The Annals of Statistics
1974, Vol. 2, No. 3, 609-613

A PROBABILITY INEQUALITY FOR LINEAR COMBINATIONS OF BOUNDED RANDOM VARIABLES[1]

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Let $Y_1, \cdots, Y_n$ be independent random variables with mean zero such that $|Y_i| \leqq i$, $i = 1, \cdots, n$, and let $\theta_1, \cdots, \theta_n$ be real numbers satisfying $\sum_1^n \theta_i^2 = 1$. Set $S_n(\theta) = \sum_1^n \theta_i Y_i$ and let $\varphi(x) = (2\pi)^{-\frac{1}{2}} \exp[-\frac{1}{2}x^2]$.

THEOREM. *For $\alpha > 0$, and for all $\theta_1, \cdots, \theta_n$,*

$$P\{|S_n(\theta)| \geqq \alpha\} \leqq 2 \inf_{0 \leqq u \leqq \alpha} \int_u^\infty \frac{(x-u)^3}{(\alpha-u)^3} \varphi(x)\, dx$$

$$\leqq 12 \frac{\varphi(\alpha)}{\alpha} \inf_{0 \leqq \delta \leqq \alpha^2} \frac{\exp[\delta/2(2-\delta/\alpha^2)]}{\delta^3(1-\delta/\alpha^2)^4}\,.$$

1. Introduction. Let $U_1, \cdots, U_n$ be independent random variables with $P\{U_i = 1\} = P\{U_i = -1\} = \frac{1}{2}$, $i = 1, \cdots, n$. Further, let $\mathscr{F}_1$ be the class of functions $f: R \to R$ such that (i) f is symmetric and has a derivative f' and (ii) $t^{-1}[f'(t+\Delta) - f'(-t+\Delta)]$ is non-decreasing in $t > 0$ for each $\Delta \geqq 0$. As in Eaton (1970), set $T_n(\theta) = \sum_1^n \theta_i U_i$ where $\theta_1, \cdots, \theta_n$ are real numbers and $\sum_1^n \theta_i^2 = 1$. With $T_n \equiv n^{-\frac{1}{2}} \sum_1^n U_i$, we have

PROPOSITION 1. *For each $f \in \mathscr{F}_1$,*

$$\mathscr{E}f(T_n(\theta)) \leqq \mathscr{E}f(T_n) \leqq \mathscr{E}f(T_{n+1}) \tag{1.1}$$

for $n = 1, 2, \cdots$.

PROOF. See Eaton (1970).

PROPOSITION 2. *If $f \in \mathscr{F}_1$ and if there exists a $\delta > 0$ and a constant M such that $\mathscr{E}|f(T_n)|^{1+\delta} \leqq M$ for all n, then*

$$\mathscr{E}f(T_n) \leqq \mathscr{E}f(Z) \tag{1.2}$$

where Z has a unit normal distribution.

PROOF. See Eaton (1970).

The purpose of this paper is to use (1.1) and (1.2) to obtain an upper bound for $P\{|\sum_1^n \theta_i Y_i| \geqq \alpha\}$ where $Y_1, \cdots, Y_n$ are independent with mean 0 and $|Y_i| \leqq 1$. The upper bound given in Theorem 2 is independent of n and $\theta_1, \cdots, \theta_n$ in contrast to a related result of Feller (1943). Feller's bound depends on n and the variances of $Y_1, \cdots, Y_n$. Consider an $f \in \mathscr{F}_1$ so that (1.2) holds, and so that

Received July 1972; revised June 1973.

[1] This research was supported in part by NSF Grant GP-34482 and NSF GP25911.

AMS 1970 *subject classification.* 6210.

Key words and phrases. Probability inequality, bounded random variables, sums.

$f \geqq 0$ and $f(x) \geqq 1$ if $|x| \geqq \alpha$. It follows immediately, using (1.1) and (1.2), that

$$P\{|T_n(\theta)| \geqq \alpha\} \leqq \mathscr{E}f(T_n(\theta)) \leqq \mathscr{E}f(T_n) \leqq \mathscr{E}f(Z) . \tag{1.3}$$

Now, to derive a probability bound, we would like to minimize the right-hand side of (1.3) for all functions f for which (1.3) is valid. However, the class $\mathscr{F}_1$ is rather difficult to describe in a manner which allows the minimization of $\mathscr{E}f(Z)$. The following lemma gives a useful sufficient condition for a symmetric function f to be in $\mathscr{F}_1$.

LEMMA 1. *Suppose $f: R \to R$ is symmetric, f''' exists and $f'''(x)$ is non-decreasing for $x \geqq 0$. Then $f \in \mathscr{F}_1$.*

PROOF. For $t > 0$ and $\Delta \geqq 0$

$$f'''(t + \Delta) - f'''(-t + \Delta) \geqq 0$$

so that

$$t[f'''(t + \Delta) - f'''(-t + \Delta)] + f''(t + \Delta) + f''(-t + \Delta) \geqq f''(t + \Delta) + f''(-t + \Delta) .$$

Hence

$$\frac{d}{dt}[t(f''(t + \Delta) + f''(-t + \Delta))] \geqq \frac{d}{dt}[f'(t + \Delta) - f'(-t + \Delta)] .$$

Therefore

$$t[f''(t + \Delta) + f''(-t + \Delta)] \geqq f'(t + \Delta) - f'(-t + \Delta) .$$

But

$$\frac{d}{dt}\left[\frac{f'(t + \Delta) - f'(-t + \Delta)}{t}\right] = \frac{t[f''(t + \Delta) + f''(-t + \Delta)] - [f'(t + \Delta) - f'(-t + \Delta)]}{t^2} \geqq 0 .$$

Thus $f \in \mathscr{F}_1$ and the proof is complete.

2. The basic inequality. To obtain a probability inequality for $P\{|T_n(\theta)| \geqq \alpha\}$, fix $\alpha > 0$ and let $\mathscr{F}_\alpha$ denote the class of functions f which are symmetric and satisfy

$$f(x) = \frac{1}{3!} \int_0^x (x - u)^3 \, dF(u) , \qquad x \geqq 0 \tag{2.1}$$

$$f(\alpha) = \frac{1}{3!} \int_0^\alpha (\alpha - u)^3 \, dF(u) = 1 .$$

Here, F is a non-decreasing function on $[0, \infty)$ with $F(0) = 0$ and $F(+\infty) < +\infty$. Define $(\cdot)_+$ by $(v)_+ = \max(0, v)$.

Then, $f \in \mathscr{F}_\alpha$ iff

$$f(x) = \frac{1}{3!} \int_0^\infty [(|x| - u)_+]^3 \, dF(u) ; \qquad x \in R \tag{2.2}$$

$$f(\alpha) = 1 ,$$

PROPOSITION 3. *If* $f \in \mathscr{F}_\alpha$, *then*

$$P\{|T_n(\theta)| \geqq \alpha\} \leqq \mathscr{E}f(T_n(\theta)) \leqq \mathscr{E}f(Z) \tag{2.3}$$

where Z *is* $N(0, 1)$.

PROOF. Since $f'''(x) = F(x)$, $x \geqq 0$, $f'''(x)$ is non-decreasing for $x > 0$. By Lemma 1, $f \in \mathscr{F}_1$. Further, $f'(x) = \frac{1}{2}\int_0^x (x-u)^2\, dF(x) \geqq 0$ for $x \geqq 0$ so $f(x)$ is increasing for $x \geqq 0$. Since $f(\alpha) = 1$, $f(x) \geqq 1$ if $|x| \geqq \alpha$. Combining the above and applying Proposition 1, we have

$$P\{|T_n(\theta)| \geqq \alpha\} \leqq \mathscr{E}f(T_n(\theta)) \leqq \mathscr{E}f(T_n)\,. \tag{2.4}$$

But,

$$\begin{aligned}\mathscr{E}|f(T_n)|^2 &= \mathscr{E}\left|\frac{1}{3!}\int_0^\infty [(|T_n| - u)_+]^3\, dF(u)\right|^2 \leqq \mathscr{E}\left[\frac{1}{3!}|T_n|^3 F(+\infty)\right]^2 \\ &= \left(\frac{F(+\infty)}{6}\right)^2 \mathscr{E}T_n^6 \leqq M\end{aligned} \tag{2.5}$$

for some constant M and for all n. By Proposition 2, $\mathscr{E}f(T_n) \leqq \mathscr{E}f(Z)$. This completes the proof.

From the above proposition, we have

$$P\{|T_n(\theta)| \geqq \alpha\} \leqq \inf_{f \in \mathscr{F}\alpha} \mathscr{E}f(Z)\,. \tag{2.6}$$

PROPOSITION 4. *For* $\alpha > 0$,

$$\inf_{f \in \mathscr{F}\alpha} \mathscr{E}f(Z) = 2 \inf_{0 \leqq u \leqq \alpha} \int_u^\infty \frac{(x-u)^3}{(\alpha - u)^3}\varphi(x)\, dx\,. \tag{2.7}$$

PROOF. For $\alpha > 0$,

$$\begin{aligned}\inf_{f \in \mathscr{F}\alpha} \mathscr{E}f(Z) &= 2 \inf_{f \in \mathscr{F}\alpha} \frac{1}{3!}\int_0^\infty \int_0^\infty [(x-u)_+]^3\, dF(u)\varphi(x)\, dx \\ &= 2 \inf_F \frac{1}{3!}\int_0^\infty w(u)\, dF(u)\end{aligned} \tag{2.8}$$

where F is non-decreasing, $F(+\infty) < +\infty$, $(1/3!)\int_0^\alpha (\alpha - u)^3\, dF(u) = 1$ and $w(u) \equiv \int_0^\infty [(x-u)_+]^3 \varphi(x)\, dx$. But

$$\begin{aligned}2 \inf_F \frac{1}{2!}\int_0^\infty w(u)\, dF(u) &\geqq 2 \inf_F \int_0^\alpha \frac{w(u)}{(\alpha-u)^3}\frac{(\alpha-u)^3}{3!}\, dF(u) \\ &\geqq 2 \inf_{0 \leqq u \leqq \alpha} \frac{w(u)}{(\alpha - u)^3}\,.\end{aligned} \tag{2.9}$$

However, it is easy to see that one has equality in both of the inequalities in (2.9) since a choice of F can be made which gives equality. Since $w(u) = \int_u^\infty (x-u)^3 \varphi(x)\, dx$, (2.7) holds.

THEOREM 1. *For* $\alpha > 0$,

$$P\{|T_n(\theta)| \geqq \alpha\} \leqq 2 \inf_{0 \leqq u \leqq \alpha} \int_u^\infty \frac{(x-u)^3}{(\alpha-u)^3}\varphi(x)\, dx\,. \tag{2.10}$$

PROOF. This follows immediately from (2.6) and Proposition 4.

The explicit minimization of the right-hand side of (2.10) has not been accomplished. The following gives some upper bounds for this minimum.

$$
\begin{aligned}
H(\alpha, u) &\equiv \int_u^\infty \frac{(x-u)^3}{(\alpha-u)^3}\varphi(x)\,dx = \int_0^\infty \frac{x^3}{(\alpha-u)^3}\varphi(x+u)\,dx \\
&= \frac{\varphi(\alpha)}{\alpha}\frac{\alpha}{(\alpha-u)^3} e^{-\frac{1}{2}(u^2-\alpha^2)} \int_0^\infty x^3 e^{-ux} e^{-\frac{1}{2}x^2}\,dx \qquad (2.11)\\
&= \frac{\varphi(\alpha)}{\alpha}\frac{\alpha}{u^4(\alpha-u)^3} e^{-\frac{1}{2}(u^2-\alpha^2)} \int_0^\infty x^3 e^{-x} e^{-\frac{1}{2}(x^2/u^2)}\,dx\,.
\end{aligned}
$$

Set $u = \alpha - (\delta/\alpha)$ for $0 \leqq \delta \leqq \alpha^2$ so

$$
H(\alpha, u) = \frac{\varphi(\alpha)}{\alpha}\frac{e^\delta}{\delta^3}\frac{e^{-\frac{1}{2}(\delta^2/\alpha^2)}}{(1-\delta/\alpha^2)^4}\int_0^\infty x^3 e^{-x} e^{-\frac{1}{2}(x^2/u^2)}\,dx\,. \qquad (2.12)
$$

Now, e^δ/δ^3 is minimized by setting $\delta = 3$ and $\int_0^\infty x^3 e^{-x} e^{-\frac{1}{2}(x^2/u^2)}\,dx \leqq \int_0^\infty x^3 e^{-x}\,dx = 6$. Thus, for $\alpha > 3^{\frac{1}{2}}$

$$
\inf_{0 \leqq u \leqq \alpha} H(\alpha, u) \leqq \frac{6e^3}{27}\frac{\varphi(\alpha)}{\alpha}\frac{e^{-\frac{1}{2}(9/\alpha^2)}}{(1-3/\alpha^2)^4}\,. \qquad (2.13)
$$

COROLLARY 1. *For* $\alpha > 3^{\frac{1}{2}}$,

$$
P\{|T_n(\theta)| \geqq \alpha\} \leqq \frac{4e^3}{9}\frac{\varphi(\alpha)}{\alpha}\frac{e^{-\frac{1}{2}(9/\alpha^2)}}{(1-3/\alpha^2)^4} \qquad (2.14)
$$

for all $\theta_1, \cdots, \theta_n$ *and* $n = 1, 2, \cdots$.

It is easy to show that $\exp[-\frac{1}{2}(9/\alpha^2)](1 - 3/\alpha^2)^{-4}$ is a decreasing function of α for $\alpha > 3^{\frac{1}{2}}$. Thus, we have

COROLLARY 2. *For* $\alpha \geqq \alpha_0 > 3^{\frac{1}{2}}$, *let* $K = K(\alpha_0) = (4e^3/9)\exp[-\frac{1}{2}(9/\alpha_0^2)(1 - 3/\alpha_0^2)^{-4}]$. *Then*

$$
P\{|T_n(\theta)| \geqq \alpha\} \leqq K\frac{\varphi(\alpha)}{\alpha}\,. \qquad (2.15)
$$

The estimates used to derive (2.14) and (2.15) are quite crude. Some numerical work indicates that for all $\alpha > 2^{\frac{1}{2}}$, $\inf_{0 \leqq u \leqq \alpha} H(\alpha, u) \leqq (6e^3/27)\varphi(\alpha)\alpha^{-1}$. However, a proof of this inequality has not yet been constructed.

3. An extension to bounded random variables. It was shown by the author (Eaton (1972)) that the inequality of Theorem 1 was valid for any independent *symmetric* random variables $X_1, \cdots, X_n$ such that $|X_i| \leqq 1$, $i = 1, \cdots, n$ and $T_n(\theta) \equiv \sum_1^n \theta_i X_i$, $\sum \theta_i^2 = 1$. After the appearance of this result, W. Hoeffding informed the author that an alternative argument could be used to establish the validity of Theorem 1 for independent random variables $Y_1, \cdots, Y_n$ such that $\mathscr{E} Y_i = 0$, $|Y_i| \leqq 1$ for $i = 1, \cdots, n$. It is this elegant argument which is presented in this section.

As above, let $Y_1, \cdots, Y_n$ be independent random variables with $\mathscr{E}Y_i = 0$ and $|Y_i| \leqq 1$, $i = 1, \cdots, n$. The following lemma due to G. A. Hunt (1955) is needed.

LEMMA 2. *Suppose* $g: \prod_{i=1}^n [-1, 1] \to R$ *is continuous and convex in each argument when the remaining* $n-1$ *arguments are held fixed. Then*

$$\mathscr{E}g(Y_1, \cdots, Y_n) \leqq \mathscr{E}g(U_1, \cdots, U_n) . \tag{3.1}$$

Now, let $\theta_1, \cdots, \theta_n$ be real numbers such that $\sum \theta_i^2 = 1$ and set $S_n(\theta) = \sum_1^n \theta_i Y_i$ and $T_n(\theta) = \sum_1^n \theta_i U_i$. For $u \geqq 0$, define $f_u: R \to [0, \infty)$ by

$$f_u(x) = [(|x| - u)_+]^3 . \tag{3.2}$$

THEOREM 2. *For each* $\alpha > 0$,

$$P\{|S_n(\theta)| \geqq \alpha\} \leqq 2 \inf_{0 \leqq u \leqq \alpha} \int_u^\infty \frac{(x-u)^3}{(\alpha - u)^3} \varphi(x)\, dx . \tag{3.3}$$

PROOF. For $0 \leqq u < \alpha$, it is clear that

$$P\{|S_n(\theta)| \geqq \alpha\} \leqq \frac{\mathscr{E}f_u(S_n(\theta))}{(\alpha - u)^3} \tag{3.4}$$

since $f_u \geqq 0$ and $f_u(x)/(\alpha - u)^{-3} \geqq 1$ if $|x| \geqq \alpha$. But $g(Y_1, \cdots, Y_n) \equiv f_u(\sum_1^n \theta_i Y_i)$ satisfies the assumption of Lemma 2. Thus $\mathscr{E}f_u(S_n(\theta)) = \mathscr{E}f_u(\sum \theta_i Y_i) \leqq \mathscr{E}f_u(\sum \theta_i U_i) = \mathscr{E}f_u(T_n(\theta))$. Using Propositions 1 and 2 on f_u, we have

$$\mathscr{E}f_u(S_n(\theta)) \leqq \mathscr{E}f_u(T_n(\theta)) \leqq \mathscr{E}f_u(T_n) \leqq \mathscr{E}f_u(Z) . \tag{3.5}$$

Combining (3.4) and (3.5) yields

$$P\{|S_n(\theta)| \geqq \alpha\} \leqq \frac{\mathscr{E}f_u(Z)}{(\alpha - u)^3} \tag{3.6}$$

for $0 \leqq u < \alpha$. Thus,

$$P\{|S_n(\theta)| \geqq \alpha\} \leqq \inf_{0 \leqq u \leqq \alpha} \frac{\mathscr{E}f_u(Z)}{(\alpha - u)^3} = 2 \inf_{0 \leqq u \leqq \alpha} \int_u^\infty \frac{(x-u)^3}{(\alpha - u)^3} \varphi(x)\, dx . \tag{3.7}$$

This completes the proof.

COROLLARY 3. *Corollaries* 1 *and* 2 *are valid with* $T_n(\theta)$ *replaced by* $S_n(\theta)$.

PROOF. This is clear from the discussion in Section 2.

Acknowledgment. I wish to thank Herman Chernoff, Grace Wahba and Coby Ward for useful discussions concerning the results in this paper.

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