The Annals of Statistics
1974, Vol. 2, No. 6, 1351-1356

ON A GEOMETRICAL METHOD OF CONSTRUCTION OF GROUP DIVISIBLE INCOMPLETE BLOCK DESIGNS

BY NOBORU HAMADA

Ehime University, Japan

In this paper it is shown that a new series of group divisible designs can be obtained by using a geometrical method.

1. Summary. It is shown that by using a geometrical method, we can obtain a new series of group divisible designs with parameters

$$\begin{aligned} v &= (q^{t+1} - q^{\pi+1})/(q-1)\,, \qquad k = (q^{\mu+1} - q^{\nu+1})/(q-1)\,, \\ b &= q^{(\pi-\nu)(\mu-\nu)}\phi(t-\pi-1, \mu-\nu-1, q)\phi(\pi, \nu, q)\,, \\ r &= q^{(\pi-\nu)(\mu-\nu-1)}\phi(t-\pi-2, \mu-\nu-2, q)\phi(\pi, \nu, q)\,, \\ \lambda_1 &= q^{(\pi-\nu)(\mu-\nu-1)}\phi(t-\pi-2, \mu-\nu-2, q)\phi(\pi-1, \nu-1, q)\,, \\ \lambda_2 &= q^{(\pi-\nu)(\mu-\nu-2)}\phi(t-\pi-3, \mu-\nu-3, q)\phi(\pi, \nu, q)\,, \\ m &= (q^{t-\pi}-1)/(q-1)\,, \qquad n = q^{\pi+1}\,, \qquad p_{11}^1 = q^{\pi+1}-2 \qquad \text{and} \\ p_{11}^2 &= 0 \end{aligned} \tag{1.1}$$

for any integers t, μ, ν and π ($\geqq 0$) such that

$$-1 \leqq \nu \leqq \pi < t-1 \qquad \text{and} \qquad \pi + \mu - t \leqq \nu < \mu < t \tag{1.2}$$

where q is a prime or a prime power and

$$\phi(t, \mu, q) = \frac{(q^{t+1}-1)(q^t-1)\cdots(q^{t-\mu+1}-1)}{(q^{\mu+1}-1)(q^\mu-1)\cdots(q-1)} \tag{1.3}$$

for any integers t and μ such that $0 \leqq \mu < t$ and $\phi(t, -1, q) = 1$ for $t \geqq -1$.

2. Introduction. Group divisible incomplete block designs are an important subclass of partially balanced incomplete block (PBIB) designs [3], [13], [4] with two associate classes and they have been investigated by many authors [1], [2], [5], [6], [7], [8], [9], [10], [11], [12], [14], [15]. They may be defined as follows.

An incomplete block design with v treatments each replicated r times in b blocks of size k is said to be group divisible (GD) if the treatments can be divided into m groups, each with n treatments, so that the treatments belonging to the same group occur together in λ_1 blocks and treatments belonging to different groups occur together in λ_2 blocks. It is known [2] that the GD designs can be divided into three exhaustive and mutually exclusive classes:

(a) Singular GD designs characterized by $r - \lambda_1 = 0$;
(b) Semi-regular GD designs characterized by $r - \lambda_1 > 0$, $rk - v\lambda_2 = 0$;
(c) Regular GD designs characterized by $r - \lambda_1 > 0$, $rk - v\lambda_2 > 0$.

Received April 1973; revised November 1973.

The purpose of this paper is to show that by using a geometrical method, we can construct, systematically, a new series of group divisible designs with parameters (1.1).

3. Points and μ-flats in PG (t, q). With the help of the Galois field GF (q), we can define a finite projective geometry PG (t, q) of t dimensions as a set of points satisfying the following conditions:

(a) A point in PG (t, q) is represented by (ν) where ν is a nonzero element of GF (q^{t+1}).

(b) Two points (ν_1) and (ν_2) represent the same point when and only when there exists a nonzero element σ of GF (q) such that $\nu_1 = \sigma\nu_2$.

(c) A μ-flat $(0 \leqq \mu \leqq t)$ in PG (t, q) is defined as a set of points

$$\{(a_0\nu_0 + a_1\nu_1 + \cdots + a_\mu\nu_\mu)\}$$

where a's run independently over the elements of GF (q), not all zero and (ν_0), $(\nu_1), \cdots, (\nu_\mu)$ are linearly independent over the coefficient field GF (q), in other words, they do not lie on a $(\mu - 1)$-flat. For convenience, we denote the empty set $\emptyset$ by (-1)-flat.

Let α be a primitive element of GF (q^{t+1}). Then, nonzero elements of GF (q^{t+1}) can be represented by α^l $(l = 0, 1, \cdots, q^{t+1} - 2)$ and every point in PG (t, q) is represented by (α^k) $(k = 0, 1, \cdots, v^* - 1)$ where $v^* = (q^{t+1} - 1)/(q - 1)$.

4. A geometrical method for the generation of GD **designs.** Let t, π, μ and ν be any integers satisfying the condition (1.2) and let W be a π-flat ($\pi = -1$ or $0 \leqq \pi < t - 1$) in PG (t, q). We denote by $B_{\pi,\nu}(t, \mu, q)$, the set of all μ-flats V in PG (t, q) such that $V \cap W$ is a ν-flat and denote by $T(t, \pi, q)$, the set of $v = (q^{t+1} - q^{\pi+1})/(q - 1)$ points in PG (t, q) which are obtained from all points in PG (t, q) by deleting $(q^{\pi+1} - 1)/(q - 1)$ points in the π-flat W. Then, we have the following

THEOREM 4.1. *By identifying the points of $T(t, \pi, q)$ with the v treatments and identifying the μ-flats in $B_{\pi,\nu}(t, \mu, q)$ with the b blocks, we obtain a* GD *design with parameters* (1.1) *for any integers t, μ, ν and π $(\geqq 0)$ satisfying the condition* (1.2). *The efficiency factors of this design are as follows*:

within groups,
$$E_1 = 1 - \frac{(q - 1)(q^{\pi+1} - q^{\nu+1})}{(q^{\pi+1} - 1)(q^{\mu+1} - q^{\nu+1})},$$

between groups,
$$E_2 = \frac{1}{1 + e} E_1 \quad \textit{where}$$

$$e = \frac{\lambda_1 - \lambda_2}{v\lambda_2} = \frac{\{(q^t - q^{\pi+1})(q^{\nu+1} - 1) - (q^\mu - q^{\nu+1})(q^{\pi+1} - 1)\}}{(q^\mu - q^{\nu+1})(q^{\pi+1} - 1)v}.$$

Note that (i) in the special case $\pi = -1$ (i.e., $W = \emptyset$), $\nu = -1$ and the above design reduces to the well-known balanced incomplete block (BIB) design

PG $(t, q):\mu$ with parameters

$$v = (q^{t+1} - 1)/(q - 1)\,, \qquad b = \phi(t, \mu, q)\,, \qquad r = \phi(t - 1, \mu - 1, q)\,,$$
$$k = (q^{\mu+1} - 1)/(q - 1) \qquad \text{and} \qquad \lambda = \phi(t - 2, \mu - 2, q)$$

and (ii) in the special case $\pi = t - 1$ (i.e., W is a hyperplane), $\nu = \mu - 1$ and the above design reduces to the well-known BIB design EG $(t, q):\mu$.

In order to prove this theorem, we prepare the following lemmas.

LEMMA 4.1. *The number, b, of μ-flats in $B_{\pi,\nu}(t, \mu, q)$ is equal to*

$$b = q^{(\pi-\nu)(\mu-\nu)}\phi(t - \pi - 1, \mu - \nu - 1, q)\phi(\pi, \nu, q) \tag{4.1}$$

for any integers t, π, μ and ν satisfying the condition (1.2).

PROOF. The following two cases must be considered.

(i) In the case where ν is a nonnegative integer, it is easy to see that the number of μ-flats V in PG (t, q) such that $V \cap W = U$, where U is a ν-flat in W, is equal to

$$\frac{(q^{t+1} - q^{\pi+1})(q^{t+1} - q^{\pi+2}) \cdots (q^{t+1} - q^{\pi+\mu-\nu})}{(q^{\mu+1} - q^{\nu+1})(q^{\mu+1} - q^{\nu+2}) \cdots (q^{\mu+1} - q^{\nu+\mu-\nu})}\,,$$

that is, $q^{(\pi-\nu)(\mu-\nu)}\phi(t - \pi - 1, \mu - \nu - 1, q)$. Since the number of ν-flats U in the π-flat W is equal to $\phi(\pi, \nu, q)$ and there is no μ-flat V in PG (t, q) such that $V \cap W = U_1$ and $V \cap W = U_2$ for ν-flats U_1 and U_2 in W unless $U_1 = U_2$, we have the required result.

(ii) Now we consider $\nu = -1$. Since the number of μ-flats in PG (t, q) is equal to $\phi(t, \mu, q)$, it follows from (i) that the number, b, of μ-flats V in PG (t, q) such that $V \cap W = \emptyset$ is equal to

$$b = \phi(t, \mu, q) - \textstyle\sum_{\nu=0}^{\pi} q^{(\pi-\nu)(\mu-\nu)}\phi(t - \pi - 1, \mu - \nu - 1, q)\phi(\pi, \nu, q) \tag{4.2}$$

where $\phi(t, \nu, q) = 0$ for any integers t and ν such that $t < \nu$ or $\nu \leqq -2$.

Since

$$\textstyle\sum_{\nu=-1}^{\pi} q^{(\pi-\nu)(\mu-\nu)}\phi(t - \pi - 1, \mu - \nu - 1, q)\phi(\pi, \nu, q) = \phi(t, \mu, q)\,, \tag{4.3}$$

we have $b = q^{(\pi+1)(\mu+1)}\phi(t - \pi - 1, \mu, q)\phi(\pi, -1, q)$. This completes the proof.

LEMMA 4.2. *Let P be any point in $T(t, \pi, q)$. Then the number, r, of μ-flats in $B_{\pi,\nu}(t, \mu, q)$ passing through the point P is equal to*

$$r = q^{(\pi-\nu)(\mu-\nu-1)}\phi(t - \pi - 2, \mu - \nu - 2, q)\phi(\pi, \nu, q) \tag{4.4}$$

for any integers t, π, μ and ν satisfying the condition (1.2).

PROOF. Let W^* be the $(\pi + 1)$-flat containing the point P and the π-flat W and let U^* be the $(\nu + 1)$-flat containing the point P and a ν-flat U in W. Since the number of μ-flats V in $B_{\pi,\nu}(t, \mu, q)$ passing through the point P such that $V \cap W = U$ is equal to the number of μ-flats V such that $V \cap W^* = U^*$, the number of such μ-flats V is equal to $q^{(\pi-\nu)(\mu-\nu-1)}\phi(t - \pi - 2, \mu - \nu - 2, q)$.

Since the number of ν-flats U in W is equal to $\phi(\pi, \nu, q)$, we have the required result.

PROOF OF THEOREM 4.1. Since $v = (q^{t+1} - q^{\pi+1})/(q - 1)$ and $k = (q^{\mu+1} - q^{\nu+1})/(q - 1)$, it follows from Lemma 4.1 and Lemma 4.2 that parameters v, b, r and k of this design are given by (1.1). It is, therefore, sufficient to show that this design is a GD design with parameters λ_1, λ_2, m, n, p_{11}^1 and p_{11}^2 given in (1.1).

Let $\phi(i)$ $(i = 1, 2, \cdots, v)$ be v integers such that each point $(\alpha^{\phi(i)})$ belongs to $T(t, \pi, q)$ and we define a relationship of association between every pair of v treatments, denoted by $\phi(1), \phi(2), \cdots, \phi(v)$, as follows: Two different treatments $\phi(i)$ and $\phi(j)$ are 1st associates or 2nd associates according to whether there does or does not exist a pair (a_1, a_2) of elements of GF (q) such that the point $(a_1\alpha^{\phi(i)} + a_2\alpha^{\phi(j)})$ belongs to W.

Let $\phi(i) \overset{1}{\leftrightarrow} \phi(j)$ and $\phi(j) \overset{1}{\leftrightarrow} \phi(k)$ where "$\phi(l_1) \overset{1}{\leftrightarrow} \phi(l_2)$" means that two treatments $\phi(l_1)$ and $\phi(l_2)$ are 1st associates. Then, there exist a pair (a_1, a_2) of nonzero elements of GF (q) such that

$$(a_1\alpha^{\phi(i)} + a_2\alpha^{\phi(j)}) \in W$$

and a pair (b_1, b_2) of nonzero elements of GF (q) such that

$$(b_1\alpha^{\phi(j)} + b_2\alpha^{\phi(k)}) \in W .$$

Since W is a π-flat $(0 \leqq \pi < t - 1)$, we have

$$(a_2^{-1}a_1\alpha^{\phi(i)} - b_1^{-1}b_2\alpha^{\phi(k)}) \in W .$$

This implies that $\phi(i) \overset{1}{\leftrightarrow} \phi(k)$. Therefore, the association defined above is the association scheme of a GD type if $0 < n_1 < v - 1$ where n_1 denotes the number of treatments being 1st associates. Let $(\alpha^{\phi(i)})$ be any point in $T(t, \pi, q)$, that is, $(\alpha^{\phi(i)}) \notin W$. Then, $(a\alpha^{\phi(i)} + \alpha^c) \notin W$ for any nonzero element a of GF (q) and any point (α^c) in the π-flat W $(\pi \geqq 0, W \neq \emptyset)$. There exists, therefore, a point $(\alpha^{\phi(l)})$ in $T(t, \pi, q)$ such that

$$(\alpha^{\phi(l)}) = (a\alpha^{\phi(i)} + \alpha^c) . \tag{4.5}$$

This implies that there exists a pair (a_1, a_2) of elements of GF (q) such that $(a_1\alpha^{\phi(i)} + a_2\alpha^{\phi(l)}) = (\alpha^c)$, that is, two treatments $\phi(i)$ and $\phi(l)$ are 1st associates. Conversely, if $\phi(i)$ and $\phi(l)$ are 1st associates, there exist a pair $(a, (\alpha^c))$ of a nonzero element a of GF (q) and a point (α^c) in W which satisfy the condition (4.5). Hence, we have $n_1 = (q - 1)\phi(\pi, 0, q) = q^{\pi+1} - 1$ $(0 < n_1 < v - 1)$. From this, we can see that parameters n, m, p_{11}^1 and p_{11}^2 are given by (1.1). It is, therefore, sufficient to show that λ_1 and λ_2 are given by (1.1).

(i) In the case where two treatments $\phi(i)$ and $\phi(j)$ are 1st associates, there exists a unique point (α^f) in the π-flat W such that

$$(\alpha^f) = (a_1\alpha^{\phi(i)} + a_2\alpha^{\phi(j)})$$

for some elements a_1 and a_2 of GF (q). Hence, any μ-flat in $B_{\pi,\nu}(t, \mu, q)$ passing through two points $(\alpha^{\phi(i)})$ and $(\alpha^{\phi(j)})$ has to contain the point (α^f). Let U be any ν-flat in W passing through the point (α^f) and let U^* be the $(\nu+1)$-flat containing the ν-flat U and the point $(\alpha^{\phi(i)})$. Since the number of μ-flats V in $B_{\pi,\nu}(t, \mu, q)$, passing through two points $(\alpha^{\phi(i)})$ and $(\alpha^{\phi(j)})$, such that $V \cap W = U$ is equal to the number of μ-flats V in $B_{\pi,\nu}(t, \mu, q)$ such that $V \cap W^* = U^*$, where W^* is the $(\pi+1)$-flat containing the π-flat W and the point $(\alpha^{\phi(i)})$, and the number of ν-flats U in W passing through the point (α^f) is equal to $\phi(\pi-1, \nu-1, q)$, the number λ_1 of μ-flats in $B_{\pi,\nu}(t, \mu, q)$ passing through two points $(\alpha^{\phi(i)})$ and $(\alpha^{\phi(j)})$ is equal to

$$\lambda_1 = q^{(\pi^*-\nu^*)(\mu-\nu^*)}\phi(t-\pi^*-1, \mu-\nu^*-1, q)\phi(\pi-1, \nu-1, q)$$

where $\pi^* = \pi + 1$ and $\nu^* = \nu + 1$.

(ii) In the case where two treatments $\phi(i)$ and $\phi(j)$ are 2nd associates, we can show that the number λ_2 of μ-flats in $B_{\pi,\nu}(t, \mu, q)$ passing through two points $(\alpha^{\phi(i)})$ and $(\alpha^{\phi(j)})$ is equal to

$$\lambda_2 = q^{(\pi^*-\nu^*)(\mu-\nu^*)}\phi(t-\pi^*-1, \mu-\nu^*-1, q)\phi(\pi, \nu, q)$$

where $\pi^* = \pi + 2$ and $\nu^* = \nu + 2$. This completes the proof.

Note that (i) for $\nu = \pi$, we get Singular GD designs from Theorem 4.1, (ii) for $\nu = \mu + \pi - t$, Semi-regular GD designs and (iii) for $\nu \neq \pi$, $\mu + \pi - t$, Regular GD designs.

Acknowledgment. The author wishes to thank the referee and the Associate Editor for their valuable comments.

REFERENCES

[1] Bose, R. C., Clatworthy, W. H. and Shrikhande, S. S. (1954). Tables of partially balanced designs with two associate classes. *North Carolina Agricultural Experiment Station Technical Bull.* No. 107.

[2] Bose, R. C. and Connor, W. S. (1952). Combinatorial properties of group divisible incomplete block designs. *Ann. Math. Statist.* **23** 367–383.

[3] Bose, R. C. and Nair, K. R. (1939). Partially balanced incomplete block designs. *Sankhyā* **4** 337–372.

[4] Bose, R. C. and Shimamoto, T. (1952). Classification and analysis of partially balanced incomplete block designs with two associate classes. *J. Amer. Statist. Assoc.* **47** 151–184.

[5] Bose, R. C., Shrikhande, S. S. and Bhattacharya, K. N. (1953). On the construction of group divisible incomplete block designs. *Ann. Math. Statist.* **24** 167–195.

[6] Clatworthy, W. H. (1955). Partially balanced incomplete block designs with two associate classes and two treatments per block. *J. Res. Nat. Bur. Standards* **54** 177–190.

[7] Clatworthy, W. H. (1956). Contributions on partially balanced incomplete block designs with two associate classes. National Bureau of Standard Applied Mathematics Series 47.

[8] Dembowski, P. (1968). *Finite Geometries.* Springer-Verlag, Berlin.

[9] Freeman, G. H. (1957). Some further methods of constructing regular group divisible incomplete block designs. *Ann. Math. Statist.* **28** 479–487.

[10] Hamada, N. (1973). On the p-rank of the incidence matrix of a balanced or partially balanced incomplete block design and its applications to error correcting codes. *Hiroshima Math. J.* **3** 153–226.

[11] KAPADIA, C. H. (1966). On the block structure of singular group divisible designs. *Ann. Math. Statist.* **37** 1398–1400.

[12] LIU, W. R. and CHANG, L. C. (1964). Group divisible incomplete block designs with parameters $v \leqq 10$ and $r \leqq 10$. *Sci. Sinica* **13** 839–840.

[13] NAIR, K. R. and RAO, C. R. (1942). A note on partially balanced incomplete block designs. *Science and Culture* **7** 568–569.

[14] RAGHAVARAO, D. (1971). *Constructions and Combinatorial Problems in Design of Experiments.* Wiley, New York.

[15] SPROTT, D. A. (1959). A series of symmetric group divisible designs. *Ann. Math. Statist.* **30** 249–251.

MATHEMATICAL INSTITUTE
HIROSHIMA UNIVERSITY
SHINONOME, HIROSHIMA
JAPAN