The Annals of Statistics
1974, Vol. 2, No. 5, 1042–1048

SOME GENERALIZATIONS OF DYNAMIC STOCHASTIC APPROXIMATION PROCESSES

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Some generalizations of Dupač's dynamic stochastic approximation have been worked out to the more general cases of time variation. Sufficient conditions for convergence in the mean square and with probability one are given in case of deterministic trend and convergence with a bound is proved for the random trend case, using the estimation scheme $x_{n+1} = g_n(x_n) + a_n(\alpha - y_{n+1}(g_n(x_n)))$. This estimation procedure seems to be of practical use to a variety of problems in estimation, prediction and control.

1. Introduction. The stochastic approximation method originated by Robbins and Monro in [6] is first applied by Dupač in [1] to the dynamic trend case where the root or the point of maximum of a regression function moves in a specified manner during the approximation process. He discussed, in his first and succeeding papers [1], [2], only the cases where the movement of the root or the point of maximum is nonrandom and expressed by a certain linear function of its present location.

In this paper, we shall be concerned with some generalization of this procedure. Namely, dynamic stochastic approximation to the nonlinear trend and/or random trend case will be discussed.

In Section 2, asymptotic convergence of the estimated value to the moving root of the nonlinear regression function in the mean square and with probability one is proved for the case where the trend is expressed by a certain nonrandom deterministic nonlinear function of the present location.

In Section 3, we show also that this procedure makes the mean square error bounded in the case where the random components are involved in the trend.

2. Asymptotic convergence to moving root. Denote by R the real line, and for each $x \in R$ and for each integer n, let $y_n(x, x^{n-1})$ be an observable random variable with conditional expectation, given $x^{n-1} = (x_1, \cdots, x_{n-1})$, $M_n(x)$, which is an (unknown) real nondecreasing function defined for all $x \in R$. That is to say,

$$E(y_n(x, x^{n-1}) \mid x^{n-1}) = M_n(x) \tag{1}$$

and

$$w_n(x, x^{n-1}) = y_n(x, x^{n-1}) - M_n(x) . \tag{2}$$

Received June 1972; revised May 1973.

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AMS 1970 *subject classifications*. Primary 62L20; Secondary 93E20.

Key words and phrases. Sequential estimation, stochastic approximation, convergence with probability one, convergence in the mean square, root searching, maximum searching.

Suppose that the equation

$$M_n(x) = \alpha \tag{3}$$

has a single root θ_n which is unknown and is to be estimated by choosing a number of x_n values and observing the corresponding $y_n(x_n)$'s.

In this paper, we assume, in general, the root θ_n moves in such a manner, that is,

$$\theta_{n+1} = g_n(\theta_n) + v_n \tag{4}$$

where $g_n(x)$ is a known real function defined for all $x \in R$ and v_n is an unknown (random or nonrandom) component and independent of x.

First we consider the case where v_n is nonrandom.

Let $\{a_n\}$ be a sequence of positive numbers and x_1 be an arbitrary number.

Define for $n = 1, 2, \cdots$

$$x_{n+1} = x_n^* + a_n(\alpha - y_n^*) \tag{5}$$

where

$$x_n^* = g_n(x_n)$$

and y_n^* is the observation of $M_{n+1}(x_n^*)$, i.e.,

$$y_n^* = M_{n+1}(x_n^*) + w_{n+1}(x_n^*, x^{n-1}) .$$

The following theorem gives us sufficient conditions for the estimation process (5) to converge to the true moving root.

THEOREM 1. *Suppose the following conditions are satisfied;*

C1 *There exist positive numbers K_1 and K_2 such that*

$$|M_n(x) - \alpha| \leqq K_1|x - \theta_n| + K_2 \quad \textit{for} \quad -\infty < x - \theta_n < \infty . \tag{6}$$

C2 *For all n and for each $0 < \varepsilon < 1$, there exists a positive number K_3 such that*

$$\inf_{\varepsilon<|x-\theta_n|} \{M_n(x)/\operatorname{sgn}(x - \theta_n)|x - \theta_n|^{K_3}\} > 0 . \tag{7}$$

C3 *There exists a sequence of positive numbers $\{\gamma_n\}$ independent of x and y such that*

$$|g_n(x) - g_n(y)| \leqq \gamma_n|x - y| \quad \textit{for} \quad -\infty < x - y < \infty , \tag{8}$$

and

$$\textstyle\sum_{n=1}^{\infty} (\gamma_n - 1)^+ < \infty \tag{9}$$

where z^+ means $(z + |z|)/2$.

C4 *$w_n(x, x^{n-1})$ is a random variable satisfying*

$$E(w_n(x, x^{n-1}) \mid x^{n-1}) = 0 , \qquad \operatorname{Var}(w_n(x, x^{n-1}) \mid x^{n-1}) \leqq \sigma_w^2 < \infty . \tag{10}$$

C5 *For the sequence $\{a_n\}$*

$$\textstyle\sum_{n=1}^{\infty} a_n = \infty , \qquad \sum_{n=1}^{\infty} a_n^2 < \infty . \tag{11}$$

C6 *For nonrandom fluctuation* v_n

(12) $$v_n = o(a_n)\ .$$

C7 *For initial estimate* x_1

(13) $$E(x_1 - \theta_1)^2 < \infty\ .$$

Then for the estimation scheme (5), *we have*

(14) $$\lim_{n\to\infty} E(x_n - \theta_n)^2 = 0\ , \qquad \Pr\{\lim_{n\to\infty}(x_n - \theta_n) = 0\} = 1\ .$$

PROOF. From (4) and (5)

$$x_{n+1} - \theta_{n+1} = g_n(x_n) - g_n(\theta_n) - a_n(M_{n+1}(g_n(x_n)) - \alpha) - v_n - a_n w_{n+1}\ .$$

Now we put

(15) $$T_n(x^n, \theta^n) = g_n(x_n) - g_n(\theta_n) - a_n(M_{n+1}(g_n(x_n)) - \alpha) - v_n$$

and

(16) $$U_n(x^n, \theta^n) = -a_n w_{n+1}\ .$$

Then

(17) $$E(U_n \mid x^n, \theta^n) = 0$$

and

(18) $$\sum_{n=1}^{\infty} E(U_n^2 \mid x^n, \theta^n) < \sigma_w^2 \sum_{n=1}^{\infty} a_n^2 < \infty\ .$$

For every sequence $\{\rho_n\}$ of positive numbers having the following properties;

$$\lim_{n\to\infty} \rho_n = 0\ , \qquad v_n = o(a_n\rho_n)\ , \qquad \sum_{n=1}^{\infty} a_n\rho_n = \infty\ ,$$

there exists a sequence $\{\eta_n\}$ of positive numbers satisfying

$$\lim_{n\to\infty} \eta_n = 0\ , \qquad \inf_{\eta_n<|x-\theta_n|} |M_n(x) - \alpha| > \rho_n\ .$$

If $|g_n(x_n) - g_n(\theta_n) - v_n| < a_n|M_{n+1}(g_n(x_n)) - \alpha|$, then there exists a finite number N_1, and for each $n > N_1$,

$$|T_n| \leqq (a_nK_1 - 1)|g_n(x_n) - g_n(\theta_n) - v_n| + a_nK_2 \leqq a_nK_2\ ,$$

while in the case where $|g_n(x_n) - g_n(\theta_n) - v_n| \geqq a_n|M_{n+1}(g_n(x_n)) - \alpha|$, we have

$$|T_n| \leqq \max\{|g_n(x_n) - g_n(\theta_n)| + |v_n| - a_n\rho_n, \eta_n\}\ .$$

Summarizing the above results, we have for each $n > N_1$,

(19) $$|T_n| \leqq \max\{\gamma_n|x_n - \theta_n| - a_n\rho_n + |v_n|, \eta_n, a_nK_2\}\ .$$

For the above inequality,

$$\sum_{n=1}^{\infty} (\gamma_n - 1)^+ < \infty\ , \qquad \sum_{n=1}^{\infty} (a_n\rho_n - |v_n|) = \infty\ ,$$
$$\lim_{n\to\infty} \eta_n = 0\ , \qquad \lim_{n\to\infty} a_nK_2 = 0$$

hold. Thus by Dvoretzky's theorem, the following results are established:

(20) $$\lim_{n\to\infty} E(x_n - \theta_n)^2 = 0$$

(21) $$\Pr\{\lim_{n\to\infty} (x_n - \theta_n) = 0\} = 1 .$$

REMARK 1. Obviously the sinusoidal function such that

$$g_n(x) = \gamma_n \sin(\omega_n x - \phi_n)$$

with

$$\sum_{n=1}^{\infty} (|\gamma_n \omega_n| - 1)^+ < \infty$$

satisfies condition C3. So, for the sinusoidal trend case, this stochastic approximation process is applicable.

REMARK 2. If $M_n(x)$ is quasi-linear, i.e.,

(6′) $$K_1'|x - \theta_n| \leqq |M_n(x) - \alpha| \leqq K_2'|x - \theta_n| ,$$

condition C3 can be weakened as

(9′) $$\gamma_n - 1 = o(a_n) .$$

Under these conditions, the polynomial trend case, where θ_n moves as $pn^q + r$, where p and r are unknown and q is positive and known, can be treated. Convergence is proved similarly to that of Dupač in [2].

REMARK 3. The following modification will remove the necessity of imposing the property on $(M_n(x) - \alpha)$ to be bounded by a linear function.

Let the modified estimation scheme be

$$x_{n+1} = x_n^* + a_n(\alpha - y_n^*)/h_{n+1}(x_n^*) \qquad \text{for} \quad n = 1, 2, \cdots$$

where $h_n(x)$ is a function which is positive and bounded in any finite interval and x_1 and x_n^* are defined as before. Then conditions C1 and C4 are weakened as follows;

C1′ *There exist positive constants* K_1'' *and* K_2'' *such that*

(6″) $$|M_n(x) - \alpha| \leqq (K_1''|x - \theta_n| + K_2'')h_n(x) .$$

C4′ *For random variable* $w_n(x, x^{n-1})$

(10′) $$E(w_n(x, x^{n-1}) \mid x^{n-1}) = 0 , \qquad \mathrm{Var}(w_n(x, x^{n-1}) \mid x^{n-1}) \leqq \sigma_w^2 h_n^2(x) < \infty .$$

Since the proof is apparent, it is omitted.

3. Convergence of the estimation error within a bound. In the case where the unknown fluctuation v_n is random, the estimation error is allowed to be bounded within some value by this procedure. Theorem 2 shows this fact.

THEOREM 2. *In this case, instead of conditions* C2, C3 *and* C6 *of Theorem* 1, *we assume*:

B2 *There exists a positive number* K_4 *such that*

(22) $$(x - \theta_n)(M_n(x) - \alpha) \geqq K_4(x - \theta_n)^2 \qquad \textit{for} \quad n = 1, 2, \cdots .$$

B3 *The positive sequence* $\{\gamma_n\}$ *is defined by* (8) *and there exists a positive integer* N_2 *and a positive constant* K_5 *such that for each* $n > N_2$,

$$\gamma_n \leqq 1 - K_5 < 1 . \tag{23}$$

B6 *For random fluctuation* v_n

$$E(v_n \mid x^n) \leqq v_n^* < \infty , \qquad \mathrm{Var}\,(v_n \mid x^n) \leqq \sigma_v^2 < \infty , \tag{24}$$
$$E(v_n w_{n+1}(x, x^n) \mid x^n) = 0$$

and there exists a positive integer N_3 *such that for each* $n > N_3$

$$|v_n^*| < 2K_5 . \tag{25}$$

Further, conditions C1, C4, C5 *and* C7 *of Theorem* 1 *hold. Then*

$$\lim_{n\to\infty} E(x_n - \theta_n)^2 < K_6 \tag{26}$$

where K_6 *is a finite positive number given in the proof.*

PROOF. As before,

$$x_{n+1} - \theta_{n+1} = g_n(x_n) - g_n(\theta_n) - a_n(M_{n+1}(x_n^*) - \alpha) - v_n - a_n w_{n+1}$$
$$\begin{aligned}(x_{n+1} - \theta_{n+1})^2 = {} & (g_n(x_n) - g_n(\theta_n) - v_n)^2 + a_n^2(M_{n+1}(x_n^*) - \alpha)^2 + a_n^2 w_{n+1}^2 \\ & - 2a_n(M_{n+1}(x_n^*) - \alpha)(x_n^* - \theta_{n+1}) \\ & + 2a_n^2(M_{n+1}(x_n^*) - \alpha)w_{n+1} - 2a_n w_{n+1}(x_n^* - \theta_{n+1}) .\end{aligned}$$

We take the conditional expectation on both sides:

$$\begin{aligned} & E((x_{n+1} - \theta_{n+1})^2 \mid x^n, \theta^n, v_n) \\ & \quad \leqq (1 - 2a_n K_4 + 2a_n^2 K_1^2)(g_n(x_n) - g_n(\theta_n) - v_n)^2 + a_n^2(\sigma_w^2 + 2K_2^2)\end{aligned}$$

and then,

$$\begin{aligned} & E((x_{n+1} - \theta_{n+1})^2 \mid x^n, \theta^n) \\ & \quad = E(E((x_{n+1} - \theta_{n+1})^2 \mid x^n, \theta^n, v_n)) \\ & \quad \leqq (1 - 2a_n K_4 + 2a_n^2 K_1^2)(g_n(x_n) - g_n(\theta_n))^2 \\ & \qquad + (1 - 2a_n K_4 + 2a_n^2 K_1^2)(\sigma_v^2 + v_n^{*2}) \\ & \qquad + 2(1 - 2a_n K_4 + 2a_n^2 K_1^2)|v_n^*|\,|g_n(x_n) - g_n(\theta_n)| \\ & \qquad + a_n^2(\sigma_w^2 + 2K_2^2) .\end{aligned}$$

Now taking unconditional expectation, we have

$$\begin{aligned} E(x_{n+1} - \theta_{n+1})^2 \leqq {} & (1 - 2a_n K_4 + 2a_n^2 K_1^2)(1 + |v_n^*|)\gamma_n^2 E(x_n - \theta_n)^2 \\ & + (1 - 2a_n K_4 + 2a_n^2 K_1^2)(\sigma_v^2 + v_n^{*2} + |v_n^*|) \\ & + a_n^2(\sigma_w^2 + 2K_2^2) .\end{aligned}$$

In this calculation we used the following inequality for random variable z with finite variance,

$$2E(|z|) \leqq K_7 + K_7^{-1}E(z^2) ,$$

which holds for any positive K_7.

Applying the above inequality successively, we have for each $n > N_4 = \max(N_2, N_3)$,

$$E(x_n - \theta_n)^2 \leqq K_8 \sum_{k=N_4}^{n-1} \prod_{j=k+1}^{n-1} (1 - 2a_j K_4 + 2a_j^2 K_1^2)(1 + |v_{N_4}^*|)(1 - K_5)^2$$

where

$$K_8 = \max\{E(x_{N_4} - \theta_{N_4})^2, \sigma_v^2 + 4K_5 + 4K_5^2 + a_{N_4}^2(\sigma_w^2 + 2K_2^2)\}.$$

Even if n goes to infinity, the last member remains in finite. In fact, let

$$u_k = \prod_{j=k+1}^{n-1} (1 - 2a_j K_4 + 2a_j^2 K_1^2)(1 + |v_{N_4}^*|)(1 - K_5)^2; \tag{27}$$

then by conditions B3 and B6, we have

$$\begin{aligned} \limsup_{k\to\infty} u_{k+1}/u_k &= \limsup_{k\to\infty} (1 - 2a_{k+1}K_4 + 2a_{k+1}^2 K_1^2)(1 + |v_{N_4}^*|)(1 - K_5)^2 \\ &< (1 + 2K_5)(1 - K_5)^2 \leqq 1, \end{aligned}$$

which indicates the convergence in finite [5].

Thus we have

$$\begin{aligned} &\lim_{n\to\infty} E(x_n - \theta_n)^2 \\ &\quad < K_6 = K_8 \sum_{k=N_4}^{\infty} \prod_{j=k+1}^{\infty} (1 - 2a_j K_4 + 2a_j^2 K_1^2)(1 + |v_{N_4}^*|)(1 - K_5)^2. \end{aligned} \tag{28}$$

REMARK 4. If condition C5 and equation (25) of condition B6 of Theorem 2 are replaced by

B5 *For the positive sequence* $\{a_n\}$, *there exists a positive number* K_9 *such that*

$$\lim_{n\to\infty} a_n \leqq K_9 < K_4/K_1^2, \tag{29}$$

$$|v_n^*| = o(a_n), \tag{25'}$$

then condition B3 can be weakened as (9′) in Remark 2. In fact, since $\limsup_{k\to\infty} u_{k+1}/u_k < 1$ also holds by these conditions, where u_k is defined by (27), convergence within a bound can be proved.

4. Concluding remarks. The results described in this paper are analogously developed to the maximum searching problem suggested by Kiefer and Wolfowitz in [4]. We, however, do not further mention this problem here.

In order to apply this dynamic stochastic approximation method to practical problems such as state estimation prediction and control, it is required to extend the present work to the multidimensional case.

Acknowledgment. The auther wishes to express his appreciation to the referee for his helpful comments with respect to the revision of the original manuscript. This work was supported in part by the Sakkokai Foundation.

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Added in proof. It is found that a similar modification to that of Remark 3 was proposed by Friedman in [7] for the stationary case.

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