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# THE UNIFORM CONVERGENCE OF AUTOCOVARIANCES

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Under general circumstances it is shown that the sample autocovariances of a discrete, stationary, ergodic process with finite covariance which is also purely nondeterministic converge, uniformly on the lag, almost surely to the true values. The result is used to prove the almost sure convergence, uniform in a parameter, of an expression relevant to the estimation of a lagged relation between two series.

Let $x(n)$ be a real, vector, stationary process of $p$ components that is ergodic and that has finite variances. For convenience we take the mean to be zero, though that requirement can easily be dispensed with. We put $\gamma_{jk}(n) = E\{x_j(m)x_k(m+n)\}$. We may consider the $x(n)$ to be random variables defined over the same probability space $(\Omega, \mathscr{A}, P)$. Let $\mathscr{M}_n$ be the sub-field of $\mathscr{A}$ generated by $x_j(m)$, $j = 1, \cdots, p$; $m \leqq n$ and put $\mathscr{M}_{-\infty} = \bigcap_{-\infty}^{\infty} \mathscr{M}_n$. We introduce the spaces $H_n = H(\mathscr{M}_n)$, of all real functions measurable with respect to $\mathscr{M}_n$ and of finite mean square. Then we call $S_n$ the orthogonal complement of $H_{n-1}$ in $H_n$, $u(n, t)$ the vector of projections of the elements of $x(n)$ onto $S_t$, $t \leqq n$ and $u(n, -\infty)$ the vector of projections onto $H_{-\infty}$. We have the decomposition

$$x(n) = \sum_{t=0}^{\infty} u(n, n-t) + u(n, -\infty) , \qquad \sum_0^{\infty} E\{u(n, n-t)'u(n, n-t)\} < \infty .$$

We finally assume that $u(n, -\infty)$ is almost surely zero. Thus $x(n)$ is to be purely nondeterministic in the strict sense of nonlinear prediction. Since it is then evidently purely nondeterministic in relation to linear prediction it has an absolutely continuous spectrum. This condition can be relaxed in the theorem below (for example by allowing $u(n, -\infty)$ to be a trigometric polynomial with random phasing) but we have not been able to perceive a useful general result without it. We introduce the autocovariances, based on $N$ observations,

$$c_{jk}(n) = N^{-1} \sum_1^{N-n} x_j(m)x_k(m+n) = c_{kj}(-n) , \qquad 0 \leqq n < N .$$

We put $c_{jk}(n) = 0$ for $n \geqq N$. Because $x(n)$ is ergodic $c_{jk}(n)$ converges almost surely to $\gamma_{jk}(n)$. We shall prove the following result.

THEOREM 1. *Under the above conditions*

$$\lim_{N\to\infty} \sup_{-\infty<n<\infty} |c_{jk}(n) - \gamma_{jk}(n)| = 0 \quad \text{a.s.} \qquad j, k = 1, \cdots, p .$$

It may be observed that the theorem would not be true if $N^{-1}$, in the definition of $c_{jk}(n)$, were replaced by $(N-n)^{-1}$. To prove the theorem let us put

$$u_j(n) = \sum_{t=0}^{r} u_j(n, n-t) , \qquad v_j(n) = \sum_{t=r+1}^{\infty} u_j(n, n-t) .$$

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Then since $x_j(n)$, and hence $v_j(n)$, is ergodic

$$\text{(1)} \qquad \lim_{N\to\infty} N^{-1} \textstyle\sum_1^N v_j(n)^2 = E\{v_j(0)^2\} = \sum_{t=r+1}^{\infty} E\{u_j(0, -t)^2\} \quad \text{a.s.}$$

The right side may be made as small as is desired by taking $r$ sufficiently large. Let $\tilde{\gamma}_{jk}(n)$, $\tilde{c}_{jk}(n)$ be the true and sample autocovariances computed from the $u_j(n)$. Then by the triangle inequality and Schwarz's inequality

$$\text{(2)} \qquad \begin{aligned} |c_{jk}(n) - \tilde{c}_{jk}(n)| \leqq\ & [\{N^{-1} \textstyle\sum_1^N u_j(m)^2\}\{N^{-1} \sum_1^N v_k(m)^2\}]^{\frac{1}{2}} \\ & + [\{N^{-1} \textstyle\sum_1^N v_j(m)^2\}\{N^{-1} \sum_1^N u_k(m)^2\}]^{\frac{1}{2}} \\ & + [\{N^{-1} \textstyle\sum_1^N v_j(m)^2\}\{N^{-1} \sum_1^N v_k(m)^2\}]^{\frac{1}{2}} \end{aligned}$$

and as $N$ increases the right-hand side, which is independent of $n$, converges almost surely to a constant that may be made arbitrarily small by taking $r$ sufficiently large. Also the same is true for the quantities $|\gamma_{jk}(n) - \tilde{\gamma}_{jk}(n)|$ by much the same proof. Thus we need only prove the theorem for $\tilde{c}_{jk}(n) - \tilde{\gamma}_{jk}(n)$, for $r$ fixed. This is

$$\text{(3)} \qquad \sum_{s,t=0}^{r}\sum \{N^{-1} \textstyle\sum_{m=1}^{N-n} u_j(m, m-s)u_k(m+n, m+n-t)\} - \tilde{\gamma}_{jk}(n)\,, \qquad 0 \leqq n < N\,.$$

There are $r + 1$ values, only, of $n$ for which $n - t$ is not always positive and we may therefore neglect these in proving the uniformity of the convergence since for $n$ fixed convergence certainly holds because $u_j(n)$ is ergodic. We may also restrict ourselves to particular pairs $(s, t)$, $(j, k)$ and to $n \geqq 0$, since $c_{jk}(-n) = c_{kj}(n)$, $n \geqq 0$. Of course for $n - t > 0$ the summand in the first term in (3) has zero expectation. We must consider, typically, the sequence

$$\text{(4)} \qquad N^{-1} \textstyle\sum_{m=1}^{N-n} \{u_j(m, m-s)u_k(m+n, m+n-t)\}\,, \qquad n = r+1, r+2, \cdots, N-1\,.$$

We now point out that we may truncate the sequences $u_j(m, m-s)$, $u_k(m+n, m+n-t)$ so as to produce, uniformly in $n$, almost surely, an arbitrarily small distortion. Indeed let $u_j' = u_j$, $|u_j| \leqq A$; $u_j' = A$, $u_j \geqq A$; $u_j' = -A$, $u_j \leqq -A$ with $u_k'$ defined in the same way. Put

$$u_j''(m, m-s) = u_j(m, m-s) - u_j'(m, m-s)\,,$$
$$u_k''(m+n, m+n-t) = u_k(m+n, m+n-t) - u_k'(m+n, m+n-t)\,.$$

Then the expressions (4) computed for $u_j$, $u_k$ and for $u_j'$, $u_k'$ differ by

$$\begin{aligned} N^{-1} \textstyle\sum_{m=1}^{N-n} \{ & u_j'(m, m-s)u_k''(m+n, m+n-t) \\ & + u_j''(m, m-s)u_k'(m+n, m+n-t) \\ & + u_j''(m, m-s)u_k''(m+n, m+n-t)\} \end{aligned}$$

and as in the proof of (2) this is dominated in modulus by an expression independent of $n$ that converges as $N$ increases almost surely to a constant that may be made arbitrarily small by taking the truncation point large enough. Finally we replace $u_j'(m, m-s)$ by

$$\bar{u}_j(m, m-s) = u_j'(m, m-s) - E\{u_j'(m, m-s) \mid \mathscr{M}_{m-s-1}\}$$

and similarly for $u_k'(m+n, m+n-t)$. Now

$$\left|\frac{1}{N} \sum_1^{N-n} E\{u_j'(m, m-s) \mid \mathscr{M}_{m-s-1}\} E\{u_j'(m+n, m+n-t) \mid \mathscr{M}_{m+n-t-1}\}\right|$$

$$= \left|-\frac{1}{N} \sum_1^{N-n} E\{u_j''(m, m-s) \mid \mathscr{M}_{m-s-1}\} E\{u_j'(m+n, m+n-t) \mid \mathscr{M}_{m+n-t-1}\}\right|$$

$$\leqq A \frac{1}{N} \sum_1^{N-n} |E\{u_j''(m, m-s) \mid \mathscr{M}_{m-s-1}\}| .$$

The last expression is dominated by

$$A \frac{1}{N} \sum_1^N E\{|u_j''(m, m-s)| \mid \mathscr{M}_{m-s-1}\}$$

which is independent of $n$ and converges almost surely to $AE\{|u_j''(m, m-s)|\}$ which may be made arbitrarily small by taking the truncation point, $A$, large enough since $u_j(m, m-s)$ has finite mean square. Similarly

$$\frac{1}{N} \sum_1^{N-n} u_j'(m, m-s) E\{u_j'(m+n, m+n-t) \mid \mathscr{M}_{m+n-t-1}\}$$

$$\frac{1}{N} \sum_1^{N-n} E\{u_j'(m, m-s) \mid \mathscr{M}_{m-s-1}\} u_j'(m+n, m+n-t)$$

may be made, almost surely and uniformly in $n$, arbitrarily small by taking the value of $A$ sufficiently large.

We put $w(m; n) = \bar{u}_j(m, m-s)\bar{u}_k(m+n, m+n-t)$, suppressing reference to $j$, $k$, $t$ for simplicity. Then we finally consider

$$\bar{c}(n) = N^{-1} \sum_{m=1}^{N-n} w(m; n) .$$

The summands $w(m; n)$ are, for each $n$, martingale differences with respect to the $\sigma$-fields $\mathscr{A}_m(n) = \mathscr{M}_{m-n-t}$ and are also (stationary, ergodic) sequences bounded uniformly in $n$. It follows immediately from Burkholder ([1] Theorem 9) that $E\{\bar{c}(n)^6\} = O(N^{-3})$, uniformly in $n$, and by Markov's inequality,

$$P\{\sup_{r+1 \leqq n < \infty} |\bar{c}(n)| > \varepsilon\} \leqq \sum_{n=r+1}^{N-1} P\{|\bar{c}(n)|^6 > \varepsilon^6\} \leqq KN^{-2}\varepsilon^{-6} , \qquad \varepsilon > 0 ,$$

so that by the Borel–Cantelli lemma the theorem is proved.

As an application of this result let us consider the statistic,

$$I_{jk}(\omega) = \frac{1}{2\pi N} \sum_1^N x_j(m) e^{im\omega} \sum_1^N x_k(m) e^{-im\omega} = \frac{1}{2\pi} \sum_{-N+1}^{N-1} c_{jk}(n) e^{-in\omega} ,$$

and form for a particular pair, $j$, $k$,

$$\hat{R}_{jk}(\tau) = \int_{-\pi}^{\pi} I_{jk}(\omega) e^{-i\tau\omega} \phi(\omega)\, d\omega . \tag{5}$$

Here $\phi(\omega)$ is a continuous weight function reflecting the relative importance of the various frequencies. If it is believed that the cross spectrum, $f_{jk}(\omega)$, is of the form $\{|f_{jk}(\omega)| \exp i(\alpha + \tau_0\omega)\}$ over a band containing the support of $\phi(\omega)$ then the maximisation of $|\hat{R}_{jk}(\tau)|^2$ will be a reasonable procedure for estimating the group delay, $\tau_0$, over the band. We call $\hat{\tau}_N$ the value of $\tau$ maximising $|\hat{R}_{jk}(\tau)|^2$. The optimal choice of $\phi(\omega)$ will depend on the nature of the spectra over the band. We do not discuss this further here. We call $R_{jk}(\tau)$ the expression (5) with $f_{jk}(\omega)$ replacing $I_{jk}(\omega)$. Let us put $\tau = \tau' + \tau''$ where $\tau' = [\tau]$. Now for any trigonometric polynomial $\psi(\omega)$ it follows from Theorem 1 that

$$(6) \qquad \lim_{N\to\infty} \sup_{-\infty<\tau<\infty} \int_{-\pi}^{\pi} \{I_{jk}(\omega) - f_{jk}(\omega)\} \exp(-i\tau'\omega)\psi(\omega)\, d\omega = 0 \quad \text{a.s.}$$

However, $\phi(\omega)\exp(-i\tau''\omega)$ may be approximated uniformly in $\tau$ by such a trigonometric polynomial, $\psi(\omega)$, except possibly in the neighbourhood of $\pm\pi$. Precisely as in the lemma in [2] it may be shown that, uniformly in $\tau$, these neighbourhoods contribute arbitrarily little to $\{\hat{R}_{jk}(\tau) - R_{jk}(\tau)\}$, if they are taken small enough, so that we have the following result.

THEOREM 2. *Under the conditions of Theorem* 1

$$\lim_{N\to\infty} \sup_{-\infty<\tau<\infty} |\hat{R}_{jk}(\tau) - R_{jk}(\tau)| = 0 \quad \text{a.s.}$$

If $|R_{jk}(\tau)|$ has a single maximum at $\tau_0$ (which is not a very restrictive requirement if $f_{jk}(\omega) = |f_{jk}(\omega)| \exp i(\alpha + \tau_0\omega)$ on the support of $\phi(\omega)$) then it follows that $\hat{\tau}_N$ converges, almost surely, to $\tau_0$.

## REFERENCES

[1] BURKHOLDER, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1504.

[2] HANNAN, E. J. and ROBINSON, P. M. (1973). Lagged regression with unknown lags. *J. Roy. Statist. Soc. Ser. B* **35**.

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