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CONTRIBUTIONS TO THE THEORY AND CONSTRUCTION OF BALANCED ARRAYS

By J. A. Rafter[1] and E. Seiden[2]

Michigan State University

Balanced arrays were introduced and first studied by I. M. Chakravarti and called partially balanced arrays. J. N. Srivastava and D. V. Chopra have recently made major contributions to the theory and construction of these arrays. They suggested dropping the adjective "partially" and we have followed their lead. Whereas their work has been concerned mainly with arrays of strength four with two symbols, we are here concerned primarily with arrays of strength two with two symbols. However, when the proofs can be carried out as easily for any strength and any number of symbols, we do state the theorems in full generality.

We are concerned here with finding bounds on the maximum possible number of rows and with the problem of constructing balanced arrays for given sets of parameters. Analogous to the problem of constructing other combinatorial configurations, we investigated whether some schemes, i.e. some subsets of columns of balanced arrays, could be extended to full balanced arrays. It is shown, analogously to orthogonal arrays, that balanced arrays of even strength, say $2u$ are extendable to arrays of strength $2u + 1$. A new technique of construction of balanced arrays with the maximum number of constraints is also described. It is shown that BIB designs with $\lambda = 1$ can be utilized for constructing balanced arrays with the number of symbols equal to the block size of the BIB design.

For completeness we include an example of the analysis of a partially balanced array of strength two with two symbols when it is used as a fractional factorial design. We exhibit in this way an explicit method of estimating main effects when higher ordered interactions are assumed to be negligible.

1. Introduction. The concept of balanced arrays as introduced by I. M. Chakravarti (1956) has served a dual purpose. First, it has served to unify the study of several areas of combinatorial theory. For example, the incidence matrices of BIB designs and doubly balanced designs as defined by D. Raghavarao and S. K. Tharthare (1967) are balanced arrays with proper specifications of the parameters involved. Also, orthogonal arrays can be studied as a special case of balanced arrays.

Unification is not the only merit of this concept. In fact, the main purpose for studying balanced arrays is their usefulness in the construction of symmetrical and asymmetrical confounded factorial experiments and fractionally repli-

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cated designs. In comparison to designs derived from orthogonal arrays, for example, designs derived from balanced arrays require fewer observations to accommodate a given number of factors. Theorems relating the estimability of interactions up to certain orders to the strength of an orthogonal array also apply to balanced arrays.

We shall begin with the definition of balanced arrays as it is known in the literature.

DEFINITION. Let A be an $m \times N$ matrix, with elements 0, 1, 2, $\cdots$, or $s-1$. Consider the s^t $(1 \times t)$ vectors, $X' = (x_1, \cdots, x_t)$, which can be formed where $x_i = 0, 1, \cdots, s-1$; $i = 1, \cdots, t$, and associate with each $(t \times 1)$ vector X a positive integer $\lambda(x_1, \cdots, x_t)$, which is invariant under permutations of $(x_1, \cdots, x_t)$. If for every t-rowed submatrix of A the s^t distinct $(t \times 1)$ vectors X occur as columns $\lambda(x_1, \cdots, x_t)$ times, then the matrix A is called a balanced array (BA) of strength t in N assemblies with m constraints, s symbols and the specified $\lambda(x_1, \cdots, x_t)$ parameters.

When one or more $\lambda(x_1, \cdots, x_t)$ is zero, A will be called degenerate. When $\lambda(x_1, \cdots, x_t) = \lambda$ for all $(x_1, \cdots, x_t)$, A is called an orthogonal array of index λ.

In view of the fact that $\lambda(x_1, \cdots, x_t)$ is invariant under permutations of $(x_1, \cdots, x_t)$, one can denote by $\lambda^{i_1, i_2, \cdots, i_r}_{x_1, x_2, \cdots, x_r}$ the number of repetitions of a fixed column of any $t \times N$ subarray of A, where the column contains $i_1 x_1$'s, $i_2 x_2$'s, $\cdots$ and $i_r x_r$'s. $(x_j = 0, 1, \cdots, s-1, \sum_{j=1}^{r} i_j = t, r = \min\{s, t\})$.

The set of all $\lambda^{i_1, i_2, \cdots, i_r}_{x_1, x_2, \cdots, x_r}$'s of an array of strength t in s symbols will be called the index set of the array and will be denoted by $\Lambda_{s,t}$. The array A will be represented as the BA (m, N, s, t) with index set $\Lambda_{s,t}$.

In case $s = 2$, we will denote λ_0^t by $\mu_0^{(t)}, \cdots, \lambda_{0,1}^{t-i,i}$ by $\mu_i^{(t)}, \cdots$, and λ_1^t by $\mu_t^{(t)}$. Where no ambiguity can arise, we will omit the superscript t from $\mu_i^{(t)}$ and write simply μ_i. Clearly, μ_i is the number of times a fixed column containing i ones occurs in any $t \times N$ submatrix of A. Finally, we will refer to a $t \times 1$ column as a t-tuple.

Several properties of BA's follow immediately from the definition. For example, it is easily shown that the number of elements in the index set $\Lambda_{s,t}$, is equal to $\binom{s+t+1}{t}$. Second, a subarray of a BA of strength t consisting of at least t rows is also a BA of strength t with the same index as the original array. This property is used occasionally to show the nonexistence of a BA with, say, m rows by establishing the nonexistence of a subarray with the same index set. A third property is that a BA of strength t is also a BA of strength less than t with appropriate index set. This observation is especially useful in establishing a recurrence relation between the elements of the indices of BA $(m, N, 2, t)$ and BA $(m, N, 2, t-1)$. Namely $\mu_i^{(t-1)} = \mu_i^{(t)} + \mu_{i+1}^{(t)}$; $i = 0, \cdots, t-1$. This is easily seen by noticing that in a $t-1$ rowed subarray a $(t-1)$-tuple containing i ones must have either i or $i+1$ ones when the subarray is extended to t rows.

By repeated use of this recurrence relation we obtain:

$$\mu_i^{(2)} = \sum_{j=1}^{t+i-2} \binom{t-2}{j-1} \mu_j^{(t)}, \qquad i = 0, 1, 2.$$

We are concerned in this paper predominately with BA's having two symbols and strength 2. When used as fractional factorial designs they enable the experimenter to estimate the main effects when interactions can be assumed to be negligible. It seems therefore worthwhile to present the analysis of a BA $(m, N, 2, 2)$. A more complete discussion can be found in Bose and Srivastava (1964).

Consider a complete 2^m factorial experiment where F_i, $i = 1, \cdots, m$ represents the ith factor and f_i represents the second of the two levels at which F_i can occur. Let f denote the column vector of all treatments, where $f' = [(1); f_1, f_2, \cdots; f_1f_2, f_1f_3, \cdots; \cdots; \cdots f_1f_2 \cdots f_m]$. Let F denote the column vector of F's in the same order where the first position represents the mean, μ. It is well known that each main effect and each interaction can be expressed as a linear contrast of all treatments. We can represent these contrasts in matrix notation as:

$$F = \mathscr{E}'f \tag{1.1}$$

where $\mathscr{E}'$ is a $2^m \times 2^m$ matrix of plus and minus ones, and $(1/2^m)\mathscr{E}\mathscr{E}' = I$. Multiplying both sides of (1.1) by $(1/2^m)\mathscr{E}$ gives

$$f = \frac{1}{2^m} \mathscr{E}F. \tag{1.2}$$

Let $A = (m, N, 2, 2,)$ be a BA with index set $\{\mu_0, \mu_1, \mu_2\}$. Each column of A represents a treatment, where if there is a one in row i, then the corresponding treatment will have factor i at the second level, and if there is a zero in row i, factor i will be at its first level. Let y be an N rowed column vector, where the ith entry in y represents the yield of the treatment which corresponds to the ith column of A.

Using A, we wish to estimate the mean and the m main effects under the assumption that no interactions of two or more factors are present. Thus $F' = (\beta', I_0')$ where $\beta' = [\mu; F_1, F_2, \cdots, F_m]$ and I_0 is a vector of all zeros. Let $\mathscr{E}_0$ be the matrix which contains the first $m + 1$ columns of $\mathscr{E}$, then from equation (1.2) we have

$$f = \frac{1}{2^m} \mathscr{E}_0 \beta.$$

It is seen from this equation that each entry in f corresponds to a row of $\mathscr{E}_0$. That is, (1) corresponds to the first row of $\mathscr{E}_0$, f_1 to the second row, and so forth. Using this correspondence, we generate an $(N \times m + 1)$ matrix X as follows.

Consider the jth column of A. Then this column corresponds to a particular treatment, t (say). We take as the jth row of X the row in $\mathscr{E}_0$ which corresponds to treatment t.

Let us make the assumption that the application of the treatments represented by the columns of A is done using a completely randomized design with one replication per column. Thus we may assume that there are no block effects. We further assume that $\operatorname{Var}(y) = \sigma^2 I_N$ and that the effects are additive. The construction of X then gives that the expected value of y, written $E(y)$, is equal to $(1/2^m)X\beta$.

The normal equations are

$$(1.3) \qquad \frac{1}{2^m} X'X\hat{\beta} = X'y ,$$

so that if $(X'X)^{-1}$ exists, the least-square estimates are given by

$$(1.4) \qquad \hat{\beta} = 2^m (X'X)^{-1} X'y .$$

Following the discussion in Bose and Srivastava (1964) one can show that

$$X'X = \begin{bmatrix} N & aj \\ aj' & (N-b)I + bj'j \end{bmatrix}^{(m+1\times m+1)}, \qquad j = (1, \cdots, 1)$$

where $a = \mu_2 - \mu_0$ and $b = \mu_0 - 2\mu_1 + \mu_2$. It is then a straightforward calculation to find $(X'X)^{-1}$. The interested reader may check to see that $(X'X)^{-1}$ is given by the following:

$$(X'X)^{-1} = \begin{bmatrix} \frac{1}{N}\left(1 + \frac{a^2}{N}(mq + m(m-1)r)\right) & -\frac{a}{N}(q + (m-1)r)j \\ -\frac{a}{N}(q + (m-1)r)j' & (q-r)I + rj'j \end{bmatrix}$$

where

$$q = \frac{N^2 - a^2(m-1) + (m-2)Nb}{(N^2 - a^2m + (m-1)Nb)(N-b)} \qquad \text{and}$$

$$r = -\frac{(Nb - a^2)}{(N^2 - a^2m + (m-1)Nb)(N-b)} .$$

The sum of squares due to error, S_e^2, is given by

$$(1.5) \qquad S_e^2 = y'y - y'X\hat{\beta} .$$

The number of degrees of freedom for error is $N - (m + 1)$. The expressions (1.3) and (1.4) can be used, for example, to carry out t-tests for hypotheses that any individual effect is zero.

2. Diophantine equations. In this section we will be concerned with a set of diophantine equations, which form a set of necessary conditions for the existence of BA's.

These equations are given by

LEMMA 2.1. *In the* BA$(m, N, 2, t)$ *with index set* $\{\mu_0, \cdots, \mu_t\}$, *let* $n_j^{(i)}$ *be the number of i-dimensional columns which contain exactly j ones*, $i = t, \cdots, m$, $j = 0, \cdots, i$. *Then*

$$\binom{i}{t}\binom{t}{l}\mu_l = \sum_{j=l}^{i-t+l} \binom{j}{l}\binom{i-j}{t-l} n_j^{(i)} ,$$

where $l = 0, 1, \cdots, t$.

Chopra (1967) has given a very simple straightforward proof for the above when $i = m$. The proof also applies when $t \leqq i < m$, so that no further work is needed. We shall be concerned with the above equations in a slightly different form and in less generality, so we give the following corollary.

COROLLARY 2.2. *In the* BA$(m, N, 2, 2)$ *with index set* $\{\mu_0, \mu_1, \mu_2\}$, *the following are true.*

$$\text{(i)} \quad \sum_{j=0}^{i} \binom{j}{2} n_j^{(i)} = \binom{i}{2} \mu_2 ;$$
$$\text{(ii)} \quad \sum_{j=0}^{i} j n_j^{(i)} = i(\mu_1 + \mu_2) ;$$
$$\text{(iii)} \quad \sum_{j=0}^{i} n_j^{(i)} = \mu_0 + 2\mu_1 + \mu_2 , \qquad i = 2, \cdots, m .$$

The proof of this corollary can be obtained independently using the usual counting procedures.

Corollary 2.2 provides information regarding the structure of BA's. We shall summarize some of this information for BA's of strength 2 and indicate further possibilities regarding arrays of higher strength.

THEOREM 2.3. *Let A be a* BA$(m, N, 2, 2)$ *with index* $\{\mu_0, \mu_1, \mu_2\}$. *For* $m \geqq 3$ *we have*

(i) $\mu_1 \leqq \mu_0 + \mu_2$ *with equality if and only if* $n_0^{(3)} = n_3^{(3)} = 0$.

(ii) $\mu_1 = \mu_2$ *if and only if* $n_1^{(3)} = 3n_3^{(3)}$. $\mu_0 = \mu_1$ *if and only if* $n_2^{(3)} = 3n_0^{(3)}$.

(iii) *Let* $\mu_1 = \mu_0 + \mu_2$. *Then* $m \leqq 4$ *with equality if and only if* $\mu_0 = \mu_2$.

(iv) *Let* $\mu_1 = \mu_0 + \mu_2$ *and* $m = 4$. *Then A is a* BA *of strength* 3 *with index* $\{\mu_0^{(3)} = 0, \mu_1^{(3)} = \mu_0^{(2)}, \mu_2^{(3)} = \mu_0^{(2)}, \mu_3^{(3)} = 0\}$.

PROOF. (i) Corollary 2.2 with $i = 3$ gives $n_0 = \mu_0 - \mu_1 + \mu_2 - n_3$, and thus $\mu_1 = \mu_0 + \mu_2$ if and only if $n_0 = n_3 = 0$.

(ii) This follows in a manner similar to (i).

(iii) Since by (i) $n_0 = n_3 = 0$, it follows that any column of A can have at most two zeros and two ones. Thus $m \leqq 4$. Now suppose $m = 4$, then using Corollary 2.2 we find $\mu_1 = 2\mu_2$. Since $\mu_1 = \mu_0 + \mu_2$, this gives $\mu_0 = \mu_2$. Conversely, suppose that $\mu_0 = \mu_2 = c$ so that $\mu_1 = 2c$, then A can be written as the juxtaposition of c arrays of the form

$$B = \begin{matrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{matrix} .$$

Clearly, $m = 4$.

(iv) Since B above is a BA of strength 3 with $\mu_0^{(3)} = 0 = \mu_3^{(3)}$ and $\mu_1^{(3)} = \mu_2^{(3)} = 1$, the juxtaposition of μ_0 B's will be a BA of strength 3 with the indicated index set.

We shall conclude this section by showing how Corollary 2.2 can be used to construct a BA step by step.

EXAMPLE 2.4. We wish to construct a BA $(m, 5, 2, 2)$ with index set $\{\mu_0 = 2, \mu_1 = 1, \mu_2 = 1\}$ with as many rows as possible.

(i) Without loss of generality we can write down the first two rows as

$$\begin{matrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{matrix}.$$

(ii) For $i = 3$, the equations in Corollary 2.2 give just two solutions. They are

	n_0	n_1	n_2	n_3
(a)	1	3	0	1
(b)	2	0	3	0.

The resulting arrays can be written as

$$\text{(a)}\ \begin{matrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{matrix}\ ,\qquad \text{(b)}\ \begin{matrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{matrix}.$$

(iii) Similarly for $i = 4$ the solutions are

	n_0	n_1	n_2	n_3	n_4
(c)	0	4	0	0	1
(d)	1	2	0	2	0.

Solution (d) is not compatible with solutions (a) and (b), and solution (c) is not compatible with solution (b). But using solution (a) we find

$$\text{(c)}\ \begin{matrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{matrix}.$$

(iv) Corollary 2.2 with $i = 5$ gives only one solution namely $n_0 = 1$, $n_1 = 1$, $n_2 = 1$, $n_3 = 1$, $n_4 = 1$ and $n_5 = 0$. But this is incompatible with the array (c). Hence the maximum possible value of m is 4, and BA $(4, 5, 2, 2)$ is unique and given by (c).

3. Bounds on the maximum number of constraints.

A BA is characterized by its index $\Lambda_{s,t}$ and the four tuple (m, N, s, t). However, the index $\Lambda_{s,t}$ determines N, s and t. We may ask, "How big can m be for a given $\Lambda_{s,t}$?" As a first step in this direction we shall investigate the upper bound for m in case $s = 2$. The justification for this is twofold. Firstly our results apply mostly to this case. Secondly these bounds will provide us with bounds, possibly rough, for any s by identifying the s elements with two distinct elements.

Notice first that $m \leqq N$ for all BA's. This can be seen from the following lemma

LEMMA 3.1. *Let A be a nondegenerate* BA $(m, N, 2, t)$ *with index set* $\Lambda_{2,t} = \{\mu_i^{(t)} \mid i = 0, \cdots, t\}$. *Then the eigenvalues of AA' are $b + mc$ with multiplicity one* *and b with multiplicity m − 1, where*

$$b = \sum_{j=1}^{t-1} \binom{t-2}{j-1} \mu_j^{(t)} \qquad c = \sum_{j=2}^{t} \binom{t-2}{j-2} \mu_j^{(t)} .$$

PROOF. Clearly any two rows of A contain b times the 2-tuple $\binom{0}{1}$ [or $\binom{1}{0}$] and c times the 2-tuple $\binom{1}{1}$. Hence

$$AA' = bI_m + cJ_m .$$

Since $A'A$ and AA' have the same m nonzero eigenvalues and $A'A$ is an $N \times N$ matrix, $m \leqq N$. When $s > 2$ the result follows by replacing each nonzero element in A with a one.

A reader familiar with factorial experiments will notice that the inequality $m \leqq N$ can be obtained from the fact that a BA(m, N, s, t) can be considered as N treatment combinations out of an s^m factorial design capable of estimating at least m parameters.

THEOREM 3.2. *Let A be a* BA$(m, N, 2, 2)$ *with index set* $\{\mu_0, \mu_1, \mu_2\}$. *If* $\mu_1^2 > \mu_0\mu_2$, *then*

$$m \leqq \frac{N\mu_1}{\mu_1^2 - \mu_0\mu_2} ,$$

with equality if and only if the number of ones in each column of A is the same.

PROOF. Let μ_j be the number of columns of A which contain j ones, and let

$$\bar{j} = \frac{1}{N} \sum_{j=1}^{m} j n_j .$$

Since $n_j \geqq 0$ for all j, it follows that

$$0 \leqq \sum_{j=0}^{m} (j - \bar{j})^2 n_j = \sum_{j=1}^{m} j^2 n_j - N(\bar{j})^2 .$$

Using Corollary 2.2, we see that

$$\bar{j} = \frac{m}{N}(\mu_1 + \mu_2)$$ and

$$\sum_{j=1}^{m} j^2 n_j = m(m-1)\mu_2 + m(\mu_1 + \mu_2) .$$

Thus

$$0 \leqq m(m-1)\mu_2 + m(\mu_1 + \mu_2) - \frac{1}{N}(m(\mu_1 + \mu_2))^2$$

$$= m^2\mu_2 + m\mu_1 - \frac{m^2}{N}(\mu_1 + \mu_2)^2 .$$

Since $m > 0$ this reduces to

$$m(\mu_1^2 - \mu_0\mu_2) \leqq N\mu_1 .$$

Since by hypothesis $\mu_1^2 - \mu_0\mu_2 > 0$,

$$m \leqq \frac{N\mu_1}{\mu_1^2 - \mu_0\mu_2} .$$

Suppose that the number of ones in each column of A is the same and equal to k. Then, $k = \bar{j}$, and replacing $\leqq$ with $=$ in the first part of this proof gives,

$$m = \frac{N\mu_1}{\mu_1^2 - \mu_0\mu_2} .$$

Suppose

$$m = \frac{N\mu_1}{\mu_1^2 - \mu_0\mu_2} .$$

Then, replacing $\leqq$ with $=$ in the first part of this proof and following the argument in reverse order, it is clear that

$$\sum_{j=0}^{m} (j - \bar{j})^2 n_j = 0 ,$$

so that the number of ones in each column of A is the same and equal to $\bar{j}$.

As mentioned in the introduction, the incidence matrix of a balanced incomplete block (BIB) design is a BA.

COROLLARY 3.3. *A* BA$(m, N, 2, t)$ *which is also the incidence matrix of a* BIB *design* (m, N, r, k, λ) *with* $k < m$ *has the maximum possible number of rows.*

PROOF. We need only show that m is a maximum when the BA is considered to be of strength 2. As an array of strength 2, the BA has index set $\{\mu_0 = N - 2r + \lambda, \mu_1 = r - \lambda, \mu_2 = \lambda\}$. Thus

$$\begin{aligned} \mu_1^2 - \mu_0\mu_2 &= (r - \lambda)^2 - \lambda(N - 2r + \lambda) \\ &= \frac{N}{m}(r - \lambda) > 0 , \end{aligned}$$

where we have used the well-known results that for a BIB design, $Nk = rm$ and $r(k - 1) = \lambda(m - 1)$. Now, since the array has k ones in each column, Theorem 3.2 implies that m is the maximum number of rows possible.

The corollary implies the following interesting property. Let A be a BA of strength t in 2 symbols. If, when A is considered to be a BA of strength 2, $\mu_1^2 \leqq \mu_0\mu_2$, then A cannot have the same number of ones in each column. This follows, since, if A had k (say) ones in each column, the proof of the corollary would give $\mu_1^2 > \mu_0\mu_2$, a contradiction.

(In the above corollary and property we exclude the case $m = k$, which would mean $\mu_0 = \mu_1 = 0$ and $\mu_2 = N$.)

THEOREM 3.4. *Let* A *be a* BA$(m, N, 2, 2)$ *with index set* $\{\mu_0, \mu_1, \mu_2\}$. *If* $\mu_1^2 = \mu_0\mu_2$, *then* $m \leqq N - 1$.

PROOF. With each column of A we associate a distinct variate. With the column of A, which contains $x_1, \cdots, x_m$ ($x_i = 0, 1; i = 1, \cdots, m$) in this order, we associate the variate $f(x_1, \cdots, x_m)$. We consider certain linear functions of these N variates.

Denote by $\sum_N$ the summation over all columns of A. Then, we define the

0th stage function to be

$$\sum_N f(x_1, \cdots, x_m),$$

the sum of all the variates.

Consider two numbers, c_0 and c_1, such that

$$(1) \qquad (\mu_0 + \mu_1)c_0 + (\mu_1 + \mu_2)c_1 = 0 \qquad \text{and}$$

$$(2) \qquad \mu_0 c_0^2 + 2\mu_1 c_0 c_1 + \mu_2 c_1^2 = 0 .$$

Choose any row, r, of A. Corresponding to this choice, we can construct the linear function

$$\sum_N c_i(r) f(x_1, \cdots, x_m) .$$

In the column of A corresponding to the variate $f(x_1, \cdots, x_m)$, the symbol occurring in row r of the array is x_r. In the linear function constructed, we make the coefficient of $f(x_1, \cdots, x_m)$ equal to c_i if $x_r = i$, $i = 0, 1$. The linear functions so defined are called first stage functions. Clearly, there are m first stage functions, one for each row of A.

Provided c_0 and c_1 are not both zero, equation (1) above implies that the first stage functions are orthogonal to the 0th stage functions. Equation (2) implies that each first stage function is orthogonal to each of the other first stage functions. Thus, the $m + 1$ functions defined above are all mutually orthogonal and therefore independent. Since the maximum number of independent linear functions of N variates is N, it follows that $N \geqq 1 + m$ or $m \leqq N - 1$.

We now show that not both c_0 and c_1 are zero. Equation (1) gives

$$c_0 = -\frac{\mu_1 + \mu_2}{\mu_0 + \mu_1} c_1 = -Kc_1 \quad \text{(say)}$$

Equation (2) gives

$$c_1^2(K^2\mu_0 + \mu_2 - 2K\mu_1) = 0 .$$

Thus, $c_1 = 0$ as well as $c_0 = 0$, unless $K^2\mu_0 + \mu_2 = 2K\mu_1$. But as the following shows $K^2\mu_0 + \mu_2 = 2K\mu_1$ if and only if $\mu_1^2 = \mu_0\mu_2$, so that by hypothesis there exist c_0 and c_1 not equal to zero.

$$2\mu_1 K = K^2\mu_0 + \mu_2$$

$$\text{iff} \qquad 2\mu_1 \frac{\mu_1 + \mu_2}{\mu_0 + \mu_1} = \mu_0 \frac{(\mu_1 + \mu_2)^2}{(\mu_0 + \mu_1)^2} + \mu_2$$

$$\text{iff} \qquad \mu_1^2 = \mu_0\mu_2 .$$

THEOREM 3.5. *Let A be a* BA$(m, N, 2, 2)$ *with index set* $\{\mu_0, 1, \mu_2\}$. *Then the maximum value of m is* $m' = \max\{\mu_0, \mu_2\} + 2$ *and* $(m', N, 2, 2)$ *exists.*

PROOF. Without loss of generality, we write the first two rows of A as

$$\begin{array}{ccccc} \overbrace{0 \quad\quad 0}^{\mu_0} & 0 & 1 & \overbrace{1 \quad\quad 1}^{\mu_2} \\ 0 \cdots 0 & 1 & 0 & 1 \cdots 1 . \end{array}$$

Since $\mu_1 = 1$, the third row must have exactly one one in a column of A which has a zero in the first row of A. Call this column c_1. Likewise, the third row must have exactly one one in a column of A which has a zero in the second row of A. Call this column c_2.

Case 1. Suppose that c_1 and c_2 are different columns. Then, without loss of generality, we write the first three rows of A as

$$\begin{array}{ccccccc} \overbrace{\quad\quad\quad}^{\mu_0} & c_1 & c_2 & c_3 & \overbrace{\quad\quad\quad}^{\mu_2 - 1} \\ 0 \quad\quad 0 & 0 & 1 & 1 & 1 \quad\quad 1 \\ 0 \quad\quad 0 & 1 & 0 & 1 & 1 \quad\quad 1 \\ 0 \cdots 0 & 1 & 1 & 0 & 1 \cdots 1 . \end{array}$$

In adding a fourth row, we note that there cannot be more than one one in the first μ_0 columns, else $\mu_1 = 1$ would be contradicted. Suppose there is one one in the first μ_0 columns. Then, since $\mu_1 = 1$, there cannot be a one in column c_1, c_2, or c_3. But this leaves only $\mu_2 - 1$ places in which to place μ_2 ones. Thus, in the fourth row we must put zeros in the first μ_0 columns and one zero and $\mu_2 + 1$ ones in the last $\mu_2 + 2$ columns.

Suppose we place the zero in one of c_1, c_2, or c_3 (c_1 say). Then the number of $\binom{0}{0}$'s occurring in the first and fourth rows of A is $\mu_0 + 1$, a contradiction. It therefore follows that the zero must be in a column which has ones in the first three rows.

Using similar arguments, we can continue to add rows as long as there are columns containing all ones. A will thus have the form of μ_0 columns of zeros and $\mu_2 + 2$ columns containing one 0 and the rest ones, where each row has exactly one zero in the last $\mu_2 + 2$ columns. Clearly the number of rows of A is $\mu_2 + 2$.

Case 2. Suppose c_1 and c_2 are the same column. Then without loss of generality, we write the first three rows of A as

$$\begin{array}{ccccc} \overbrace{\quad\quad\quad}^{\mu_0 - 1} & & & & \overbrace{\quad\quad\quad}^{\mu_2} \\ 0 \quad\quad 0 & 0 & 0 & 1 & 1 \quad\quad 1 \\ 0 \quad\quad 0 & 0 & 1 & 0 & 1 \quad\quad 1 \\ 0 \cdots 0 & 1 & 0 & 0 & 1 \cdots 1 . \end{array}$$

Using similar arguments to those used in Case 1, we see that A has the first $\mu_0 + 2$ columns with one one and the rest zeros, where each row has exactly one one. The remaining μ_2 columns contain all ones. Clearly, the number of rows of A is $\mu_0 + 2$.

Since Cases 1 and 2 are the only ones possible, the theorem follows.

EXAMPLE. Recall Example 2.4. A is a BA (4, 5, 2, 2) with index set {2, 1, 1} and was shown to have the form

$$\begin{array}{ccccc} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 . \end{array}$$

THEOREM 3.6. *Let A be a* BA$(m, N, 2, 2)$ *with index set* $\{\mu_0, \mu_1, \mu_2\}$. *If l is the number of ones in some column of A, we have*

$$0 \leqq m^2(\mu_0\mu_2 - \mu_1^2 - \mu_2) + m\mu_1(N-1) + 2ml(2\mu_1^2 - 2\mu_0\mu_2 + \mu_2 - \mu_1) + l^2(4\mu_0\mu_2 - 4\mu_1^2 - \mu_0 + 2\mu_1 - \mu_2)\,.$$

PROOF. Pick a column with l ones. Count the number of 2-tuples belonging to the remaining columns which coincide with each 2-tuple of the chosen column. The details of the proof are available in Rafter's thesis.

COROLLARY 3.7. *Let A be a* BA$(m, N, 2, 2)$ *with index set* $\{\mu_0, \mu_1, \mu_2\}$ *then*:

(i) *If* $l = 0$,

$$m \leqq \frac{(N-1)\mu_1}{\mu_1^2 - \mu_2(\mu_0 - 1)} \quad \textit{provided} \quad \mu_1^2 - \mu_2(\mu_0 - 1) > 0\,.$$

(ii) *If* $l = m$,

$$m \leqq \frac{(N-1)\mu_1}{\mu_1^2 - \mu_0(\mu_2 - 1)} \quad \textit{provided} \quad \mu_1^2 - \mu_0(\mu_2 - 1) > 0\,.$$

EXAMPLE. Let A be the BA$(m, 9, 2, 2)$ with index $\{3, 2, 2\}$. Then Theorem 3.6 gives

$$0 \leqq 16m - 8ml + 7l^2\,.$$

It is seen that for $l = 0, 1, 2$ the inequality holds for any m. For $l = 3, 4, 5, 6$, $m \leqq 7$ and for $l = 7, 8, 9$, m may exceed 7. Hence if we could construct an array with more than 7 rows the columns of the array must have the number of ones either not exceeding 2 or exceeding 7. Suppose now that we could construct such an array with $m = 8$ rows. From the index of the array we see that each row has to have 4 ones. Thus the total number of ones of the array is 32. Consequently the array must contain at least two columns with $l > 7$. This in turn implies that a four-rowed subarray contains two columns of all ones, which contradicts the equations of Corollary 2.2 with $i = 4$.

We shall present an array with $m = 7$ satisfying the above conditions, showing that the maximum can in fact be achieved.

$$A = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{matrix}\,.$$

We conclude this section with a theorem and an example related to BA with $s \geqq 2$ symbols and strength $t \geqq 2$.

THEOREM 3.8. *Let A be the* BA(m, N, s, t) *with index set* $\Lambda_{s,t}$. *Then A contains s* BA's, *where each array is of strength $t-1$ in s symbols.*

PROOF. Choose any row, r, of A. Divide the columns of A into s sets, so that each column in a given set has the same number in row r. Exclude row r, and call the $m-1$ rows of the set, which has j in row r, A_j. Let N_j be the number of j's in row r, then A_j has N_j columns. We show A_j is a BA$(m-1, N_j, s, t-1)$.

Excluding r, choose any $t-1$ rows of A. By definition, every possible $(t-1)$-tuple must occur in these $t-1$ rows with the j's of row r. Moreover, for A_j,

$$\lambda_{x_1,\cdots,x_p}^{(t-1)i_1,\cdots,i_p} = \lambda_{j,x_1,\cdots,x_p}^{(t),1,i_1,\cdots,i_p},$$

where $p = \min\{s, t-1\}$, $x_j = 0, \cdots, s-1$, $i_j = 0, 1, \cdots, t-1$, and $\sum_{j=1}^{p} i_j = t-1$. (If $j = x_k \in \{x_1, \cdots, x_p\}$),

$$\lambda_{j,x_1,\cdots,x_p}^{(t),1,i_1,\cdots,i_p} = \lambda_{x_1,\cdots,x_k,\cdots,x_p}^{(t)i_1,\cdots,i_k+1,\cdots,i_p}.$$

Thus A_j, $j = 0, \cdots, s-1$, is a BA.

This theorem can also be established from the usual partitioning of the total degrees of freedom in the ANOVA table of the specific design BA(m, N, s, t).

The bounds derived in Section 3 can also be useful when directly applied to BA's of strength 2 in more than 2 symbols. For example, let A be the BA$(m, 20, 3, 2)$ with $\lambda_{00} = 4$, $\lambda_{01} = 3$, $\lambda_{02} = 3$, $\lambda_{11} = 1$, $\lambda_{12} = 1$ and $\lambda_{22} = 1$. Replacing all nonzero elements in A with 1 gives a BA$(m, 20, 2, 2)$ with $\mu_0 = 4$, $\mu_1 = 6$, $\mu_2 = 4$. By Theorem 3.2 $m \leqq 20(6)/(36 - 16) = 6$. Thus A can have at most 6 rows.

Indeed A can be expressed as follows:

$$A = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \end{matrix}.$$

4. Construction. The problem of actual construction of BA's is very complex. The algebraic conditions described thus far can be used to analyze some properties of BA's. However, the algebraic conditions neither ensure the existence of BA's nor give any clue as to how to construct them in case they do exist. The main difficulty in construction is due to the fact that the number of columns of the array determined by the index $\Lambda_{s,t}$ grows very fast with s or t.

One possible attack on the problem which arises naturally is to construct some smaller number of columns which generate the whole array. We call such columns a scheme for the array and shall describe presently one way of searching for a scheme.

This terminology was introduced by R. C. Bose (1939) and used by many other researchers, e.g. Seiden (1954). Recently some authors Webb (1968) and Margolin (1969a, 1969b) called such techniques fold-over techniques. It will be shown here that this method works also for balanced arrays.

Let S be an ordered set of s elements, $e_0, e_1, \cdots, e_{s-1}$. For any positive integer t, consider the s^t different ordered t-tuples of the elements of S. Then these can be divided into s^{t-1} sets, each set consisting of s t-tuples and closed under cyclic permutations of the elements of S. We denote these sets by S_i, $i = 1, 2, \cdots, s^{t-1}$.

We can define the sets S_i, $i = 1, \cdots, s^{t-1}$, as follows. Consider the s^{t-1} distinct $(t-1)$-tuples formed from elements of S. Let the first t-tuple of each S_i be $(e, e_{i_1}, \cdots, e_{i_{t-1}})'$, where e is a fixed element arbitrarily chosen from S, and $(e_{i_1}, \cdots, e_{i_{t-1}})'$ is one of the distinct $(t-1)$-tuples formed from elements of S. The additional $s-1$ t-tuples of each of the sets S_i are obtained from the first by cyclic permutation of the elements of S.

A matrix T of m rows with elements in S will form a scheme if in every t-rowed submatrix of T the number of columns belonging to an S_i is a positive integer, with the restriction that if S_j contains a column which is a row wise permutation of a column in S_i, then the number of columns occurring in a t-rowed submatrix from S_j is the same as the number of columns occurring in a t-rowed submatrix from S_i.

We may now state the following theorem.

THEOREM 4.1. *A matrix T which forms a scheme as described above can be used to construct a* BA(m, N, s, t).

PROOF. Append to the columns of the scheme T all transformations of these columns consisting of cyclic permutations of the elements of S. The result will clearly be a BA(m, N, s, t) with the index $\Lambda_{s,t}$ determined by T.

EXAMPLE. Let $S = \{0, 1\}$ and $t = 3$. The four distinct 2-tuples, which can be formed from S are $\binom{0}{0}\binom{0}{1}\binom{1}{0}\binom{1}{1}$. Pick 0 as the element e of the above discussion, then

$$S_1 = \begin{Bmatrix} 0 & 1 \\ 0 & 1 \\ 0, & 1 \end{Bmatrix}, \quad S_2 = \begin{Bmatrix} 0 & 1 \\ 0 & 1 \\ 1, & 0 \end{Bmatrix}, \quad S_3 = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0, & 1 \end{Bmatrix} \quad \text{and} \quad S_4 = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 1, & 0 \end{Bmatrix}.$$

Let

$$T = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{matrix},$$

and let s_i be the number of times columns from S_i occur in any three rows of T. Then one can check that

$$s_1 = 1 \quad \text{and} \quad s_2 = s_3 = s_4 = 3 .$$

Thus we have a BA of strength 3 given by

$$A = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{matrix}.$$

A close look at T in the above example reveals that it is a BA of strength 2. We may thus ask whether it is always true that a BA of strength 2 forms a scheme for the construction of a BA of strength 3. This is answered by the following.

THEOREM 4.2. *Let $t = 2u$. Then a* BA$(m, N, 2, t)$ *with index set* $\{\mu_i^{(t)} \mid i = 0, \cdots, t\}$ *forms a scheme for the construction of a* BA$(m, 2N, 2, t+1)$ *with index set* $\{\mu_i^{(t+1)} \mid i = 0, \cdots, t+1\}$, *where*

$$\mu_k^{(t+1)} = \mu_{t+1-k}^{(t+1)} = \sum_{i=0}^{t-2k} (-1)^i \mu_{k+i}^{(t)}, \qquad k \leqq u .$$

PROOF. A set S_i is composed of vectors

$$v = (x_1, \cdots, x_{t+1})' \qquad \text{and}$$
$$v^* = (x_1^*, \cdots, x_{t+1}^*) ,$$

where if v has k ones, then v^* has $t + 1 - k$ ones.

Suppose v contains k ones, then, v will appear in $t + 1$ rows of $(m, N, 2, t)$ m_k times, and v^* will appear in those $t + 1$ rows m_{t+1-k} times. Moreover,

$$m_k = \sum_{i=0}^{t-2k} (-1)^i \mu_{k+i}^{(t)} + (-1)^{t-2k+1} m_{t+1-k} , \qquad \text{where } 2k \leqq t .$$

Since $t - 2k + 1 = 2(u - k) + 1$ is odd, we have

$$m_k + m_{t+1-k} = \sum_{i=0}^{t-2k} (-1)^i \mu_{k+i}^{(t)} ,$$

which is independent of the $t + 1$ rows chosen from BA$(m, N, 2, t)$.

Suppose S_j contains a permutation of v. Call it u. Then u^* is a permutation of v^*, and, since m_k and m_{t+1-k} are independent of order, the number of columns of S_j occurring in any $t + 1$ rows of BA$(m, N, 2, t)$ is

$$\sum_{i=0}^{t-2k} (-1)^i \mu_{k+i}^{(t)} , \qquad 2k \leqq t ,$$

the same as for S_i. Thus BA$(m, N, 2, t)$ forms a scheme satisfying Theorem 4.1.

It is clear from the construction of BA$(m, 2N, 2, t + 1)$ that a $(t + 1)$-tuple containing k ones will appear $m_k + m_{t+1-k}$ times. Likewise, a $(t + 1)$-tuple containing $t + 1 - k$ ones will appear $m_{t+1-k} + m_k$ times. Thus, where $k \leqq t/2 = u$, we have

$$\mu_k^{(t+1)} = \mu_{t+1-k}^{(t+1)} = \sum_{i=0}^{t-2k} (-1)^i \mu_{k+i}^{(t)} .$$

COROLLARY 4.3. *A* BA$(m, N, 2, 2)$ *with index set* $\{\mu_0^{(2)}, \mu_1^{(2)}, \mu_2^{(2)}\}$ *forms a scheme for the construction of a* BA$(m + 1, 2N, 2, 3)$ *if either* $\mu_1^{(2)} = \mu_2^{(2)}$ or $\mu_0^{(2)} = \mu_1^{(2)}$. *Furthermore, if m' is the maximum number of constraints of* BA$(m, N, 2, 2)$, *then $m' + 1$ will be the maximum number of constraints of* BA$(m + 1, 2N, 2, 3)$.

PROOF. Without loss of generality, we assume $\mu_0^{(2)} = \mu_1^{(2)}$. (If $\mu_1^{(2)} = \mu_2^{(2)}$, we can interchange zeros and ones in BA$(m, N, 2, 2)$ and obtain an array where $\mu_0^{(2)} = \mu_1^{(2)}$.)

Let T be the BA$(m, N, 2, 2)$ and T^* be the array obtained from T by interchanging zeros and ones. Let TT^* represent the juxtaposition of T and T^*, then, by Theorem 4.2, TT^* is a BA $(m, 2N, 2, 2)$ with index $\{\mu_0^{(3)} = \mu_2^{(2)}, \mu_1^{(3)} = \mu_0^{(2)}, \mu_2^{(3)} = \mu_0^{(2)}, \mu_3^{(3)} = \mu_2^{(2)}\}$.

We add the $(m + 1)$st row to TT^* by placing a zero in each column of T^* and a one in each column of T. Call the resulting array A. It is easy to see that A is a BA $(m + 1, 2N, 2, 3)$. The remainder of the theorem follows from Theorem 3.8.

EXAMPLE. Consider A in the example following Theorem 4.1. A is a BA (4, 20, 2, 3) with index set {1, 3, 3, 1}. By placing ones under the first 10 columns of A and zeros under the remaining 10 columns, we obtain a BA (5, 20, 2, 3) with index set {1, 3, 3, 1}.

Corollary 4.3 has application in the following kind of situation. Suppose that, after an experiment has been performed, it becomes desirable to include an additional factor, where the original design was a BA $(m, N, 2, 2)$. Then instead of performing an entirely new experiment, we consider the original experiment to be half of an array, BA $(m + 1, 2N, 2, 2)$ and add the remaining half in which the new factor will appear constant at the 1 level.

The problem may arise that, while, in the original experiment, there were no interaction effects, the introduction of a new factor makes this assumption questionable. The corollary shows that the additional treatment combinations may be designed so that the augmented experiment is of strength 3, which will allow estimation of main effects in the presence of first order interactions.

We now turn to the construction of BA's in more than two symbols. Since we will be concerned with arrays of strength two, we write $\lambda_{i,j}^{1,1}$ as $\lambda_{i,j}$, $i, j = 0, 1, \cdots, 2s - 1$.

We have shown previously that the incidence matrix of a BIB design is a BA with the maximum possible number of rows. We now show the following:

THEOREM 4.1. *Consider a* BA *which is also the incidence matrix of a* BIB *design* $(v, b, r, k, \lambda = 1)$. *The existence of this array is equivalent to the existence of a* BA $A = (m, N, s, 2)$ *with index set* $\Lambda_{s,2}$, *where* $m = r$, $N = b - r$, $s = k$, $\lambda_{00} = b - r - (k - 1)(2r - k - 1)$, $\lambda_{0i} = r - k$, $i = 1, \cdots, k - 1$, $\lambda_{ij} = 1$, $i, j = 1, \cdots, k - 1$.

PROOF. Let T be the incidence matrix of the BIB design $(v, b, r, k, 1)$. Then T is a BA $(v, b, 2, 2)$ with index set $\{\mu_{0'} = b - 2r + 1, \mu_1 = r - 1, \mu_2 = 1\}$.

Interchanging columns and rows as necessary, we can put T into the following form. For $i = 1, \cdots, r$, the ith column of T contains a one in its first row, ones in rows $i(k - 1) - (k - 3)$ through $i(k - 1) + 1$, and zeros elsewhere. Next, let K_i represent the $k - 1$ rows from row $i(k - 1) - (k - 3)$ through row $i(k - 1) + 1$, $i = 1, \cdots, 1, r$. Consider column $c_j, j = r+1, \cdots, b$. Since $\mu_2 = 1$, c_j can have at most one in the rows of K_i, where the other entries are zero. If the one occurs in the first row of K_i, enter a one in row i and column $j - r$ of A. If the one occurs in the second row of K_i, enter a two in row i and column $j - r$, and so on. If no one occurs in the rows of K_i, enter a zero in row i column

$j - r$ of A. Clearly, $m = r$, $N = b - r$, and $s = k$, so that we must now show that A is a BA of strength 2 with the indicated index set.

Consider K_u and K_v, $u, v = 1, \cdots, r$, $u \neq v$. Each row of K_u (K_v) contains $r - 1$ ones in the columns from $r + 1$ through b. Since $\mu_2 = 1$, each row of K_u must have exactly one of these ones in common with each row of K_v. Thus for rows u and v of A,

$$\lambda_{i,j} = 1 \qquad i, j = 1, \cdots, k - 1$$

and

$$\begin{aligned} \lambda_{0i} &= r - 1 - (k - 1) \\ &= r - k , \qquad i = 1, \cdots, k - 1 . \end{aligned}$$

Moreover,

$$\begin{aligned} \lambda_{00} &= (b - r) - 2 \textstyle\sum_{i=1}^{k-1} \lambda_{0i} - \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \lambda_{ij} \\ &= b - r - (k - 1)(2r - k - 1) . \end{aligned}$$

Since u and v were arbitrary, the λ's are independent of the two rows of A. Thus A is a BA with the indicated index set.

Given the BA, A, we can reverse the construction and derive T.

EXAMPLE. Consider the BIB design (13, 26, 6, 3, 1), then we write T in the indicated form and A below T, where K_1 contains rows two and three, K_2 contains rows four and five, $\cdots$, and K_6 contains rows 12 and 13.

$$T = \begin{array}{cccccccccccccccccccccccccc}
1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&1&1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&1&1&1&0&0&0&0\\
0&0&1&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1&1&0&0\\
0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0&1&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1&0&0&0&1&0&1&0\\
0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&1&0&0&0&1&0&1\\
0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1&1&0\\
0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0&1\\
0&0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&1&0&0&0&1&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0&1&1&0&0&0
\end{array}$$

$$A = \begin{array}{cccccccccccccccccccc}
1&1&1&1&1&2&2&2&2&2&0&0&0&0&0&0&0&0&0&0\\
1&2&0&0&0&1&2&0&0&0&1&1&1&2&2&2&0&0&0&0\\
0&1&2&0&0&0&0&1&2&0&1&2&0&2&0&0&1&1&2&0\\
0&0&0&1&2&2&1&0&0&0&0&0&1&2&0&0&1&2&1&2\\
1&0&2&0&0&0&0&2&0&1&0&0&2&0&1&2&0&1&1&2\\
0&0&0&1&2&0&0&0&1&2&1&2&0&0&1&2&2&0&0&1
\end{array} .$$

Clearly A is a BA(6, 20, 3, 2) with $\lambda_{00} = 4$, $\lambda_{0i} = 3$, $i = 1, 2$ and $\lambda_{ij} = 1$, $i, j = 1, 2$.

COROLLARY 4.5. *A* BA *constructed by the method of Theorem* 4.4 *has the maximum number of rows.*

PROOF. Replace all nonzero elements by 1. The resulting array will then represent an incidence matrix of BIB design $[r, b-r, (k-1)(r-1), k, (k-1)^2]$, where r, b, and k relate to the array with $\lambda = 1$ which was used for the construction. Hence the conclusion.

REMARK. It is worthwhile noticing that $\lambda_{ij} = 1$, hence a constant for all i and j different from zero. λ_{0i} and λ_{00} depend on the parameters of the BIB used for construction. Setting $r = k$, $\lambda_{00} = 0$ and $\lambda_{0i} = 0$ for all i will yield $N = b - k = (k-1)^2$. For $k = s + 1$, $b = s^2 + s + 1$. Hence if we start with a $v \times b$ incidence matrix of a projective plane, A will become a $(s+1) \times s^2$ orthogonal array with rows forming the $s + 1$ orthogonal squares of s symbols. This is an alternative proof of the equivalence of the two representations of projective planes.

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STATISTICAL AND COMPUTER
OPERATIONS RESEARCH
JOHNSON AND JOHNSON
NEW BRUNSWICK, N.J. 08903

DEPARTMENT OF STATISTICS AND PROBABILITY
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48823