

ASYMPTOTIC RESULTS FOR INFERENCE PROCEDURES BASED ON THE r SMALLEST OBSERVATIONS¹

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We consider procedures for statistical inference based on the smallest r observations from a random sample. This method of sampling is of importance in life testing. Under weak regularity conditions which include the existence of a q.m. derivative for the square root of the ratio of densities, we obtain an approximation to the likelihood and establish the asymptotic normality of the approximation. This enables us to reach several important conclusions concerning the asymptotic properties of point estimators and of tests of hypotheses which follow directly from recent developments in large sample theory. We also give a result for expected values which has importance in the theory of rank tests for censored data.

1. Introduction and summary. Suppose that n items are placed on life test and that the test is censored at the time the r th failure occurs. Below, we consider asymptotic properties of statistical procedures based on the first r order statistics Y_1, \dots, Y_r from a random sample of size n from a cdf F_θ where $\theta \in R^k$. The law $P_{n,\theta}$ of the r order statistics is related to a pdf

$$(1.1) \quad \frac{n!}{(n-r)!} f_\theta(y_1) \cdots f_\theta(y_r) [1 - F_\theta(y_r)]^{n-r},$$

where f_θ is the population pdf. All of our results follow directly from an expansion of the likelihood ratio

$$(1.2) \quad \Lambda_{r,n}[\theta_n, \theta_0] = \sum_{j=1}^r \log \frac{f_{\theta_n}(Y_j)}{f_{\theta_0}(Y_j)} + (n-r) \log \frac{1 - F_{\theta_n}(Y_r)}{1 - F_{\theta_0}(Y_r)},$$

for sequences $\{\theta_n\}$ with $\theta_n = \theta_0 + h_n n^{-\frac{1}{2}}$, $h_n \rightarrow h$. In particular, Theorem 3.1 establishes that, as $n \rightarrow \infty$, $r/n \rightarrow p$

$$(1.3) \quad \Lambda_{r,n}[\theta_n, \theta_0] - h' \Delta_n(\theta_0) \rightarrow_{P_{n,\theta_0}} -\frac{1}{2} h' \Gamma_p(\theta_0) h,$$

where $\Gamma_p(\theta_0)$ is the Fisher information for the censored case (see 3.16) and

$$(1.4) \quad \Delta_n(\theta_0) = \sum_{j=1}^r \frac{2}{n^{\frac{1}{2}}} \dot{\varphi}(Y_j) - (n-r) \frac{\int_{-\infty}^{Y_r} 2\dot{\varphi} f_{\theta_0}}{1 - F_{\theta_0}(Y_r)},$$

with φ defined in assumptions (A). Corollary 3.2 shows that $\mathcal{L}[\Delta_n | P_{n,\theta_0}]$ converges to a normal distribution with zero mean vector and covariance Γ_p .

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Several important conclusions can be drawn from the convergence (1.3) and the asymptotic normality of $\Delta_n(\theta_0)$. These in turn employ assumptions (A) and (B) below together with the assumption that $\Gamma_p(\theta_0)$ is positive definite.

The first set of conclusions, dealing with asymptotic properties of point estimators, follows immediately from the representation theorem of Hájek (1970) (see also Roussas and Soms (1971)). Hájek's proof requires only (1.3) and the asymptotic normality of $\Delta_n(\theta_0)$. The main conclusions for estimators are given by (E1), (E2) and (E3).

Asymptotic efficiency of point estimators. These results apply to any sequence of estimators $T_n = T_n(Y_1, \dots, Y_r)$ such that $\mathcal{L}[n^{1/2}(T_n - \theta_0) - h | P_{n,\theta_n}] \rightarrow L(v)$, for all $h \in R^k$, at the continuity points of $L(v)$. $L(v)$ need not be a normal distribution.

(E1) $L(v)$ has the representation $L(v) = \int \Phi_{\Gamma_p}(v - u) dG(u)$ where $G(u)$ is a distribution in R^k and Φ_{Γ_p} is the normal cdf with zero mean vector and covariance $\Gamma_p^{-1}(\theta_0)$.

$$(E2) \quad \limsup P_{n,\theta_0}[n^{1/2}(T_n - \theta_0) \in C] \leq \int_C d\Phi_{\Gamma_p}$$

for all convex symmetric sets C in R^k .

$$(E3) \quad \liminf E[n^{1/2}h'(T_n - \theta_0)]^2 \geq h'\Gamma_p^{-1}(\theta_0)h, \quad \text{all } h \in R^k,$$

so that the limit covariance D , if it exists, satisfies $D - \Gamma_p^{-1}$ nonnegative definite.

The conclusions regarding asymptotically optimal tests of hypotheses follow from the development in Johnson and Roussas (1969), (1970), (1971), since the relevant proofs there do not use the Markovian character of the observations. Because this is not in as convenient a form for our purposes as Hájek (1970), we review the main steps in the development before presenting the main conclusions on testing hypothesis.

It is first shown in Section 4 of Johnson and Roussas (1970) that $\rho(\Delta_n) \rightarrow \rho(\Delta)$, in law, where Δ is distributed as $N(0, \Gamma_p)$, under P_{θ_0} , and $\rho(\cdot)$ is the usual Euclidean norm. A truncated version Δ_n^* of Δ_n is then constructed, using $\rho(\Delta_n)$, in such a way that $P_{\theta_0}[\Delta_n^* \neq \Delta_n] \rightarrow 0$ and the moment generating function of $\rho(\Delta_n^*)$ converges to that of $\rho(\Delta)$.

Based on the sequence Δ_n^* , an exponential family

$$dR_{n,h} = e^{-B_n(h)} e^{h\Delta_n^*} dP_{n,\theta_0}$$

is constructed which approximates P_{n,θ_n} . Theorem 5.1 shows that $\sup\{\|R_{n,h} - P_{n,\theta_n}\|, h \in \text{bounded set}\} \rightarrow 0$ where $\|\cdot\|$ is the total variation. This exponential family approximation is also crucial to Hájek's proof of the representation theorem for point estimators.

Next, Theorem 6.1 shows that $E_{\theta_0}[Z_n | \Delta_n]$ has the same local power, asymptotically, as any test function Z_n . Furthermore, Theorem 6.3 establishes that test functions based on Δ_n and Δ_n^* , respectively, have the same asymptotic local power functions. Hence, asymptotically, we can restrict ourselves to tests based on Δ_n^* or, if we prefer, Δ_n .

The method of establishing the optimality of tests based on $\Delta_n(\theta_0)$ is by assuming the contrary holds and then obtaining a contradiction for the test based on Δ_n^* rather than Δ_n . This conclusion employs the fact that Δ_n^* is sufficient for the approximating exponential family $R_{n,h}$. The conclusions below for one-sided tests can be obtained from the Neyman–Pearson lemma as in Johnson and Roussas (1969). Alternatively, these could be derived in the same manner as those for two-sided tests using the exponential approximation and the fact that a test based on Δ_n^* is UMP unbiased. The conclusions for tests concerning multi-dimensional parameters, in turn, use the admissibility of the class of tests based on Δ_n^* which reject outside of convex sets. The complete details appear in Johnson and Roussas (1971). All of this development uses only (1.3) and the limit law of Δ_n .

We now state our main conclusions regarding statistical tests of hypotheses based on censored data.

Asymptotically optimal tests of hypotheses. The following are established for local alternatives of order \sqrt{n} away from θ_0 . To obtain global results, we would need an assumption like (A5) of Johnson and Roussas (1969).

(T1) Let $\theta \subset R$ and let η_α denote the upper α th point of a standard normal. Then, the test ϕ_n which rejects $H_0: \theta = \theta_0$ for $\Delta_n(\theta_0) > \eta_\alpha \Gamma_p^{1/2}$ is asymptotically most powerful for local alternatives in that for any other sequence of tests $\{\lambda_n\}$, with $E_{\theta_0} \lambda_n \rightarrow \alpha$,

$$\limsup [\sup_{0 \leq h < c} (E_{\theta_n} \lambda_n - E_{\theta_n} \phi_n)] = 0 .$$

(T2) The test ϕ_{1n} which rejects for $|\Delta_n(\theta_0)| > \eta_{\alpha/2} \Gamma_p^{1/2}$ is asymptotically most powerful unbiased. For any other sequence $\{\lambda_n\}$ of tests which is asymptotically of level α and $\liminf \{\inf E_\theta \lambda_n\} \geq \alpha$, $\limsup \{\sup [E_\theta \lambda_n - E_\theta \phi_{1n}]\} \leq 0$ where inf and sup are over bounded sets of h and $\theta = \theta_0 + hn^{-1/2}$.

(T3) For testing $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$ when $\theta \subset R^k$, the test which rejects for $\Delta_n' \Gamma_p^{-1} \Delta_n$ large has asymptotically best average power over certain ellipsoids and is asymptotically most stringent (see Johnson and Roussas (1971) for the relevant definitions).

The regularity conditions imposed here are much weaker than those imposed in previous papers on life testing. For instance, they include the normal, log normal, Weibull, exponential and gamma. In the latter cases, the location parameter fixes the support and this must be known, otherwise n is not necessarily the correct normalization. See David (1970), Chapter 6, for a survey of more applications. Chernoff, Gastwirth and Johns (1967) established lower bounds for the variance of point estimators of location and scale parameters. They require that the partials of $\log f$ exist everywhere and a condition on f''_{θ_0} . This is slightly stronger than our assumptions even for this special case. The results here extend their optimal estimator to a wider class than those which are asymptotically normal.

Incidentally, it can be readily observed from the discussion in Section 3 that

the limiting covariance $\Gamma_p(\theta_0)$, given by (3.16), is also the covariance for fixed time censoring.

Section 4 contains a lemma which shows the equality of the expected value of the last term in Δ_n and the expected value of the scores evaluated at the unobserved order statistics. This result is also of importance in the derivation of locally most powerful rank tests for a general parameter. It is also shown that $E[\Delta_n(\theta_0)]$ is the zero vector for each n . We conclude with an application to the double exponential with $p = \frac{1}{2}$.

2. Assumptions and preliminary results. We first make some smoothness assumptions regarding the law of the univariate distributions. These are similar to those employed in Johnson and Roussas (1970) except that they are further specialized to Lebesgue measure. Although most of the results hold without this specialization, there would be difficulty in defining the censoring scheme without it.

Let Θ be an open subset of R^k , $(\mathcal{X}, \mathcal{A})$ a measurable space and, for each $\theta \in \Theta$, Q_θ a probability on $(\mathcal{X}, \mathcal{A})$ such that $X_1, X_2, \dots, X_n, \dots$ are independent and identically distributed with X_n taking values in the Borel real line (R, \mathcal{B}) . Set \mathcal{A}_n equal to the σ -field induced by (X_1, \dots, X_n) and let $Q_{n,\theta}$ denote the restriction of Q_θ to \mathcal{A}_n . We can, if we wish, transfer to the coordinate space.

Assumptions (A).

(A1) The law of X_1 has pdf $f_\theta(x)$ with respect to Lebesgue measure and the set where it is positive does not depend on θ .

(A2) Set $\varphi(\theta, \theta^*) = [f_{\theta^*}(x)/f_\theta(x)]^{\frac{1}{2}}$. Then

(i) For each $\theta \in \Theta$, $\varphi(\theta, \theta^*)$ is differentiable in q.m. at (θ, θ) when $P_{1,\theta}$ is employed. Denote this derivative by $\dot{\varphi}(\theta)$.

(ii) $\dot{\varphi}(\theta)$ is $X_1^{-1}(\mathcal{B}) \times \mathcal{C}$ measurable where \mathcal{C} is the class of Borel subsets of Θ .

(iii) For every $\theta \in \Theta$, $4E_\theta[\dot{\varphi}(\theta)\dot{\varphi}'(\theta)]$ is positive definite.

Although $\dot{\varphi} = \dot{\varphi}(x, \theta)$, we will write $\dot{\varphi}(\theta)$ when we want to emphasize the parameter and $\dot{\varphi}(X_1)$ when we want to consider it as a function of a random variable. We allow a similar abuse of notation with $\varphi = \varphi(X, \theta, \theta_n)$ and sometimes write $\varphi(X_1)$.

Under assumptions (A), we have the following result when the X_n are governed by Q_{n,θ_0} and the alternatives are of the form $\theta_n = \theta_0 + h_n n^{-\frac{1}{2}}$ with $h_n \rightarrow h$.

$$(2.1) \quad \max_{i \leq n} |\varphi(X_i) - 1| \rightarrow_{Q_{n,\theta_0}} 0,$$

$$(2.2) \quad n^{\frac{1}{2}}(\varphi(X_1) - 1) \rightarrow_{q.m.} h' \dot{\varphi}(\theta_0),$$

$$(2.3) \quad n^{\frac{1}{2}}(\varphi^2(X_1) - 1) \rightarrow 2h' \dot{\varphi}(\theta_0) \quad \text{in 1st mean},$$

$$(2.4) \quad E_{\theta_0} \dot{\varphi}(X_1) = 0 \quad (k \times 1 \text{ column vector}).$$

These are (3.1.3), (3.1.2), Lemma 3.1.3 and Lemma 3.1.4 (i) in Roussas (1965).

The existence of a q.m. derivative for φ implies the existence of a pointwise partial for the cdf $F_\theta(x)$. For notational convenience, we sometimes write f for f_{θ_0} .

LEMMA 2.1. *Under assumptions (A), uniformly on bounded sets of $h \in R^k$ and z ,*

$$(2.5) \quad n^{\frac{1}{2}} \left[F_{\theta_n}(z) - F_{\theta_0}(z) - \frac{h'}{n^{\frac{1}{2}}} \dot{F}(z) \right] \rightarrow 0,$$

where

$$(2.6) \quad h' \dot{F}(z) = \int_{-\infty}^z 2h' \phi f.$$

PROOF. We write $F_{\theta_n}(z) - F_{\theta_0}(z) = \int_{-\infty}^z (\varphi^2 - 1)f$, so that the difference is

$$n^{\frac{1}{2}} \int_{-\infty}^z \left[(\varphi^2 - 1) - \frac{2h'}{n^{\frac{1}{2}}} \phi \right] f$$

which, by (2.2), goes to zero uniformly in bounded sets of h .

We now describe the sampling scheme. Only the first r order statistics (Y_1, \dots, Y_r) are observed where r is selected so that

$$(2.7) \quad \left| \frac{r}{n} - p \right| < \frac{1}{n} \quad \text{with} \quad 0 < p < 1.$$

We further assume that f_{θ_0} is positive in a neighborhood of the p th percentile ξ_p , so that it is unique.

The next result is a specialization of Bahadur (1966) to the uniform order statistics $F(Y_i)$, $i = 1, \dots, n$. Let

$$(2.8) \quad Z_n(t) = \#X_1, \dots, X_n \leq \xi_t, \quad 0 < t < 1.$$

LEMMA 2.2. *With r given by (2.7) and $Z_n \equiv Z_n(p)$ by (2.8),*

$$n^{\frac{1}{2}} \left[\frac{Z_n}{n} - p + F_{\theta_0}(Y_r) - p \right] \rightarrow_{Q_{n,\theta_0}} 0.$$

Below, we investigate the behavior of certain functions over the random set

$$(2.9) \quad A_{r,n} = \{(Y_r, \xi_p] \text{ if } Y_r < \xi_p \text{ and } (\xi_p, Y_r] \text{ if } Y_r \geq \xi_p\}.$$

We also have the following property for the $Z_n(t)$.

LEMMA 2.3. *Let $Z_n(p)$ be defined by (2.8), then*

$$\sup_{\xi_s, \xi_t \in A_{r,n}} n^{\frac{1}{2}} \left| \frac{Z_n(s)}{n} - \frac{Z_n(t)}{n} - [F_{\theta_0}(\xi_s) - F_{\theta_0}(\xi_t)] \right| \rightarrow_{Q_{n,\theta_0}} 0.$$

PROOF. Since $Z_n(s) = \#F_{\theta_0}(X_i) \leq p$ and $F_{\theta_0}(X_i)$ is uniform, it is sufficient to show convergence in probability for uniform variables. It is well known that $V_n(t) = n^{\frac{1}{2}}(Z_n(t)/n - t)$ converges weakly to a Brownian bridge. Therefore, by a characterization of tightness in C (see Theorem 8.2, Billingsley (1968)) the modulus of continuity of a continuous piecewise linear approximation to $V_n(t)$ is small with high probability and the same holds true for $V_n(t)$. In particular,

given any ε, η there exists a $\delta, 0 < \delta < 1$, such that

$$Q_{n,\theta} [\sup_{t \leq s \leq t+\delta} |V_n(t) - V_n(s)| > \varepsilon] \leq \eta \quad \text{all sufficiently large } n.$$

For a direct verification see (13.16) of Billingsley (1968). Since $\{(s, t) : \xi_s, \xi_t \in A_{r,n}\} \subset \{(s, t) : |s - t| \leq \delta\}$ for sufficiently large n with probability one, the result follows.

In order to establish our main results, we have to know that φ and ψ act smoothly at ξ_p . This does not seem to follow from the quadratic mean calculus and we must make additional assumptions. We will first state these in the form needed later and then prove a lemma which leads to sufficient conditions which are easy to verify.

The assumptions may be expressed in terms of the indicator function $I_{A_{r,n}}$.

Assumptions (B).

$$(B1) \quad \sum_1^n (\varphi - 1)I_{A_{r,n}} - n \int_{A_{r,n}} (\varphi - 1)f \rightarrow_{Q_{n,\theta_0}} 0,$$

$$(B2) \quad \sum_1^n \frac{h'\psi}{n^{\frac{1}{2}}} I_{A_{r,n}} - n^{\frac{1}{2}} \int_{A_{r,n}} (h'\psi)f \rightarrow_{Q_{n,\theta_0}} 0.$$

In order to see what conditions on the pdf's would imply (B1) and (B2), we prove the following.

LEMMA 2.4. *For all sufficiently large n , let ψ_n be the difference of two non-decreasing functions over the interval $\xi_p \pm n^{-\frac{1}{2}}(\log \log n)^{\frac{1}{2}}$ where each is essentially bounded by M . Then*

$$\sum_1^n \frac{\psi_n}{n^{\frac{1}{2}}} I_{A_{r,n}} - n^{\frac{1}{2}} \int_{A_{r,n}} \psi_n f \rightarrow_{Q_{n,\theta_0}} 0.$$

PROOF. Without loss of generality, we assume that ψ_n is non-decreasing on the interval and we define ψ_n^{-1} by $\inf \{x : \psi_n(x) \geq a\}$. For fixed n and arbitrary ε , consider a partition $\{a_i\}$ of $[-M, M]$ with norm less than ε and not more than $2M\varepsilon^{-1} + 3$ terms. Set $b_i = \psi_n^{-1}(a_i)$ and

$$(2.10) \quad M_l = \text{ess sup}_{[b_l, b_{l+1}]} \psi_n(x), \quad m_l = \text{ess inf}_{[b_l, b_{l+1}]} \psi_n(x).$$

Then,

$$(2.11) \quad M_l - m_l \leq \varepsilon, \quad \text{each } l.$$

Furthermore, if $Z_{nl}^* = \#X_i$ belonging to $[b_l, b_{l+1}]$, then

$$(2.12) \quad \begin{aligned} \sum_1^n \frac{\psi_n}{n^{\frac{1}{2}}} I_{[b_l, b_{l+1}]} - n^{\frac{1}{2}} \int_{b_l}^{b_{l+1}} \psi_n f' & \\ & \leq M_l Z_{nl}^* n^{-\frac{1}{2}} - m_l n^{\frac{1}{2}} [F_{\theta_0}(b_{l+1}) - F_{\theta_0}(b_l)] \\ & \leq M_l \{Z_{nl}^* n^{-\frac{1}{2}} - n^{\frac{1}{2}} [F_{\theta_0}(b_{l+1}) - F_{\theta_0}(b_l)]\} \\ & \quad + \varepsilon n^{\frac{1}{2}} [F_{\theta_0}(b_{l+1}) - F_{\theta_0}(b_l)], \quad \text{all } l. \end{aligned}$$

The lower bound has M_l replaced by m_l and ε by $-\varepsilon$. Define $Z_n(t)$ by (2.8) so that, setting $b_l = \hat{\xi}_{p+\Delta_l}$,

$$(2.13) \quad Z_{nl}^* = Z_n(p + \Delta_{l+1}) - Z_n(p + \Delta_l).$$

Then, for all l such that $b_l, b_{l+1} \in A_{r,n}$,

$$n^{\frac{1}{2}} \left| \frac{Z_{nl}^*}{n} - [F_{\theta_0}(b_{l+1}) - F_{\theta_0}(b_l)] \right| \leq \sup_{\xi_s, \xi_t \in A_{r,n}} n^{\frac{1}{2}} \left| \frac{Z_n(s)}{n} - \frac{Z_n(t)}{n} - [F_{\theta_0}(\xi_s) - F_{\theta_0}(\xi_t)] \right|,$$

and the r.h.s., which does not depend on the partition within $A_{r,n}$, converges in probability to zero by Lemma 2.3. Now Y_r belongs to the interval $\xi_p \pm n^{-\frac{1}{2}}(\log \log n)^{\frac{1}{2}}$, for all sufficiently large n , with probability one. If for each n , we include ξ_p and Y_r in the partition, the result follows if we add the inequalities that correspond to the interval between ξ_p and Y_r , and employ the asymptotic normality of $n^{\frac{1}{2}}[F_{\theta_0}(Y_r) - p]$ and the fact that ϵ is arbitrary.

COROLLARY 2.A. *If, for all sufficiently large n ,*

(B'1) $n^{\frac{1}{2}}(\varphi - 1)$ *is the difference of two non-decreasing functions on the interval $\xi_p \pm n^{-\frac{1}{2}}(\log \log n)^{\frac{1}{2}}$ and each is essentially bounded by M , then (B1) is satisfied. The two functions and M may depend on h . If there is a version of φ such that φ has one-sided limits at ξ_p , then (B2) is satisfied for each $h \in R^k$.*

PROOF. Inspection of the previous proof shows that (2.11) and (2.12) can be established, under (B'2), with a single interval $(Y_r, \xi_p]$ or $(\xi_p, Y_r]$.

REMARK. For a location or scale parameter, a simple sufficient condition for (B'1) and (B'2) is that f'_{θ_0} is continuous at ξ_p . The existence of a continuous derivative insures one-sided monotonicity, for sufficiently large n , and the mean value theorem gives uniform boundedness. This includes most one-parameter applications.

In the remaining sections, we will often employ the joint distribution $P_{n,\theta}$ of the first r order statistics since the probabilities can be computed under $Q_{n,\theta}$ or $P_{n,\theta}$.

3. Proof of main results. In this section, we employ the previous results to obtain the expansion of the likelihood and its asymptotic distribution. We first note that, from Lemma 2.1, $|(1 - F_{\theta_n}(Y_r))/(1 - F_{\theta_0}(Y_r)) - 1| \rightarrow 0$ in probability since $Y_r \rightarrow \xi_p$ in probability. The expansion

$$(3.1) \quad \log Z = (Z - 1) - \frac{1}{2}(Z - 1)^2 + c(Z - 1)^3, \quad |c| \leq 3 \text{ for } |Z - 1| \leq \frac{1}{2}$$

is then applied to each term of $\Lambda_{r,n}$.

LEMMA 3.1. *Under Assumptions (A),*

$$(3.2) \quad \Lambda_{r,n} = \left\{ \sum_{j=1}^r 2[\varphi(Y_j) - 1] + (n - r) \left[\frac{F_{\theta_0}(Y_r) - F_{\theta_n}(Y_r)}{1 - F_{\theta_0}(Y_r)} \right] \right\} - \frac{1}{2} \left\{ \sum_{j=1}^r 2[\varphi(Y_j) - 1]^2 + (n - r) \left[\frac{F_{\theta_0}(Y_r) - F_{\theta_n}(Y_r)}{1 - F_{\theta_0}(Y_r)} \right]^2 \right\} + W_n,$$

where W_n converges in probability to zero.

PROOF. The expansion follows from (3.1) and the result for W_n from (2.1) and the next two lemmas which show that the terms in the second bracket converge to constants.

LEMMA 3.2. Under Assumptions (A),

$$(3.3) \quad (n - r) \left[\frac{F_{\theta_n}(Y_r) - F_{\theta_0}(Y_r)}{1 - F_{\theta_0}(Y_r)} \right]^2 \rightarrow_{P^{n, \theta_0}} \frac{(h' \dot{F}(\xi_p))^2}{1 - p} = \frac{[2 \int_{-\infty}^{\xi_p} h' \phi f]^2}{1 - p}.$$

PROOF. Since $Y_r \rightarrow p$ in probability and F_{θ_0} is continuous, it is sufficient to show that $n^{\frac{1}{2}}[F_{\theta_n}(Y_r) - F_{\theta_0}(Y_r)] \rightarrow h' \dot{F}(\xi_p)$; but this follows from Lemma 2.1.

LEMMA 3.3. Under Assumptions (A),

$$(3.4) \quad \sum_{j=1}^r [\phi(Y_j) - 1]^2 - n^{-1} \sum_{j=1}^r [h' \phi(Y_j)]^2 \rightarrow_{P^{n, \theta_0}} \mathbf{0},$$

$$(3.5) \quad n^{-1} \sum_{j=1}^r [h' \phi(Y_j)]^2 \rightarrow_{P^{n, \theta_0}} \int_{-\infty}^{\xi_p} [h' \phi]^2 f.$$

PROOF. The Markov inequality gives the bound

$$\varepsilon^{-1} \sum_{j=1}^r E \left| [\phi(Y_j) - 1]^2 - \left[\frac{h'}{n^{\frac{1}{2}}} \phi(Y_j) \right]^2 \right| \leq \varepsilon^{-1} n E \left| [\phi(X_1) - 1]^2 - \left[\frac{h'}{n^{\frac{1}{2}}} \phi(X_1) \right]^2 \right|$$

for the probability that the r.h.s. of (3.4) exceeds $\varepsilon > 0$ and this bound converges to zero (see Roussas (1965), equation (3.1.19)).

Next, (3.5) follows from the law of large numbers since

$$n^{-1} \sum_{j=1}^r [h' \phi(Y_j)]^2 - n^{-1} \sum_{i=1}^n [h' \phi(X_i) I_{(-\infty, \xi_p]}]^2 \rightarrow 0 \quad \text{in probability.}$$

This last difference is dominated by $\sum_{i=1}^n n^{-1} h' \phi I_{(\xi_p - \delta, \xi_p + \delta)}$ for all sufficiently large n with δ arbitrary.

For notational convenience, we set $B_p = (-\infty, \xi_p]$ and introduce two statistics corresponding to the case of censoring at a fixed percentile ξ_p .

$$(3.6) \quad S_p = \sum_{i=1}^n \{2(\varphi - 1) I_{B_p} + (1 - p)^{-1} [F_{\theta_0}(\xi_p) - F_{\theta_n}(\xi_p)] I_{B_p^c}\},$$

$$(3.7) \quad \dot{S}_p = \sum_{i=1}^n \left\{ \frac{2h'}{n^{\frac{1}{2}}} \phi I_{B_p} - \frac{h' \dot{F}(\xi_p)}{n^{\frac{1}{2}}(1 - p)} I_{B_p^c} \right\}.$$

These will later be compared with the statistics for censoring at the r th order statistic. Namely,

$$(3.8) \quad S_r = \sum_{j=1}^r 2[\varphi(Y_j) - 1] + (n - r) \left[\frac{F_{\theta_0}(Y_r) - F_{\theta_n}(Y_r)}{1 - F_{\theta_0}(Y_r)} \right],$$

$$(3.9) \quad \dot{S}_r = \sum_{j=1}^r \frac{2h'}{n^{\frac{1}{2}}} \phi(Y_j) - \frac{(n - r)}{n^{\frac{1}{2}}} \frac{h' \dot{F}(Y_r)}{1 - F_{\theta_0}(Y_r)}.$$

LEMMA 3.4. Under Assumptions (A),

$$E_{\theta_0}[S_p - \dot{S}_p] = -nE[(\varphi(X_1) - 1)^2 I_{B_p}] \rightarrow -E(h' \phi)^2 I_{B_p},$$

$$\text{Var}[S_p - \dot{S}_p] \rightarrow 0.$$

PROOF. Set $p_n = F_{\theta_n}(\xi_p)$ and consider the identity

$$\varphi^2 - p_n/p = (\varphi - 1)^2 + 2(\varphi - 1) + 1 - p_n/p.$$

Multiplying both sides by I_{B_p} and taking expected values gives $E[2(\varphi - 1)I_{B_p}] = -E(\varphi - 1)^2 I_{B_p} + p_n - p$ so that $E[S_p - \dot{S}_p] = -nE(\varphi - 1)^2 I_{B_p}$, and this converges to $-E(h'\dot{\varphi})^2 I_{B_p}$ since $n^{1/2}(\varphi - 1) \rightarrow h'\dot{\varphi}$ in q.m. by (2.2). Also,

$$\begin{aligned} \text{Var}[S_p - \dot{S}_p] &= n \text{Var} \left\{ \left[2(\varphi - 1) - \frac{2h'}{n^{1/2}} \dot{\varphi} \right] I_{B_p} + (1 - p)^{-1} \left[p - p_n + \frac{h'}{n^{1/2}} \dot{F} \right] I_{B_p} \right\} \\ &\leq 16E\{[n^{1/2}(\varphi - 1) - h'\dot{\varphi}]^2 I_{B_p}\} + 4(1 - p)^{-1}\{n^{1/2}(p - p_n) + h'\dot{F}\}^2, \end{aligned}$$

which converges to zero by (2.2) and Lemma 2.1.

We now employ the statistics (3.6), (3.7) and the result for first moments to obtain an approximation to S_r in the expansion of $\Lambda_{r,n}$. Here we require the extra smoothness assumptions on the pdf's at ξ_p .

LEMMA 3.5. *Under Assumptions (A) and Assumptions (B) (or (B'1) and (B'2)),*

$$S_r - \dot{S}_r - (S_p - \dot{S}_p) \rightarrow_{P_{n,\theta_0}} 0.$$

PROOF. Let $Z_n \equiv Z_n(p)$ be defined by (2.8). First, we have

$$(3.10) \quad [(n - Z_n) - (n - r)] \left[F_{\theta_0}(\xi_p) - F_{\theta_n}(\xi_p) + \frac{h'}{n^{1/2}} \dot{F}(\xi_p) \right] \rightarrow_{P_{n,\theta_0}} 0$$

by Lemma 2.1 and the asymptotic normality of the binomial variable Z_n . Next, we write

$$F_{\theta_0}(z) - F_{\theta_n}(z) + \frac{h'}{n^{1/2}} \dot{F}(z) = -\int_{-\infty}^z \left[(\varphi^2 - 1) - \frac{2h'}{n^{1/2}} \dot{\varphi} \right] f$$

and employ the asymptotic normality of $F_{\theta_0}(Y_r)$ to give

$$(3.11) \quad [1 - F_{\theta_0}(Y_r)]^{-1} = (1 - p)^{-1} \{1 + (1 - p)^{-1} [F_{\theta_0}(Y_r) - p] + o_p(n^{-1/2})\}.$$

Together, (3.10) and (3.11) give

$$\begin{aligned} &\frac{(n - r)}{1 - F_{\theta_0}(Y_r)} \left[F_{\theta_0}(Y_r) - F_{\theta_n}(Y_r) + \frac{h'}{n^{1/2}} \dot{F}(Y_r) \right] \\ &- \sum_1^n (1 - p)^{-1} \left[p - F_{\theta_n}(\xi_p) + \frac{h'}{n^{1/2}} \dot{F}(\xi_p) \right] I_{B_p^c} \\ (3.12) \quad &= \frac{(n - r)/n}{1 - p} \left[-n \int_{\xi_p}^{Y_r} \left[(\varphi^2 - 1) - \frac{2h'}{n^{1/2}} \dot{\varphi} \right] f \right] + o_p(n^{-1/2}) + o_p(1) \\ &- \frac{(n - r)/n}{1 - p} \cdot n^{1/2} \left[\frac{F_{\theta_0}(Y_r) - p}{1 - p} \right] \cdot n^{1/2} \int_{-\infty}^{Y_r} \left[(\varphi^2 - 1) - \frac{2h'}{n^{1/2}} \dot{\varphi} \right] f \\ &= -n \int_{\xi_p}^{Y_r} \left[(\varphi^2 - 1) - \frac{2h'}{n^{1/2}} \dot{\varphi} \right] f + o_p(1). \end{aligned}$$

The last equality follows since $F_{\theta_0}(Y_r)$ is asymptotically normal and $n^{1/2}(\varphi^2 - 1)$ converges in first mean to $2h'\dot{\varphi}$ by (2.3).

From the definitions (3.6)—(3.9) of the statistics, we now see that it is sufficient to show that

$$(3.13) \quad \sum_{Y_i \in A_{r,n}} \left[2(\varphi - 1) - \frac{2h'}{n^{\frac{1}{2}}} \dot{\varphi} \right] - n \int_{A_{r,n}} \left[(\varphi^2 - 1) - \frac{2h'}{n^{\frac{1}{2}}} \dot{\varphi} \right] f \rightarrow_{P_{n,\theta_0}} \mathbf{0},$$

where $A_{r,n}$ is defined by (2.9). To this end, let B_ε be an interval about ξ_p such that $\int_{B_\varepsilon} (h'\dot{\varphi})^2 f < \varepsilon$. Then, $\limsup \int_{A_{r,n}} n(\varphi - 1)^2 f < \varepsilon$ by the first mean convergence of $n(\varphi - 1)^2$. Since $(\varphi - 1)^2 = (\varphi^2 - 1) - 2(\varphi - 1)$, we see that it remains to show that

$$(3.14) \quad \sum_{Y_i \in A_{r,n}} 2(\varphi - 1) - 2n \int_{A_{r,n}} (\varphi - 1) \rightarrow_{P_{n,\theta_0}} \mathbf{0},$$

$$\sum_{Y_i \in A_{r,n}} \frac{2h'}{n^{\frac{1}{2}}} \dot{\varphi} - 2n^{\frac{1}{2}} \int_{A_{r,n}} h'\dot{\varphi} f \rightarrow_{P_{n,\theta_0}} \mathbf{0}.$$

This follows directly from Corollary 2.4 under the assumptions (B'1) and (B'2) on $\varphi - 1$ and $\dot{\varphi}$.

THEOREM 3.1. *Under the Assumptions (A) and (B) (or (B'1) and (B'2)), for each alternative $h_n \rightarrow h$,*

$$(3.15) \quad \Lambda_{r,n} - \frac{h'}{n^{\frac{1}{2}}} \left[\sum_{i=1}^r 2\dot{\varphi}(Y_i) - \frac{(n-r)}{1 - F_{\theta_0}(Y_r)} \dot{F}(Y_r) \right] \rightarrow -\frac{1}{2} h' \Gamma_p(\theta_0) h,$$

where

$$(3.16) \quad \Gamma_p(\theta_0) = 4 \int_{\xi_p}^{\xi_p} \dot{\varphi} \dot{\varphi}' f + \frac{1}{1-p} \dot{F}(\xi_p) \dot{F}(\xi_p)'$$

is the Fisher information for the censored case.

PROOF. We write \sim when the difference converges in probability to zero. Thus, from Lemma 3.1 together with Lemmas 3.2 and 3.3, we have

$$\Lambda_{r,n} \sim S_r - \frac{1}{2}(1-p)^{-1} [h' \dot{F}(\xi_p)]^2 - \int_{\xi_p}^{\xi_p} (h'\dot{\varphi})^2 f$$

and Lemma 3.4 combined with Lemma 3.5 gives

$$S_r \sim \hat{S}_r + S_p - \hat{S}_p \sim \hat{S}_r - \int_{\xi_p}^{\xi_p} (h'\dot{\varphi})^2 f.$$

The next result yields the asymptotic normality of the statistic which approximates the likelihood in (3.15). That is, of

$$(3.17) \quad \Delta_n(\theta_0) = n^{-\frac{1}{2}} \left[\sum_{j=1}^r 2\dot{\varphi}(Y_j) - \frac{(n-r)\dot{F}(Y_r)}{1 - F_{\theta_0}(Y_r)} \right],$$

which is central to the derivation of the main result.

THEOREM 3.2. *Under Assumptions (A) and (B2) or (B'2),*

$$\hat{S}_p - \hat{S}_r \rightarrow_{P_{n,\theta_0}} \mathbf{0},$$

where the h , which enters the definitions (3.7) and (3.9) of \dot{S}_p and \dot{S}_r , respectively, is arbitrary.

PROOF. Employing (3.11), we expand the last term of \dot{S}_r as

$$(3.18) \quad n^{-\frac{1}{2}}(n-r)[1 - F_{\theta_0}(Y_r)]^{-1} \int_{-\infty}^{Y_r} 2h'\dot{\phi}f \\ = n^{\frac{1}{2}} \int_{-\infty}^{Y_r} 2h'\dot{\phi}f + n^{\frac{1}{2}} \left[\frac{F_{\theta_0}(Y_r) - p}{1-p} \right] \int_{-\infty}^{Y_r} 2h'\dot{\phi}f + o_p(1).$$

Again, setting Z_n equal to the number of observations $\leq \xi_p$, we expand the corresponding term of \dot{S}_p as

$$(3.19) \quad n^{-\frac{1}{2}}(n - Z_n)(1-p)^{-1} \int_{-\infty}^{\xi_p} 2h'\dot{\phi}f \\ = \frac{(1 - Z_n/n) - (1-p)}{1-p} n^{\frac{1}{2}} \int_{-\infty}^{\xi_p} 2h'\dot{\phi}f + n^{\frac{1}{2}} \int_{-\infty}^{\xi_p} 2h'\dot{\phi}f.$$

Subtracting (3.18) from (3.19) gives

$$(3.20) \quad n^{\frac{1}{2}} \int_{Y_r}^{\xi_p} 2h'\dot{\phi}f - n^{\frac{1}{2}} \left\{ \frac{Z_n}{n} - p + [F_{\theta_0}(Y_r) - p] \right\} \\ \times (1-p)^{-1} \int_{-\infty}^{\xi_p} 2h'\dot{\phi}f + o_p(1),$$

since $F_{\theta_0}(Y_r)$ is asymptotically normal and $Y_r \rightarrow \xi_p$ in probability. Furthermore, Lemma 2.2 establishes that the second term in (3.20) is $o_p(1)$.

From the definitions of \dot{S}_r and \dot{S}_p it is clearly sufficient to show that

$$n^{\frac{1}{2}} \sum_{Y_i \in A_{r,n}} \frac{2h'\dot{\phi}}{n^{\frac{1}{2}}} - n^{\frac{1}{2}} \int_{A_{r,n}} 2h'\dot{\phi}f \rightarrow_{P_{n,\theta_0}} 0,$$

where $A_{r,n}$ is given by (2.9). However, this follows directly from Corollary 2.4 under the Assumption (B'2).

Applying the central limit theorem to \dot{S}_p , for each h , we obtain

COROLLARY 3.3. Under the assumptions of the theorem,

$$(3.21) \quad \mathcal{L}[\Delta_n | P_{n,\theta_0}] \rightarrow \mathcal{N}[0, \Gamma_p(\theta_0)]$$

where $\Gamma_p(\theta_0)$ is defined by (3.16), $\Delta_n(\theta_0)$ by (3.17) and $h'\Delta_n(\theta_0) = \dot{S}_r$.

Summarizing, we have the asymptotic normality of Δ_n from (3.21), and Theorem 3.1 states that

$$\Lambda_{r,n} + h'\Delta_n \rightarrow_{Q_{n,\theta_0}} -\frac{1}{2}h'\Gamma_p(\theta_0)h, \quad h \in R^k.$$

These two results lead to an exponential approximation of the sequence of alternatives from which asymptotic optimality properties may be obtained. In particular, the results for sequences of point estimators are just the conclusions from the theorem of Hájek (1970) which requires only these two results.

As far as testing problems are concerned, the results follow from the approximation of P_{n,θ_n} by the exponential family

$$R_{n,h}(A) = \exp[-B_n(h)] \int_A \exp[h'\Delta_n^*] dP_{n,\theta_0}$$

based on a truncated version Δ_n^* of Δ_n , as in Section 5 of Johnson and Roussas (1970), since none of the results from Section 4 onward require anything but Assumptions (A) and the approximation of $\Lambda_{r,n}$ by Δ_n which converges to a normal distribution (see Section 7 for conclusions). The results on multiparameter testing follow from Johnson and Roussas (1971).

Besides the conclusions stated at the start of the paper, just as in Theorem 6.2 of Johnson and Roussas (1970), we also conclude that the alternative distribution satisfies

$$(3.22) \quad \mathcal{L}[\Delta_n(\theta_0) | P_{n,\theta_n}] \rightarrow \mathcal{N}(\Gamma_p h, \Gamma_p).$$

As in Proposition 3.1, the measures $\{P_{n,\theta_0}\}$ and $\{P_{n,\theta_n}\}$ are contiguous. The result (3.22) enables one to calculate asymptotic power.

Finally, we note that, from Theorem 3.2, that \dot{S}_p could be used instead of \dot{S}_r to obtain the exponential approximation to P_{n,θ_n} . The statistic \dot{S}_p is the one that approximates the likelihood, under Assumptions (A), for the sampling scheme which observes only the lifetimes which are not greater than ξ_p . The above lemma states that the asymptotically sufficient statistics for each of the two cases are asymptotically equivalent. See Kendall and Stuart (1967), page 523, for a statement that maximum likelihood estimation is the same for the two schemes although they do not state their conditions.

4. An interpretation of \dot{S}_r . We consider two special cases when Θ is a subset of the real line. For location families with $f_\theta(x) = f(x - \theta)$, Assumptions (A) are satisfied if $0 < \int_{-\infty}^{\infty} [f'(x)/f(x)]^2 f(x) dx < \infty$. In this case, $2\dot{\varphi} = -f'(x - \theta_0)/f(x - \theta_0)$ so that

$$(4.1) \quad \frac{-h\dot{F}(Y_r)}{1 - F_{\theta_0}(Y_r)} = \frac{hf_{\theta_0}(Y_r)}{1 - F_{\theta_0}(Y_r)},$$

which is the hazard rate evaluated at the r th order statistic. For scale alternatives, $f_\theta(x) = \theta^{-1}f(x/\theta)$, $\theta > 0$, it is sufficient for Assumptions (A) to require that $\int_{-\infty}^{\infty} [1 + x(f'(x)/f(x))]^2 f(x) dx < \infty$. Then $2\dot{\varphi} = \theta_0^{-1}[-1 - (x/\theta_0)(f(x/\theta_0)/f(x/\theta_0))]$ and

$$(4.2) \quad \frac{-h\dot{F}(Y_r)}{1 - F_{\theta_0}(Y_r)} = \frac{(Y_r/\theta_0)f_{\theta_0}(Y_r)}{1 - F_{\theta_0}(Y_r)}.$$

Equations (4.1) and (4.2) show the manner in which the hazard rate carries the information on the unobserved order statistics in the case of scale or location alternatives. We also have the following general result which gives another interpretation.

LEMMA 4.1. *If Assumptions (A) hold, then*

$$(4.3) \quad (n - r)E\left[\frac{-h'\dot{F}(Y_r)}{1 - F_{\theta_0}(Y_r)}\right] = \sum_{j=r+1}^n E[2h'\dot{\varphi}(Y_j)], \quad \text{for } 1 \leq r < n.$$

PROOF. Set $b_{n,r} = (n - r)E[-h'F(Y_r)/1 - F_{\theta_0}(Y_r)]$. Now $b_{n,n} = 0$ and a direct

evaluation of $b_{n,n-1}$, using integration by parts with $dU = F^{n-2}f$ and $h'\dot{F}(y) \rightarrow 0$ as $y \rightarrow \infty$ ($-\infty$) (according to (2.4)) gives $b_{n,n-1} = E[2h'\dot{\phi}(Y_n)]$. Furthermore, an integration by parts with $dU = -f[1 - F]^{n-r-1}$ establishes that

$$(4.4) \quad \begin{aligned} b_{n,r} &= c_{n,r}[h'\dot{F}(y)]F^{r-1}(y)[1 - F(y)]^{n-r}|_{-\infty}^{\infty} + b_{n,r-1} - E[2h'\dot{\phi}(Y_r)] \\ &= b_{n,r-1} - E[2h'\dot{\phi}(Y_r)] \end{aligned} \quad 1 < r < n,$$

where $c_{n,r}$ is the constant for the pdf of Y_r . Thus, a general solution of (4.4) is given by

$$b_{n,r} = \sum_{j=r+1}^n E[2h'\dot{\phi}(Y_j)].$$

The result above shows that the "hazard" fate term has an expected value equal to the sum of the expected values of the unobserved scores, which themselves appear in the uncensored version in the general case. Besides giving some intuitive feeling for the manner in which the unobserved scores enter, this relationship is exactly what is needed to obtain the statistic for the locally most powerful rank test for the two sample case. It extends Johnson and Mehrotra (1972) to a general parameter. Furthermore, we also have the exact moments

$$(4.5) \quad \begin{aligned} E\dot{S}_p &= 0 \\ E\dot{S}_r &= nE[2h'\dot{\phi}(x_1)] = 0, \end{aligned}$$

which follow from the lemma and (2.4).

We conclude with a particular example, the double exponential $f_\theta(x) = \frac{1}{2} \exp[-|x - \theta|]$.

EXAMPLE. It is well known that the pdf $\frac{1}{2} \exp[-|x - \theta|]$ satisfies Assumptions (A) with $\dot{\phi} = \frac{1}{2} \operatorname{sgn}(x - \theta_0)$. Since θ_0 is a location parameter, we take $\theta_0 = 0$. Then $\varphi = \exp\{\frac{1}{2}|x| - \frac{1}{2}|x - \theta|\}$. We see that φ is monotone in x . Consider the scheme for $p = \frac{1}{2}$ and $\theta_n = h_n n^{-\frac{1}{2}}$, $h_n \rightarrow h$. Then $n^{\frac{1}{2}}|\varphi - 1| \leq |h_n|e^{|\theta_n|}$ is uniformly bounded. Thus (B'1) and (B'2) are satisfied. Here the censored tests are the same for all $p \geq \frac{1}{2}$.

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