infty} P[X^- + Y \leqq Z \mid W(T) - \tfrac{1}{2}T = u] \times \frac{1}{(2\pi T)^{\frac{1}{2}}} \exp -\left[\frac{1}{2T}(u + \tfrac{1}{2}T)^2\right] du\,.$$

The case $T < 0$ follows from symmetry.

Let

$$\text{(5)}\qquad G(T, u) = P[X^- + Y \leqq Z \mid W(T) - \tfrac{1}{2}T = u]\,.$$

We shall derive a computable expression for $G(T, u)$.

Let us first quote some well-known properties of $W(t)$, defined by (1) and (2), which we shall use in the derivation and which are given in any standard text-book on stochastic processes.

Extrapolation property.

$$\text{(6)}\qquad W(t_1 + t_2) = W(t_1) + W^*(t_2)\,, \qquad \text{for } t_1 t_2 \geqq 0\,,$$

where $W(t_1)$ and $W^*(t_2)$ are independent Wiener processes with the same structure as $W(t)$. In fact $W(t_1)$ is the same as $W(t_1 + t_2)$ on the left side.

Interpolation property. The conditional density of $W(t)$, $t_1 < t < t_2$, $t_1 t_2 \geqq 0$, given $W(t_1) = A$ and $W(t_2) = B$ is a normal density with mean

$$\text{(7)}\qquad \text{(i)}\quad A + \frac{B - A}{t_2 - t_1}(t - t_1)$$

and the variance

$$\text{(ii)}\quad (t_2 - t)(t - t_1)/(t_2 - t_1)\,.$$

We shall now state a result which is not well known. Rubin [3] proved a related result. Later Fox and Rubin [1] proved the same result for general stochastic processes. The following result can be proved by the techniques used in the latter part of this paper.

RESULT 1. Let X be as defined in (3-ii). Then, the distribution of $1/X$ is an exponential distribution with a scale factor two, that is, the Laplace transform of $1/X$ is given by

$$\text{(8)}\qquad E[\exp(-\lambda/X)] = 2/(\lambda + 2)\,, \qquad \lambda \geqq 0\,.$$

RESULT 2. Let X and X^- be as defined in (3). Then $X/(X + X^-)$ (or $X^-/(X + X^-)$) is uniformly distributed over [0, 1] and is independent of $X + X^-$.

3. Derivation. Let us start by looking back to the definition of the random variables Y and Z. Note, we could replace $W(t) - \frac{1}{2}t = V(t)$, where $V(t)$ is a Wiener process with mean $-\frac{1}{2}t$, $t \geqq 0$ and same variance-covariance structure as $W(t)$. Then

$$Y = \int_0^T e^{V(t)}\, dt\,, \qquad Z = \int_T^\infty e^{V(t)}\, dt = e^{V(T)} Z'$$

where $V(T)$ and Z' are independent and Z', X are identically distributed. Now let us condition $V(T) = u$. Therefore, by using the interpolation property (7), we have

$$E\{V(t)\} = t \cdot u/T\,, \qquad 0 \leqq t \leqq T\,, \tag{9}$$
$$\operatorname{Var}\{V(t)\} = (T - t) \cdot t/T\,, \qquad 0 \leqq t \leqq T\,.$$

Let

$$F(a, T, u) = P[Y \leqq a \mid T, V(T) = u]\,, \qquad a \geqq 0\,, \tag{10}$$

$$f(a, T, u) = \frac{\partial}{\partial a} F(a, T, u)\,, \qquad a \geqq 0\,. \tag{11}$$

Lemma 1. *$F(a, T, u)$, defined by (10), satisfies the following partial differential equation,*

$$\left(-1 - \frac{u}{T}a + \tfrac{1}{2}a\right) F_a - F_T - \frac{u}{T} F_u + \tfrac{1}{2}a^2 F_{aa} + aF_{au} + \tfrac{1}{2}F_{uu} = 0 \tag{12}$$

where the subscripts stand for the corresponding partial differentiations of $F(a, T, u)$.

Proof. Let us write

$$Y = \int_0^\varepsilon e^{V(t)}\, dt + \int_\varepsilon^T e^{V(t)}\, dt\,, \qquad \varepsilon > 0\,,$$
$$\sim \varepsilon + e^{V(\varepsilon)} V^*$$

where $V(\varepsilon)$ and V^* are independent.

$$F(a, T, u) \sim EP[V^* \leqq ae^{-V(\varepsilon)} - \varepsilon \mid T - \varepsilon, V(T - \varepsilon) = u - V(\varepsilon)]$$
$$= EF(ae^{-V(\varepsilon)} - \varepsilon, T - \varepsilon, u - V(\varepsilon))\,,$$

where the expectation is with respect to the random variable $V(\varepsilon)$.

Using the Taylor's series expansion, omitting all the terms of order equal or higher than ε^2, and taking expectations, one can arrive at expression (12).

Lemma 2. *$F(a, T, u)$ also satisfies the following backward partial differential equation*

$$-e^u F_a - F_T - \frac{u}{T} F_u + \tfrac{1}{2}F_{uu} = 0\,. \tag{13}$$

Proof. The only difference in this proof from the previous proof is that we write

$$Y = \int_0^{T-\varepsilon} e^{V(t)}\, dt + \int_{T-\varepsilon}^T e^{V(t)}\, dt$$

and proceed as before to get (13).

If (13) is subtracted from (12) we get the following corollary.

COROLLARY 1. *$F(a, T, u)$ satisfies the following partial differential equation*

$$\left(e^u - 1 - \frac{u}{T}a + \tfrac{1}{2}a\right)F_a + \tfrac{1}{2}a^2F_{aa} + aF_{au} = 0\,. \tag{14}$$

The above equation can be written in $f(a, T, u)$ as

$$\left(e^u - 1 - \frac{u}{T}a + \tfrac{1}{2}a\right)f + \tfrac{1}{2}a^2f_a + af_u = 0\,. \tag{15}$$

LEMMA 3. *The general solution of the partial differential equation* (15) *is*

$$f(a, T, u) = \exp\left[\frac{1}{2}\frac{u^2}{T} - \tfrac{1}{2}n - \frac{4\cosh\frac{1}{2}u}{ae^{-u}} + \psi(ae^{-\frac{1}{2}u})\right] \tag{16}$$

where ψ depends on a and u only through $ae^{-\frac{1}{2}u}$.

PROOF. Let $h(a, T, u) = \ln f(a, u)$; then (15) is the same as

$$\left(e^u - 1 - \frac{u}{T}a + \tfrac{1}{2}a\right) + \tfrac{1}{2}a^2h_a + ah_u = 0\,. \tag{17}$$

The general solution of the homogeneous equation of (17) is $\psi(ae^{-u})$. A particular solution of (17) is obtained by inspection. Note if $(\frac{1}{2})(u^2/T) - (\frac{1}{2})u$ is included in h then $a \cdot h_u$ will cancel out the term $a(\frac{1}{2} - u/T)$. Now consider the equation

$$(e^u - 1) + \tfrac{1}{2}a^2h_a + ah_u = 0\,.$$

Observe that $K(u)/a$ is a solution of this equation if $K(u)$ satisfies the following differential equation.

$$(e^u - 1) + K'(u) - \tfrac{1}{2}K(u) = 0$$

where $K'(u)$ stands for the differentiation of $K(u)$. The proof of Lemma 3 is completed by observing that $-4\cosh\frac{1}{2}u/e^{-\frac{1}{2}u}$ is a solution of the above differential equation.

COROLLARY 2. *The density function, $f(a, T, u)$, is given by* (16) *where ψ is a function of $ae^{-\frac{1}{2}u}$ only (possibly depending on T) such that*

$$\int_0^\infty \exp\left[\frac{1}{2}\frac{u^2}{T} - \tfrac{1}{2}u - \frac{4\cosh\frac{1}{2}u}{ae^{-\frac{1}{2}u}} + \psi(ae^{-\frac{1}{2}u})\right]da = 1\,. \tag{18}$$

Equation (18) is the condition that $f(a, T, u)$ is a probability density function in a.

COROLLARY 3. *If $Y_1 = e^{\frac{1}{2}u}/Y$, then the Laplace transform of Y_1, viz., $E[\exp(-\lambda Y_1)] = \phi(\lambda)$, is given by*

$$\phi(\lambda) = \exp\left[\tfrac{1}{2}u^2/T\right]\exp -\frac{2}{T}\operatorname{arc\,cosh}^2\left(\cosh\tfrac{1}{2}u + \frac{\lambda}{4}\right), \qquad \lambda \geqq 0\,. \tag{19}$$

PROOF. The probability density function of Y, say $f_1(a, T, u)$, is given by

$$f_1(a, T, u) = \frac{1}{a^2}\exp\left[\frac{1}{2}\frac{u^2}{T} - 4a\cosh u + \psi(a)\right].$$

$\psi(\cdot)$ is a function such that

$$\int_0^\infty \frac{1}{a^2} \exp[-4a \cosh \tfrac{1}{2}u + \psi(a)]\, da = \exp - \frac{1}{2}\frac{u^2}{T}$$
$$= \exp - \frac{2}{T} \operatorname{arc\,cosh}^2 (\cosh \tfrac{1}{2}u) .$$

$$\therefore \quad \phi(\lambda) = \int_0^\infty \frac{1}{a^2} \exp\left(\frac{1}{2T}u^2\right) \exp[-4a \cosh u - \lambda a + \psi(a)]\, da$$
$$= \int_0^\infty \frac{1}{a^2} \exp \frac{1}{2T} u^2 \exp\left[-4\left(\cosh u + \frac{\lambda}{4}\right) a + \psi(a)\right] da$$
$$= \exp \frac{1}{2T} u^2 \exp - \frac{2}{T} \operatorname{arc\,cosh}^2 \left(\cosh \tfrac{1}{2}u + \frac{\lambda}{4}\right) .$$

This completes the proof of Corollary 3.

We are now in a position to find $G(T, u)$ as defined by (5).

LEMMA 4.

$$(20) \qquad G(T, u) = k - \int_0^k \exp \frac{1}{2T} [u^2 - 4 \operatorname{arc\,cosh}^2 (\cosh \tfrac{1}{2}u + \tfrac{1}{2}c)]$$
$$\times \left[1 + \frac{2c}{T} \frac{\operatorname{arc\,cosh} (\cosh \frac{1}{2}u + \frac{1}{2}c)}{\{(\cosh \frac{1}{2}u + \frac{1}{2}c)^2 - 1\}^{\frac{1}{2}}}\right] da$$

where $k = 1/(1 + e^{-u})$, $c = (e^{\frac{1}{2}u}/a) - (e^{-\frac{1}{2}u}/(1 - a))$.

PROOF. Note we can write $G(T, u)$ as

$$G(T, u) = P[X^- + Y \leqq e^u Z' \mid V(T) = u]$$

where X^- and Z' are identically distributed.

We shall omit the condition $V(T) = u$ and the prime of Z in the above definition of $G(T, u)$ for notational convenience. If

$$W = \frac{1}{Z} + \frac{1}{X^-} \qquad A = X^-/(Z + X^-)$$

it follows from Result 2 that A is a uniform random variable on [0, 1], W is a gamma random variable with the parameter two, and W, A, are independent.

$$\therefore \quad G(T, u) = P[e^{-\frac{1}{2}u} Y \leqq e^{\frac{1}{2}u} Z - e^{-\frac{1}{2}u} X^-]$$
$$= P\left[\frac{W}{Y_1} \leqq (e^{\frac{1}{2}u}/A) - (e^{-\frac{1}{2}u}/(1 - A))\right],$$

where Y_1 is as defined in Corollary 3.

$$\therefore \quad G(T, u) = \int P\left[\frac{W}{Y_1} \leqq (e^{\frac{1}{2}u}/a) - (e^{-\frac{1}{2}u}/(1 - a))\right] da ,$$

where the range of the integration is such that $(e^{\frac{1}{2}u}/a) - (e^{-\frac{1}{2}u}/(1 - a)) \geqq 0$, i.e., for $0 \leqq a \leqq k = 1/(1 + e^{-u})$.

$$\therefore \quad G(T, u) = k - \int_0^k \int_0^\infty P[W \geqq c y_1] f(y_1)\, dy_1\, da ,$$

where

(i) k is as defined above,

(ii) c is as defined in Lemma 4,

(iii) $f(y_1)$ is the density function of the random variable Y_1, whose Laplace transform is given by (19).

Now (20) is proved by using $P[W \geqq cy_1] = (1 + 2cy_1)e^{-2cy_1}$ and (19). The direct substitution of (20) in (4) and some simplifications prove the formula (22) in the following theorem.

THEOREM. *Let t be the non-stationarity point of a Gaussian process $X(t)$, $-\infty < t < \infty$, with the Lebesgue measure on the real line as the a priori distribution for t. Let the process $X(t)$ be such that the likelihood ratio $L(t)$ of $X(t)$, at t against $t = 0$, admits the following representation*

$$L(t) = \exp(W(t) - \tfrac{1}{2}|t|)\,, \qquad -\infty < t < \infty\,, \tag{21}$$

where $W(t)$ is a standard Wiener process. Then the distribution of the median of the a posteriori distribution, which is proportional to (21), *is given by*

$$\begin{aligned} &P(\text{median} \geqq T) \\ &= \frac{e^{-T/8}}{(2\pi T)^{\frac{1}{2}}} \int_0^\infty e^{-\frac{1}{2}u^2/T} \operatorname{sech} \tfrac{1}{2}u \, du \\ &\quad - \frac{e^{-T/8}}{(2\pi T)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u} \int_0^k \exp - \frac{2}{T} (\operatorname{arc\,cosh}^2 (\tfrac{1}{2}u + \tfrac{1}{2}c)) \\ &\quad \times \left[1 + \frac{2c}{T} \frac{\operatorname{arc\,cosh} (\cosh \frac{1}{2}u + \frac{1}{2}c)}{\{(\cosh \frac{1}{2}u + \frac{1}{2}c)^2 - 1\}^{\frac{1}{2}}} \right] da \, du\,, \end{aligned} \tag{22}$$

where $k = 1/(1 + e^{-u})$, $c = (e^{\frac{1}{2}u}/a) - (e^{-\frac{1}{2}u}/(1 - a))$.

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