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STOPPING TIMES OF SOME ONE-SAMPLE SEQUENTIAL RANK TESTS[1]

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Two models for modified, one-sample, sequential probability ratio tests based on Lehmann alternatives are considered, one developed by Weed and Bradley and one by Govindarajulu. It is shown how they are related. Sure termination of the SPRT's is established under very general conditions.

1. Introduction. Weed and Bradley (1971, 1973) and Weed (1968) developed a model (Model I) for a modified, one-sample, sequential probability ratio test (SPRT) based on ranks and Lehmann alternatives and reported on Monte Carlo studies of its properties. The research paralleled work on the two-sample sequential rank test by Wilcoxon, Rhodes and Bradley (1963) and Bradley, Merchant and Wilcoxon (1966).

Govindarajulu (1968) proposed an alternative model (Model II) for the one-sample SPRT using Lehmann alternatives in different form. Model I was chosen by its authors to provide continuity at the origin for the probability density function of the random variable under consideration under alternative hypotheses. I. R. Savage has given a transformation that shows Model I as a special case of Model II.

The authors of both models considered termination properties of the resulting SPRT's using an extension of the techniques of Savage and Sethuraman (1966) for the two-sample problem. Because of the similarity of the research, this joint paper was developed.

In this paper, we shall show that the SPRT's under both models terminate with probability one and that the moments of stopping times are finite for all alternatives within the classes defined by the models.

2. Notation and formulation of the problem. Let $Z_1, Z_2, \cdots$ be independent and identically distributed random variables observed sequentially and having

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a continuous cdf F. We wish to test the hypothesis,

$$H_0: F(z) + F(-z) = 1 \qquad \text{for all} \quad z\,, \tag{1}$$

that is, F is symmetric about zero. Two models for F are considered leading to alternatives to (1). Let $H(z) = P(Z \leqq z \mid Z \geqq 0) = \{F(z) - F(0)\}/\{1 - F(0)\}$ and $G(z) = P(|Z| \leqq z \mid Z < 0) = \{F(0) - F(-z)\}/F(0)$ for $z \geqq 0$ and $H(z), G(z) = 0$, for $z < 0$. We can rewrite (1) as

$$H_0: H(z) = G(z) \qquad \text{for all} \quad z \quad \text{and} \quad F(0) = \tfrac{1}{2}$$

and take

$$H_a: H(z) \neq G(z) \qquad \text{for some} \quad z\,.$$

We postulate structure for two models under alternative hypotheses:

$$\text{Model I:} \quad H_{a\mathrm{I}}: H(z) = 1 - \{1 - G(z)\}^A \quad \text{for all} \quad z\,, \quad A > 0\,, \quad A \neq 1\,, \quad A \quad \text{specified}, \quad F(0) = A/(1+A)\,, \tag{2}$$

and

$$\text{Model II:} \quad H_{a\mathrm{II}}: H(z) = G^A(z) \qquad \text{for all} \quad z\,, \quad A > 0\,, \quad A \neq 1\,, \quad A \quad \text{specified}, \quad F(0) = \lambda_0\,, \quad \lambda_0 \quad \text{specified.} \tag{3}$$

The two models are not the same, but see Theorem 1, and the discussion by Weed and Bradley (1971). They have also given examples of cdf's $F(z)$ and associated pdf's $f(z) = F'(z)$ for Model I; it is easy to generate examples for Model II in the same way.

When the experiment has reached stage t, $Z_1, \cdots, Z_t$ have been observed. Let $X_1, \cdots, X_m$ denote the absolute values of those Z's that are negative and let $Y_1, \cdots, Y_n$ denote the positive Z's, $m + n = t$. Note that m is binomially distributed with parameters t and $\lambda = F(0)$, $0 < \lambda < 1$. Let the ordered combined sample of X's and Y's be denoted by $W_1, \cdots, W_t$. Let G_m and H_n respectively denote the empirical cdf's of $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$. Further, following Savage's (1959) definition, let $\Delta = (\Delta_1, \cdots, \Delta_t)$ where $\Delta_i = 1$ or 0 according as W_i corresponds to a negative or positive Z respectively. Also let

$$L_t(A, \delta) = P_t(\Delta = \delta \mid A)/P_t(\Delta = \delta \mid A = 1) = 2^t P_t(\Delta = \delta \mid A) \tag{4}$$

where δ represents a realization of the random rank order and $P_t(\Delta = \delta \mid A)$ denotes the probability of the rank order δ when either (2) or (3) holds. The SPRT for testing H_0 against $H_{a\mathrm{I}}$ or $H_{a\mathrm{II}}$ is given by:

$$\begin{aligned} &\text{(i)} \quad \text{Take one more observation if} \quad a < L_t < b\,, \\ &\text{(ii)} \quad \text{Accept} \quad H_0 \quad \text{if} \quad L_t \leqq a\,, \\ &\text{(iii)} \quad \text{Reject} \quad H_0 \quad \text{if} \quad L_t \geqq b\,, \quad t = 1, 2, \cdots \end{aligned} \tag{5}$$

where $0 < a < 1 < b$ are suitable constants (independent of t).

The number of stages before termination T is defined as follows:

$$T = r \quad \text{if} \quad a < L_t < b \quad \text{for} \quad t = 1, \cdots, r-1 \quad \text{and} \quad L_r \geqq b \quad \text{or} \quad L_r \leqq a\,, \quad r = 1, 2, \cdots. \tag{6}$$

We investigate properties of the distribution of T.

3. Preliminary results. We obtain explicit expressions for $L_t(A, \delta)$.

LEMMA 1. *Under Model* II,

$$L_t(A, \delta) = 2^t t!\, \lambda_0^m (1-\lambda_0)^n A^n \prod_{i=1}^t \{mG_m(W_i) + AnH_n(W_i)\}^{-1}\,. \tag{7}$$

PROOF. It follows at once that

$$P(\Delta = \delta \mid A) = \binom{t}{m} \lambda_0^m (1-\lambda_0)^n P(\Delta = \delta \mid A, m)\,, \qquad \lambda_0 = F(0)\,,$$

where $P(\Delta = \delta \mid A, m)$ is the conditional probability that $\Delta = \delta$ given m. Following Savage (1959), we have

$$\begin{aligned} P(\Delta = \delta \mid A, m) &= m!\, n! \int \cdots \int_{0<w_1<\cdots<w_t<\infty} \prod_{j=1}^t \{dG(w_j)\}^{\delta_j} \{dH(w_j)\}^{1-\delta_j} \\ &= m!\, n!\, A^n \prod_{i=1}^t \{mG_m(W_i) + AnH_n(W_i)\}^{-1} \end{aligned}$$

where G_m and H_n are the sample cdf's of $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ respectively. The result (7) follows with use of (4).

An expression similar to (7) for $L_t(A, \delta)$ under Model I was obtained by Weed and Bradley (1971), formula (2.18) of the reference.

The transformation suggested by Savage yields the following theorem.

THEOREM 1. *If Z^* is a random variable with* cdf'*s F^*, G^*, H^* and parameter A^* satisfying Model* I *in* (2), *then $Z = -1/Z^*$ has* cdf'*s F, G, H and parameters $A = 1/A^*$ and $\lambda_0 = 1/(1 + A^*)$ satisfying Model* II *in* (3).

PROOF. The proof follows easily with use of (2) and (3) and demonstration that $G(t) = 1 - H^*(1/t)$, $H(t) = 1 - G^*(1/t)$, $t \geqq 0$, and $F(t) = F^*(-1/t) - F^*(0)$, $t < 0$, $F(t) = \lambda_0 + F^*(-1/t)$, $t \geqq 0$. The converse of the theorem is also true.

COROLLARY 1. *Given samples of independent observations $Z_1^*, \cdots, Z_t^*$ and $Z_1, \cdots, Z_t$, $Z_i = -1/Z_i^*$, $i = 1, \cdots, t$, on Z and Z^* of Theorem* 1, *the probability ratios for the two models and corresponding samples are identical.*

The corollary follows from the theorem. It may be demonstrated also through use of $L_t(A, \delta)$ of Lemma 1 for Model II and the corresponding form for Model I. Reason for the forms of G_m^* and H_n^* will be apparent.

Theorem 1 and Corollary 1 demonstrate that Model I may be taken as a special case of Model II. The remainder of this paper will deal with Model II and all results will apply also to Model I.

The following notation is required:

$$(8)\qquad \begin{aligned} S_t &= \{\log L_t(A, \delta)\}/t \\ &= \log 2 - 1 + \log\{A(1-\lambda_0)\} - \lambda_t \log\{A(1-\lambda_0)/\lambda_0\} - B_t \\ &\quad + O(t^{-1}\log t)\,, \end{aligned}$$

$$(9)\qquad \begin{aligned} S_\lambda(A, \lambda_0, G, H) &= \log 2 - 1 + \log\{A(1-\lambda_0)\} \\ &\quad - \lambda \log\{A(1-\lambda_0)/\lambda_0\} - B_\lambda(A, G, H)\,, \end{aligned}$$

where

$$(10)\qquad \lambda_t = m/t\,,$$

$$(11)\qquad B_t = t^{-1}\textstyle\sum_{i=1}^t \log\{\lambda_t G_m(W_i) + A(1-\lambda_t)H_n(W_i)\}\,,$$

$$(12)\qquad B_\lambda(A, G, H) = \textstyle\int_0^\infty \log\{\lambda G(z) + A(1-\lambda)H(z)\}d\{\lambda G(z) + (1-\lambda)H(z)\}\,.$$

Consider

$$(13)\qquad \begin{aligned} S_t - S_\lambda(A, \lambda_0, G, H) &= (\lambda_t - \lambda)\log\{\lambda_0/A(1-\lambda_0)\} \\ &\quad - \{B_t - B_\lambda(A, G, H)\} + O(t^{-1}\log t)\,. \end{aligned}$$

The main theorem of this section follows; the proof is delayed until needed lemmas are developed.

THEOREM 2. *For every* $\varepsilon > 0$ *and for* t *sufficiently large, there exists* $0 \leqq \rho(\varepsilon) < 1$ *such that*

(i) $P\{|B_t - B_\lambda(A, G, H)| \geqq \varepsilon\} \leqq \rho^t(\varepsilon)$,
(ii) $P\{|S_t - S_\lambda(A, \lambda_0, G, H)| \geqq \varepsilon\} \leqq \rho^t(\varepsilon)$.

LEMMA 2. *For every* $\varepsilon > 0$ *and sufficiently large* t, *there exists* $0 \leqq \rho(\varepsilon) < 1$ *such that*

$$(14)\qquad P\{|\lambda_t - \lambda| \geqq \varepsilon\} \leqq \rho^t(\varepsilon)\,.$$

COROLLARY 2. *For every* $\varepsilon > 0$, *sufficiently large* t, *and* $0 < \lambda < 1$, *there exists* $0 \leqq \rho(\varepsilon, \lambda) < 1$ *such that*

$$(15)\qquad P\{\log \lambda_t^{\lambda_t}/\lambda^\lambda) \geqq \varepsilon\} \leqq \rho^t(\varepsilon, \lambda)\,,$$

$$(16)\qquad P[\log\{(1-\lambda_t)^{1-\lambda_t}/(1-\lambda)^{1-\lambda}\} \geqq \varepsilon] \leqq \rho^t(\varepsilon, \lambda)\,.$$

COROLLARY 3. *Let* $\Omega_1(m) = \sup_z |G_m(z) - G(z)|$, $\Omega_2(n) = \sup_z |H_n(z) - H(z)|$ *and* $\Omega(t) = (1 + A)|\lambda_t - \lambda| + \lambda\Omega_1(m) + A(1-\lambda)\Omega_2(n)$. *For each* $0 < \varepsilon < \min(\lambda, 1-\lambda)$ *and for sufficiently large* t, *there exists* $0 \leqq \rho(\varepsilon) < 1$ *such that*

$$P\{\Omega(t) \geqq \varepsilon\} \leqq \rho^t(\varepsilon)\,.$$

PROOFS. Lemma 2 is well known; it follows, for example, from Theorem 1 of Chernoff (1952). Corollary 2 gives easy consequences of Lemma 2. For Corollary 3, write as two sums with the same argument

$$P\{\Omega_1(m) \geqq \varepsilon\} = (\textstyle\sum_{|r/t-\lambda|\leqq\varepsilon} + \sum_{|r/t-\lambda|>\varepsilon})P\{\Omega_1(m) \geqq \varepsilon \,|\, m = r\}P(m = r) \leqq \rho^t(\varepsilon)\,.$$

In reaching this result we have used Theorem 1 of Sethuraman (1964) in the first summation and Lemma 2 in the second. A similar result holds for $\Omega_2(n)$. Then these results with a second use of Lemma 2 yield Corollary 3.

Lemma 2 is used to bound the first term in the right-hand side of (13) and, together with Corollary 3, in Lemma 3 below. We consider the second term in the right-hand side of (13) and Part (i) of Theorem 2. Let

$$B_t^{(1)} = \lambda_t m^{-1} \sum_{i=1}^m \log \{\lambda G(X_i) + A(1-\lambda)H(X_i)\} + (1-\lambda_t)n^{-1} \sum_{j=1}^n \log \{\lambda G(Y_j) + A(1-\lambda)H(Y_j)\} \tag{17}$$

and

$$B_t^{(2)} = t^{-1} \sum_{i=1}^t \log \left\{ \frac{\lambda_t G_m(W_i) + A(1-\lambda_t)H_n(W_i)}{\lambda G(W_i) + A(1-\lambda)H(W_i)} \right\}. \tag{18}$$

It is easily checked from (11), (17), (18) that

$$B_t = B_t^{(1)} + B_t^{(2)}. \tag{19}$$

LEMMA 3. *For every* $\varepsilon > 0$ *and for sufficiently large* t, *there exists* $0 \leqq \rho(\varepsilon) < 1$ *such that*

(i) $P\{|B_t^{(1)} - B_\lambda(A, G, H)| \geqq \varepsilon\} \leqq \rho^t(\varepsilon)$,
(ii) $P\{|B_t^{(2)}| \geqq \varepsilon\} \leqq \rho^t(\varepsilon)$.

PROOF. (i) Let $V_i = \log \{\lambda G(X_i) + A(1-\lambda)H(X_i)\}$, $i = 1, \cdots, m$, and $V_j^* = \log \{\lambda G(Y_j) + A(1-\lambda)H(Y_j)\}$, $j = 1, \cdots, n$. For fixed m, the V_i are independent and identically distributed random variables having a finite moment generating function. Application of Theorem 1 of Chernoff (1952) yields the first inequality of (20) below and use of Lemma 2 in a manner similar to that of the proof of Corollary 3 yields the second. We have

$$P\{|m^{-1} \sum_{i=1}^m V_i - E(V_1)| \geqq \varepsilon\} \leqq \rho_1^m(\varepsilon) \leqq \rho^t(\varepsilon), \qquad 0 \leqq \rho_1(\varepsilon), \rho(\varepsilon) < 1. \tag{20}$$

An analogous result holds for the V^*'s. Since $|E(V_1) - E(V_1^*)| \leqq |E(V_1)| + |E(V_1^*)| \leqq 2|\log(1+A)|$ and since then

$$\begin{aligned} P\{|B_t^{(1)} - B_\lambda(A, G, H)| \geqq \varepsilon\} \leqq\ & P\{|m^{-1} \textstyle\sum_{i=1}^m V_i - E(V_1)| \geqq \varepsilon/4\} \\ & + P\{|n^{-1} \textstyle\sum_{j=1}^n V_j^* - E(V_1^*)| \geqq \varepsilon/4\} \\ & + P(|\lambda_t - \lambda| \geqq \varepsilon_1), \end{aligned}$$

$\varepsilon_1 = \varepsilon/4|\log(1+A)|$, use of (20), its analogue for the V^*'s and Lemma 2 lead to Part (i) of this lemma.

(ii) From (18) and the definitions of Corollary 3,

$$\begin{aligned} B_t^{(2)} &\leqq t^{-1} \sum_{i=1}^t \log \left\{ 1 + \frac{\lambda\Omega_1(m) + A(1-\lambda)\Omega_2(n) + (G_m - AH_n)(\lambda_t - \lambda)}{AG(W_i) + A(1-\lambda)H(W_i)} \right\} \\ &\leqq m^{-1} \sum_{i=1}^m \log \{1 + \Omega(t)/\lambda G(X_i)\} \\ &\quad + n^{-1} \sum_{j=1}^n \log \{1 + \Omega(t)/A(1-\lambda)H(Y_j)\}. \end{aligned} \tag{21}$$

Let $0 < \delta < \lambda$. Then

$$\begin{aligned} &P[m^{-1} \sum_{i=1}^{m} \log\{1 + \Omega(t)/\lambda G(X_i)\} \geqq \varepsilon] \\ &\quad \leqq P[\Omega(t) \geqq \delta] + \sum_{r=0}^{t} P[m^{-1} \sum_{i=1}^{m} \log\{1 + \delta/\lambda G(X_i)\} \geqq \varepsilon \mid m = r] P(m = r) \\ &\quad \leqq P[\Omega(t) \geqq \delta] + P[|\lambda_t - \lambda| \geqq \delta] \\ &\qquad + \sum_{|r/t - \lambda| < \delta} P[m^{-1} \sum_{i=1}^{m} \log\{1 + \delta/\lambda G(X_i)\} \geqq \varepsilon \mid m = r] \\ &\quad \leqq \rho^t(\varepsilon), \qquad 0 \leqq \rho(\varepsilon) < 1, \end{aligned}$$

for sufficiently large t. The last result follows after applying Corollary 3, Lemma 2, and a result due to Savage and Sethuraman (1966, equation 10). An analogous result holds for the second term in the right-hand side of (21). Hence

$$P(B_t^{(2)} \geqq \varepsilon) \leqq \rho^t(\varepsilon), \qquad 0 \leqq \rho(\varepsilon) < 1. \tag{22}$$

Savage and Sethuraman (1968) filled a gap in their earlier paper. Following their method, with consideration of λ_t as a random variable as done several times above, one may show that $P(-B_t^{(2)} \geqq \varepsilon) \leqq \rho^t(\varepsilon)$, $0 \leqq \rho(\varepsilon) < 1$, for sufficiently large t and Part (ii) of Lemma 3 follows.

Part (i) of Theorem 2 follows at once from Lemma 3. Part (ii) of Theorem 2 follows from (13), Lemma 2, and Part (i) of Theorem 2.

4. The basic results. We are ready to give the main theorems.

THEOREM 3. *From* (9) *let* $S_\lambda(A, \lambda_0, G, H) \neq 0$ *and let* T *denote the number of stages before termination of the* SPRT *under Model* II. *Then*

(i) $P(T > t) < \rho^t$ *for sufficiently large* t *and some* $0 \leqq \rho < 1$,
(ii) $P(T < \infty) = 1$,
(iii) $E(e^{\theta T}) < \infty$ *for* θ *in some interval* $(-\infty, \gamma)$, $\gamma > 0$.

PROOF. Parts (ii) and (iii) immediately follow from (i). If $S_\lambda(A, \lambda_0, G, H) \neq 0$ and L_t is as defined in (7),

$$\begin{aligned} P(T \leqq t) &\geqq P(L_t \leqq a \text{ or } L_t \geqq b) \\ &= P\{S_t \leqq (\log a)/t \text{ or } S_t \geqq (\log b)/t\} \\ &\geqq P\{|S_t - S_\lambda(A, \lambda_0 G, H)| \leqq \varepsilon\} \geqq 1 - \rho^t(\varepsilon), \quad 0 \leqq \rho(\varepsilon) < 1 \end{aligned} \tag{23}$$

for sufficiently large t. Hence (i) follows from Theorem 2. This completes the proof of Theorem 3.

REMARK 1. When H_0 or $H_{a\text{II}}$ is true, $S_\lambda(A, \lambda_0, G, H) \neq 0$ provided that $A \neq 1$ and/or $\lambda_0 \neq \frac{1}{2}$ and the SPRT terminates with probability 1. For example, under H_0 we have $H = G$ and $\lambda = \frac{1}{2}$ and

$$S_{\frac{1}{2}}(A, \lambda_0, G, G) = (\tfrac{1}{2}) \log\{16A\lambda_0(1 - \lambda_0)/(1 + A)^2\}. \tag{24}$$

In view of Theorem 1, the similar expression for Model I is, say,

$$\tilde{S}_{\frac{1}{2}}(A, G, G) = \log\{4A/(1 + A)^2\}.$$

REMARK 2. We have assumed throughout that $0 < \lambda < 1$. Consider Model II and $\lambda = 0$. Then $m = 0$ and $n = t$ and, from (4),

$$L_t(A, \delta) = 2^t(1 - \lambda_0)^t \qquad \text{or} \qquad S_t = \log\{2(1 - \lambda_0)\}\,.$$

Hence $P\{S_t \leqq (\log a)/t\} = 1$ for $\lambda_0 > \frac{1}{2}$ and sufficiently large t and $P\{S_t \geqq (\log b)/t\} = 1$ for $\lambda_0 < \frac{1}{2}$ and sufficiently large t. Analogously we can cover the case $\lambda = 1$.

In the following lemma we list some of the properties of $S_\lambda(A, \lambda_0, G, H)$.

LEMMA 4. (i) $S_\lambda(1, \lambda_0, G, G) = \log 2 + (1 - \lambda)\log(1 - \lambda_0) + \lambda \log \lambda_0$.

(ii) $S_\lambda(1, \frac{1}{2}, G, G) = 0$.

(iii) $S_\lambda\{A, \lambda_0, G, l(G)\}$ *is independent of* G *where* $l(\cdot)$ *denotes a distribution function on* $[0, 1]$.

(iv) $S_{1-\lambda}(1/A, 1 - \lambda_0, H, G) = S_\lambda(A, \lambda_0, G, H)$.

(v) $S_{\frac{1}{2}}(A, \lambda_0, G, G) = \log[4\{A\lambda_0(1 - \lambda_0)\}^{\frac{1}{2}}/(1 + A)]$ *and is zero only when* $\lambda_0 = \frac{1}{2}$ *and* $A = 1$.

(vi) *For each* A *there exists a unique* $C(A)$ *lying between* 1 *and* A *such that* $S_{\frac{1}{2}}(A, \frac{1}{2}, G, G^C) = 0$. *Further* $1/C(1/A) = C(A)$.

PROOF. Properties (i)—(v) follow from the definition (9).

In (vi) we assume that the Lehmann alternatives as well as the sampling distribution have zero medians. Thus, when $\lambda = \lambda_0 = \frac{1}{2}$, $S_\lambda(A, \lambda_0, G, H)$ is equal to one half the parameter studied by Savage and Sethuraman (1966). Hence $S_{\frac{1}{2}}(A, \frac{1}{2}, G, H)$ enjoys all the properties given in Lemma 4 of Savage and Sethuraman. In particular, the pairs of values of (A, C) for which $S_{\frac{1}{2}}(A, \frac{1}{2}, G, G^C) = 0$ will be those values tabulated in the reference. Use of Theorem 1 yields similar properties for Model I.

5. Discussion and concluding remarks. Theorems 1 and 3 establish the sure termination of the SPRT's for Models I and II under very general conditions.

Model I was devised for a sequential test of location within the family of distributions indexed by A in (2); as A departs from unity, location change and asymmetry are induced in the distribution. Weed and Bradley (1973) showed through Monte Carlo studies that the sequential test was adequate for applied purposes for the test of location for a normal population.

Model II is more difficult to interpret. The density $f(z)$ has a discontinuity at the origin except in the null case with $\lambda_0 = \frac{1}{2}$, $A = 1$. Specification of $\lambda_0 \neq \frac{1}{2}$ in (3) leads to a location change in terms of the median for all $A > 0$. If it is specified that $\lambda_0 = \frac{1}{2}$, $A \neq 1$ in $H_{a\text{II}}$, an alternative hypothesis is provided within the class of distributions given by (3) with A indexing departure from symmetry. Inversion of Model II through the transformation $Z^* = -1/Z$ produces a class of distributions broader than that of Model I.

The methods of this paper in investigation of the stopping times of the specified one-sample sequential rank tests can be extended to related test statistics. As an example, Govindarajulu (1968) considered substitution of $\lambda_t = m/t$ for λ_0 in

(7) to obtain $L_t^*(A, \delta)$ and $tS_t^* = \log L_t^*(A, \delta)$ where

$$S_t^* = \log 2 - 1 + \log\{A(1-\lambda_t)\} - \lambda_t \log\{A(1-\lambda_t)/\lambda_t\} - B_t + O(t^{-1}\log t)$$

with B_t defined in (11). Then, parallel with $S_\lambda(A, \lambda_0, G, H)$ in (9),

$$S_\lambda^*(A, G, H) = \log 2 - 1 + \log\{A(1-\lambda)\} - \lambda \log\{A(1-\lambda)/\lambda\} - B_\lambda(A, G, H)$$

is defined where $B_\lambda(A, G, H)$ is given in (12). It follows that

$$\begin{aligned} S_t^* - S_\lambda^*(A, G, H) = (\lambda - \lambda_t)\log A &+ \log(\lambda_t^{\lambda_t}/\lambda^\lambda) \\ &+ \log\{(1-\lambda_t)^{1-\lambda_t}/(1-\lambda)^{1-\lambda}\} \\ &- \{B_t - B_\lambda(A, G, H)\} + O(t^{-1}\log t)\,. \end{aligned}$$

Corollary 2 is needed now, but otherwise proofs follow as before. It is clear that, if T^* denotes the number of observations before termination of the test based on $L_t^*(A, \delta)$ or S_t^*, Theorem 3 applies with T^* replacing T and $S_\lambda^*(A, G, H)$ replacing $S_\lambda(A, \lambda_0, G, H)$. Remarks analogous to those following Theorem 3 may be made and properties similar to those of Lemma 4 may be noted. Research is needed before use of any test based on $L_t^*(A, \delta)$ or S_t^*. The test is not an SPRT. No information is available on how to specify bounds a and b for specified Type I and Type II error probabilities, but it is anticipated that the effective error probabilities will be close to the nominal error probabilities when Wald's bounds are used.

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