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ON CONSTRUCTION OF SOME FAMILIES OF GENERALIZED YOUDEN DESIGNS

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Wald (1943) formulated criteria of optimality relative to experimental designs and proved that some known intuitively attractive designs are in fact optimal. Kiefer (1958), (1959), (1961), (1971) extended Wald's work on criteria of optimality and suggested some generalizations of experimental designs discussing their merits in respect to the formulated criteria. The purpose of this paper is to exhibit constructions of some families of Generalized Youden Designs (GYD) introduced by Kiefer. In his published work Kiefer gave some small examples of Generalized Youden Designs mainly for the purpose of illustrating the theory on criteria of optimality.

1. Introduction. In the design setting when the problem is to obtain best linear estimators of contrasts between variety effects the optimality criteria are usually formulated in terms of the covariance matrix of the estimators V_d in Kiefer's (1961) notation.

For the GYD two criteria of optimality are of most interest.

(a) D-optimality; minimizing the generalized variance V_d.
(b) E-optimality; minimizing the largest eigenvalue of V_d.

Kiefer (1958) showed that GYD are E-optimal and under some conditions D-optimal. He also obtained recently (1972) some remarkable results regarding the optimality properties of GYD showing:

(i) GYD are D-optimal unless the number of varieties v equals 4.
(ii) If $v = 4$ any GYD is still A-optimal (minimizing tr V_d).
(iii) If $v = 4$ and $b = k$ a GYD is never D-optimal unless v divides b or k, where b is the number of blocks and k the block size.
(iv) If $v = 4$ and b/k is sufficiently large or small a GYD is again D-optimal.

We describe here a construction of an infinite class of GYD with $v = 4$, $b = k = 6t$, t odd which according to the latest results of Kiefer cannot be D-optimal. We retain, nevertheless, this example in the present paper together with the direct proof of its non D-optimality, because the method of construction and the proof in this case seem to be attractive and interesting per se.

We shall construct some classes of D-optimal GYD which satisfy the original more stringent conditions for D-optimality established by Kiefer and some which

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do not satisfy them. In the latter case we shall show how to utilize the method of construction to obtain GYD which satisfy the original conditions.

2. Some properties of GYD.

DEFINITION. A $(v, b, k, r, \lambda_1, \lambda_2)$ Generalized Youden Design (GYD) is a $k \times b$ matrix on a set of v varieties, b blocks and k varieties per block such that the following conditions are satisfied:

(a) Every variety occurs r times.

(b) Every variety occurs either m or $m + 1$ times in each row, as well as either n or $n + 1$ times in each column, where m is the integer part of b/v and n is the integer part of k/v.

(c) Every two distinct varieties occur together λ_1 times in the same row and λ_2 times in the same column.

Kiefer (1958) showed that when either b or k is divisible by v, the GYD are D-optimal.

The row-incidence matrix of a GYD is a $v \times k$ matrix $A = (a_{ij})$, where a_{ij} is the number of times that the ith variety appears in the jth row; of course $a_{ij} \in \{m, m + 1\}$.

Similarly, the column-incidence matrix of a GYD is a $v \times b$ matrix $B = (b_{ij})$, where b_{ij} is the number of times that the ith variety appears in the jth column; evidently, $b_{ij} \in \{n, n + 1\}$.

Notation. The quotient and remainder of the division of an integer a by another b will be written $[a/b]$ and $a_{(b)}$ respectively.

PROPOSITION 2.1. *In a* GYD

(i) *The number of rows containing a given variety $m + 1$ times is the same for all the varieties, and equals $r_{(k)}$.*

(ii) *The number of columns containing a given variety $n + 1$ times is the same for all the varieties, and equals $r_{(b)}$.*

(iii) *The number of varieties occurring $m + 1$ times in a given row is the same for all rows, and equals $b_{(v)}$.*

(iv) *The number of varieties occurring $n + 1$ times in a given column is the same for all columns, and equals $k_{(v)}$.*

(v)
$$r = mk + r_{(k)} \qquad r = nb + r_{(b)}$$
$$b = mv + b_{(v)} \qquad k = mv + k_{(v)} .$$

(vi)
$$vr_{(k)} = kb_{(v)} \qquad br_{(k)} = rb_{(v)} .$$

The proof follows immediately from the definition of GYD. Note that (vi) is easily obtainable from (v) together with the equation $vr = bk$.

PROPOSITION 2.2. *Let (v_i, v_l) be a pair of distinct varieties; let $\alpha_0, \alpha_1, 2\alpha$ be the number of rows containing the pair $(v_i, v_l)m^2, (m + 1)^2, m(m + 1)$ times respectively, and similarly let $\beta_0, \beta_1, 2\beta$ be the number of columns containing the pair $(v_i, v_l)n^2$,*

$(n+1)^2$, $n(n+1)$ *times respectively. Then* $\alpha_0, \alpha_1, \alpha, \beta_0, \beta_1, \beta$ *are independent of the pair* (v_i, v_l) *and*

$$\begin{aligned} \alpha &= m(r + r_{(k)}) + r_{(k)} - \lambda_1 & \beta &= n(r + r_{(b)}) + r_{(b)} - \lambda_2 \\ \alpha_1 &= \lambda_1 - m(r + r_{(k)}) & \beta_1 &= \lambda_2 - n(r + r_{(b)}) \\ \alpha_0 &= k - 2r_{(k)} - m(r + r_{(k)}) + \lambda_1 & \beta_0 &= b - 2r_{(b)} - n(r + r_{(b)}) + \lambda_2 \,. \end{aligned}$$

PROOF. Looking at the row-incidence matrix we easily establish the following relations among a, a_0, a_1:

$$\begin{aligned} 2\alpha + \alpha_0 + \alpha_1 &= k \\ \alpha + \alpha_1 &= r_{(k)} \qquad (1) \\ \alpha_0 m^2 + \alpha_1(m+1)^2 + 2\alpha m(m+1) &= \lambda_1 \,. \end{aligned}$$

The first equation expresses the fact that the row-incidence matrix has k columns, the second gives the number of rows which contain the ith variety $m+1$ times, which we know to be $r_{(k)}$, while the third equation expresses the fact that any pair of varieties occurs together λ_1 times in the same row.

The first and the second equations give $\alpha + \alpha_0 = k - r_{(k)}$. The third equation can be rewritten in the form

$$(\alpha_0 + \alpha)m^2 + (\alpha_1 + \alpha)(m+1)^2 = \lambda_1 + \alpha \qquad \text{or}$$

$$(k - r_{(k)})m^2 + r_{(k)}(m+1)^2 = \lambda_1 + \alpha \,.$$

Simplifying and making use of (v) in Proposition 2.1 we obtain $\alpha = m(r + r_k) + r_k + \lambda_1$ and using the first two equations of (1) we get the expression for α_0 and α_1. The β's are obtained in the same fashion replacing k by b and λ_1 by λ_2.

PROPOSITION 2.3. *If* $A(B)$ *is the row (column) incidence matrix of a* GYD, *then* $A - mJ_{v,k}(B - nJ_{k,v})$ *is an incidence matrix of a* BIB *design with parameters* $(v', b', k', r', \lambda')$, $(v'', b'', k'', r'', \lambda'')$ *where* $v' = v$, $b' = k$, $k' = b_{(v)}$, $r' = r_{(k)}$, $\lambda' = [r_{(k)}(b_{(v)} - 1)/(v-1)]$, $v'' = v$, $b'' = b$, $k'' = k_{(v)}$, $r'' = r_{(b)}$, $\lambda'' = [r_{(b)}(k_{(v)} - 1)/(v-1)]$ *and* $J_{s,t}$ *is an* $s \times t$ *matrix of all ones. This is a consequence of the definition of* GYD.

COROLLARY 2.4.

$$\lambda_1 = m(r + r_{(k)}) + \frac{r_{(k)}(b_{(v)} - 1)}{v - 1}$$

$$\lambda_2 = m(r + r_{(k)}) + \frac{r_{(b)}(k_{(v)} - 1)}{v - 1} \,.$$

This follows from Propositions 2.2 and 2.3 since λ' and λ'' are equal to α_1 and β_1 respectively.

REMARK. Notice that this corollary yields a necessary condition for the existence of GYD namely

$$\frac{r_{(k)}(b_{(v)} - 1)}{v - 1} \quad \text{and} \quad \frac{r_{(b)}(b_{(v)} - 1)}{v - 1}$$

have to be integers.

PROPOSITION 2.5.

$$\frac{r_{(k)}(v - b_{(v)})}{v - 1} = rb - \lambda_1 v$$

$$\frac{r_{(b)}(v - k_{(v)})}{v - 1} = rk - \lambda_2 v .$$

These equations follow from the definitions of the quantities involved. It seems interesting to mention them since they imply that $rb \geqq \lambda_1 v$ ($rk \geqq \lambda_2 v$) and equality holds if and only if $b(k)$ are multiples of v.

COROLLARY 2.6.

$$AA^T = (rb - \lambda_1 v)I_v + \lambda_1 J_v$$

$$BB^T = (rk - \lambda_2 v)I_v + \lambda_2 J_v .$$

This is a simple consequence of Propositions 2.3 and 2.5.

3. Construction of some families of GYD. The first two families of GYD are obtained by trivial applications of well-known combinatorial structures. We state them here for the sake of completeness since they do not seem to appear explicitly elsewhere.

PROPOSITION 3.1. *There exist* GYD *with* $b = mv$, $k = nv$ *for any positive integers* m, n *and* v.

PROOF. Let $\{L_{i,j} | i = 1, \cdots, n; j = 1, \cdots, m\}$ be a collection of mn Latin squares of order v, not necessarily different. Then the $nv \times mv$ matrix

$$D = \begin{bmatrix} L_{11} L_{12} \cdots L_{1m} \\ L_{n1} L_{n2} \cdots L_{nm} \end{bmatrix}$$

is a GYD, since clearly every variety occurs m times in each row and n times in each column, and every pair of distinct varieties occur together in the same row m^2k times and n^2b times in the same column.

PROPOSITION 3.2. *The existence of a* BIB *design with parameters* $(v', b', k', r', \lambda')$ *and* $b' = tv'$ *implies the existence of a* GYD *with the parameters*:

$$v = v' , \qquad b = mtv , \qquad k = nv + k' , \qquad r = mtk ,$$

$$\lambda_1 = m^2t^2k , \qquad \lambda_2 = m[\lambda' + nt(k + k')]$$

for any positive integers m, n.

The construction can be carried out as follows: Take any m BIB designs satisfying the condition of the proposition and rearrange the elements within each block in such a way that a juxtaposition of the blocks written vertically contains each variety t times. This can be done as shown by Agrawal (1966). For the actual construction one may use e.g. the algorithm of Hall (1956). Let the first k' rows of the GYD consist of a juxtaposition of these m BIB designs, and the remaining nv rows of n repetitions of a juxtaposition of mt Latin squares of order v.

There are many well-known series of BIB which satisfy the condition of Proposition 3.2.

We shall present next three families of GYD constructed by a method which we believe was not noticed before. These designs do not satisfy the divisibility conditions required in Kiefer's proof for proving the D-optimality. However, we shall show that the same method can be used to construct three corresponding families of GYD with a modified set of parameters which do satisfy at least one of the divisibility conditions. Hence they are D-optimal.

We shall frequently make use of the following conventions and notation.

s will designate a power of a prime number, $s = p^n$; GF (s) will stand for the Galois Field with s elements; EG $(2, s)$ will designate the Euclidean plane based on GF (s).

Let $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2, \cdots, \alpha_{s-1}$ be the s elements of GF (s) in some order; let l_i be the line with equation $x = \alpha_i$, $i = 0, 1, \cdots, s-1$ and similarly let $l_{j,i}$ be the line with equation $\alpha_j x + y = \alpha_i$, $i, j = 0, 1, \cdots, s-1$; the s parallel lines l_i, $i = 0, 1, \cdots, s-1$ form a pencil X, and for each $\alpha_j \in$ GF (s) the s parallel lines $l_{j,i}$, $i = 0, 1, \cdots, s-1$, form also a pencil Y_j; the order in GF (s) induces an order of the lines within each pencil as follows: for any α_i, α_j, $\alpha_u \in$ GF (s),

$$l_i < l_u \qquad \text{if and only if} \quad \alpha_i < \alpha_u$$

$$l_{j,i} < l_{j,u} \qquad \text{if and only if} \quad \alpha_i < \alpha_u \,.$$

The lines l_i and $l_{j,i}$ will be referred to as the ith lines of pencils X and Y_j respectively.

Any point P of EG $(2, s)$ is uniquely determined as the intersection of a line of the pencil X and a line of the pencil Y_0. We can therefore order the points of EG $(2, s)$ as follows:

Let P, P' be two distinct points of EG $(2, s)$ given by

$$P = l_i \cap l_{0,j} \,, \qquad P' = l_{i'} \cap l_{0,j'} \,;$$

then $P < P'$ if and only if $l_i < l_{i'}$ or $i = i'$ and $l_{0,j} < l_{0,j'}$.

We will assign the numbers $0, 1, \cdots, s^2 - 1$ to the s^2 points of EG $(2, s)$ in that order.

Lines will be viewed as s-tuples of their points enumerated in increasing order, and pencils as square matrices of points whose ith row is the ith line of the pencil, $i = 0, 1, \cdots, s-1$.

We will use the $n \times n$ permutation matrices τ_n and ζ_n defined as follows:

$$\tau_n = \begin{bmatrix} 0_{n-1,1} & I_{n-1} \\ 1 & 0_{1,n-1} \end{bmatrix}, \qquad \zeta_n = \tau_n{}^T \,.$$

By premultiplying an $m \times n$ matrix A by τ_m we achieve a cyclic permutation of its rows; by postmultiplying A by ζ_n we achieve a cyclic permutation of its columns. The subindices will be dropped whenever the dimensions of the matrices involved are clear.

We will also introduce the transformation σ defined on the points of EG (2, s) as follows:

$$\sigma(x, y) = (y, x) \qquad \text{for all } (x, y) \in \text{EG}(2, s) .$$

Y will denote the $s^2 \times s$ matrix

$$Y = \begin{bmatrix} Y_0 \\ Y_1\zeta \\ \vdots \\ Y_{s-1}\zeta^{s-1} \end{bmatrix}$$

and G will be the $(s^2 + s) \times s$ matrix

$$G = \begin{bmatrix} X \\ Y \end{bmatrix} .$$

THEOREM 3.1. *There exist* GYD *with parameters* $v = s^2$, $b = k = s(s + 1)$,

$$r = (s + 1)^2 , \qquad m = n = 1 , \qquad \lambda_1 = \lambda_2 = s^2 + 3s + 3$$
$$\alpha = s , \qquad \alpha_0 = s^2 - s - 1 , \qquad \alpha_1 = 1 .$$

PROOF. We take the varieties of the design to be the points of EG (2, s).

Each column of the matrix Y is a permutation of the set of the s^2 points.

Suppose that the point "a" appears twice in the jth column of Y for some j; then we must have

$$\{a\} = l_{\alpha,i} \cap l_{j+\alpha} = l_{\beta,k} \cap l_{j+\beta}$$

for some α, β, i, k, $\alpha \neq \beta$, which is impossible since the lines $l_{j+\alpha}$ and $l_{j+\beta}$ are different and parallel.

Similarly each row of σY^T is also a permutation of the points of EG (2, s).

It can be seen that the matrix

$$D = \begin{bmatrix} X & \sigma Y^T \\ Y & L \end{bmatrix}$$

where L is any Latin square of order s^2, is the desired GYD.

First note that $\sigma X^T = X$; therefore the first s rows of D are the lines of EG (2, s) written vertically, and we have natural one-to-one correspondence between the lines of EG (2, s) and the rows and the columns of D.

Note that a point occurs twice in a row or column of D if and only if it belongs to the corresponding line; consequently since no two lines have more than one point in common any two rows or columns will have at most one point occurring twice in common. Therefore $\alpha_1 = \beta_1 = 1$ and we conclude that D is a GYD.

EXAMPLE. For $s = 3$ we have

$$X = \begin{matrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{matrix} \qquad Y_0 = \begin{matrix} 0 & 3 & 6 \\ 1 & 4 & 7 \\ 2 & 5 & 8 \end{matrix} \qquad Y_1 = \begin{matrix} 0 & 5 & 7 \\ 1 & 3 & 8 \\ 2 & 4 & 6 \end{matrix} \qquad Y_2 = \begin{matrix} 0 & 4 & 8 \\ 1 & 5 & 6 \\ 2 & 3 & 7 \end{matrix}$$

$$D = \begin{matrix} 0 & 1 & 2 & 0 & 3 & 6 & 7 & 1 & 4 & 8 & 2 & 5 \\ 3 & 4 & 5 & 1 & 4 & 7 & 5 & 8 & 2 & 0 & 3 & 6 \\ 6 & 7 & 8 & 2 & 5 & 8 & 0 & 3 & 6 & 4 & 7 & 1 \\ 0 & 3 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 4 & 7 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \\ 2 & 5 & 8 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 \\ 5 & 7 & 0 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 \\ 3 & 8 & 1 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 \\ 4 & 6 & 2 & 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 \\ 8 & 0 & 4 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 \\ 6 & 1 & 5 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 2 & 3 & 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix}.$$

REMARK. It may be worthwhile to describe the construction verbally. We numbered the points in such a way that the σ transformation amounts to reflection upon the diagonal of the X or Y_0 matrix. Moreover, σ transforms X into Y_0 and vice versa. Hence the construction consists of the following steps. First we compute the X and the Y matrix using the properties of the GF (s) in question and assigning numbers to the points as described. Then we put the X matrix in the upper left corner followed to the right and below by the Y_0 matrix. Then we complete the vertical part of the design by writing below the Y_0 matrix a cyclic permutation of the columns of Y_1 starting with the second column. We continue in this manner starting cyclic permutation of the Y_ith matrix with the $i + 1$th column. To complete the horizontal border we may use the juxtaposition of the transpose of each row of the vertical border in the same order after changing the elements to their images reflected upon the diagonal of the X matrix.

To complete the design we fill the empty space with any Latin square of order s^2. Thus, in fact, we can get by this construction as many non-isomorphic designs as they are non-isomorphic Latin squares.

LEMMA 3.2. *There exist Latin squares of order s^2 which can be split into s groups of s columns in such a way that every row in each group is a line of* EG $(2, s)$.

PROOF. We claim that

$$L = \begin{bmatrix} Y_0 & \tau Y_0 & \cdots \tau^{s-1} Y_0 \\ Y_1 \zeta & \tau Y_1 \zeta & \cdots \tau^{s-1} Y_1 \zeta \\ \vdots & \vdots & \vdots \\ Y_{s-1} \zeta^{s-1} & \tau Y_{s-1} \zeta^{s-1} & \cdots \tau^{s-1} Y_{s-1} \zeta^{s-1} \end{bmatrix}$$

is the desired Latin square.

We have already shown that each column of Y is a permutation of the s^2 points; therefore so is every column of L.

We must show now that each row of L is also a permutation of the s^2 points; but since τ^i is not the identity if $0 < i < s - 1$ each row of L is made out of s different lines belonging to the same parallel pencil and therefore no point can occur twice in the same row.

EXAMPLE. We have already constructed EG (2, 3). The Latin square can now be exhibited as follows:

$$L = \begin{matrix} 0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 \\ 1 & 4 & 7 & 2 & 5 & 8 & 0 & 3 & 6 \\ 2 & 5 & 8 & 0 & 3 & 6 & 1 & 4 & 7 \\ 5 & 7 & 0 & 3 & 8 & 1 & 4 & 6 & 2 \\ 3 & 8 & 1 & 4 & 6 & 2 & 5 & 7 & 0 \\ 4 & 6 & 2 & 5 & 7 & 0 & 3 & 8 & 1 \\ 8 & 0 & 4 & 6 & 1 & 5 & 7 & 2 & 3 \\ 6 & 1 & 5 & 7 & 2 & 3 & 8 & 0 & 4 \\ 7 & 2 & 3 & 8 & 0 & 4 & 6 & 1 & 5 \end{matrix} .$$

THEOREM 3.3. *There exist* GYD *with parameters* $v = s^2$, $b = s(s^2 - 1)$, $k = s(s + 1)$.

$$r = (s + 1)^2(s - 1) \qquad m = s - 1 \qquad n = 1$$
$$r_{(b)} = s^2 - 1 \qquad r_{(k)} = s^2 - 1$$
$$\lambda_1 = (s - 1)(s^2 - 1)(s + 2) + (s^2 - s - 1)$$
$$\lambda_2 = (s^2 - 1)(s + 2) + (s - 1)$$
$$\alpha = s \qquad \alpha_0 = 1 \qquad \alpha_1 = s^2 - s - 1$$
$$\beta = s^2 - s \qquad \beta_0 = s^3 - 2s^2 + 1 \qquad \beta_1 = s - 1 .$$

PROOF. Let L be the Latin square of order s^2 constructed as in the previous lemma. For every point a, let $p_L(a)$ be the transpose of the column vector of L whose first component is a with that first component missing. This notation is consistent since each row of L is a permutation of the points of EG (2, s). Thus $p_L(a)$ is a $(s^2 - 1)$-tuple of distinct points and it does not contain the point a; p_L is a mapping defined through the Latin square L; in matrix notation

$$p_L(a) = c_L(a)^T \begin{bmatrix} 0_{1-s^2-1} \\ I_{s^2-1} \end{bmatrix}$$

where $c_L(a)$ is the column of L whose first element is a.

For any $m \times n$ matrix $A = (a_{ij})$, $p_L(A)$ will be naturally understood as the $m \times n(s^2 - 1)$ matrix $p_L(A) = (p_L(a_{ij}))$.

Now let $G = [{}^X_Y]$ and consider the $s(s + 1) \times s(s^2 - 1)$ matrix $D = p_L(G)$.

We will prove first that the rows of D satisfy the requirements for a GYD.

Any row of D contains every point of the geometry s times, except for the s

points in the corresponding row of G, which will occur $s-1$ times. Furthermore, since the rows of G are the lines of EG $(2, s)$ the two elements of every pair of distinct points occur $s-1$ times in the same row of D exactly once. Therefore $\alpha_0 = 1$ and the row conditions are satisfied.

Let $x_{i,j}$, $y_{i,j}$ be the (i, j) entries in the matrices X and Y respectively; let G_j, $j = 0, 1, \cdots, s-1$, be the $s \times (s^2-1)$ matrix whose ith row is $p_L(x_{ij})$, $i = 0, 1, \cdots, s-1$, and similarly let L_j, $j = 0, 1, \cdots, s-1$, be the $s^2 \times (s^2-1)$ matrix whose ith row is $p_L(y_{ij})$, $i = 0, 1, \cdots, s^2-1$. Note that there are no repeated points in any row or column of L_j, $j = 0, \cdots, s-1$, but it is not a Latin square since each row has only s^2-1 points.

The matrix D can be written

$$D = p_L(G) = \begin{bmatrix} G_0 & G_1 \cdots G_{s-1} \\ L_0 & L_1 \cdots L_{s-1} \end{bmatrix}.$$

Observe that since $X^T = Y_0$,

$$G_j = \begin{bmatrix} \tau^j Y_0 \\ \tau^j Y_1 \zeta \\ \vdots \\ \tau^j Y_{s-1} \zeta^{s-1} \end{bmatrix}^T \begin{bmatrix} 0_{1,s^2-1} \\ I_{s^2-1} \end{bmatrix} \qquad j = 0, 1, \cdots, s-1,$$

that is, the matrix G_j is the transpose of the jth block of s columns of L with the first row missing, and that missing first row is $l_{0,j}$, the jth line of the pencil Y_0. Therefore the columns of G_j are the lines of EG $(2, s)$ written vertically except for the line $l_{0,j}$ and the s lines l_i, $i = 0, 1, \cdots, s-1$, of the pencil X. Hence in each G_j there are $s+1$ missing lines.

The idea of the construction is to use one of the matrices G_j consisting of $s^2 - 1 = (s+1)(s-1)$ s-tuple columns to complete each of the remaining $s-1$ G_j's to a full geometry. We shall show that this can be achieved by permuting the elements within each row of chosen G_j and keeping the rows constant, which will preserve the already established GYD property for the rows.

The lines to be recovered by the chosen G_j are the s lines of the pencil X each replicated $s-1$ times plus the lines of the pencil Y_0 except $l_{0,j}$, a total of

$$s(s-1) + s - 1 = s^2 - 1 \text{ lines.}$$

Let the lines of the pencil X be written vertically. Since $X^T = Y_0$, if we apply the cyclic permutation τ^{s-i} to the ith line of the pencil X, each row of the resulting matrix X^* will contain one point from each line of Y_0; indeed

$$X^* = [l_0, \tau l_1^{s-i}, \cdots, \tau l_{s-1}]$$

where l_i, $i = 0, 1, \cdots, s-1$, is the ith line $x = \alpha_i$ of X written vertically. Consequently each row of the $s \times s(s-1)$ matrix

$$[X^*, \tau X^*, \cdots, \tau^{s-2} X^*]$$

will contain $s-1$ points from each line of Y_0.

We shall add to each row of the above matrix $s - 1$ points chosen in such a way that all the lines except $l_{0,j}$ will be completed. Notice that this must be done in a unique way since each of the lines had exactly one point missing. We obtain this way the $s \times (s^2 - 1)$ matrix $G_j{}^*$ which is characterized by the fact that only the line $l_{0,j}$ of Y_0 is missing.

It is clear from the way $G_j{}^*$ was constructed that the ith point of $l_{0,j}$ will appear in the $j + i$th ($j + i$ taken mod s) row of $G_j{}^*$ as well as in the $s - 2$ preceding rows $u + i - 1$ (mod s), $\cdots, j + 1 - (s - 2)$(mod s), but not in the following row $j + i + 1$ (mod s), $i = 0, 1, \cdots, s - 1$. Therefore the matrix $\tau^{j+1}G_j{}^*$ is such that its ith row does not contain the ith point of $l_{0,j}$, $i = 0, 1, \cdots, s - 1$, which is also the case with G_j. Thus the ith rows of $\tau^{j+1}G_j{}^*$ and of G_j contain the same points, but in a different order.

Substituting $\tau^{j+1}G_j{}^*$ for G_j in D we obtain

$$D_j{}^* = \begin{bmatrix} G_0 & \cdots & \tau^{j+1}G_j{}^* & \cdots & G_{s-1} \\ L_0 & \cdots & L_j & \cdots & L_{s-1} \end{bmatrix}$$

which we claim is a GYD.

We need only to verify the conditions regarding the columns.

Since every column of L_i, $i = 0, 1, \cdots, s - 1$ is a row of a Latin square, and since each column of G_i, $i = 0, 1, \cdots, s - 1$ and $G_j{}^*$ is a line of EG $(2, s)$, we see that a point occurs twice in a column as many times as it appers in a line; since each point belongs to $s + 1$ lines in the geometry and we have $s - 1$ replicated geometries, we conclude that any given point occurs twice in $(s + 1)(s - 1) = s^2 - 1 = r_{(b)}$ columns.

Two distinct points will appear each twice in the same column if they belong to the same line; since a pair of distinct points determine a unique line and there are $s - 1$ replicated geometries, $\beta = s - 1$ and we can conclude that $D_j{}^*$ is a GYD.

EXAMPLE. For $s = 3$ we have

$$v = 9\,, \qquad b = 24\,, \qquad k = 12\,, \qquad r = 32\,, \qquad m = 2\,, \qquad n = 1\,,$$
$$\lambda_1 = 85\,, \qquad \lambda_2 = 42\,.$$

Using X, Y_0, Y_1, Y_2 and L as computed before we get

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{matrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \\ 0 & 3 & 6 \\ 1 & 4 & 7 \\ 2 & 5 & 8 \\ 5 & 7 & 0 \\ 3 & 8 & 1 \\ 4 & 6 & 2 \\ 8 & 0 & 4 \\ 6 & 1 & 5 \\ 7 & 2 & 3 \end{matrix} \qquad \begin{matrix} G_0 = \begin{matrix} 1 & 2 & 5 & 3 & 4 & 8 & 6 & 7 \\ 4 & 5 & 7 & 8 & 6 & 0 & 1 & 2 \\ 7 & 8 & 0 & 1 & 2 & 4 & 5 & 3 \end{matrix} \\ \\ G_1 = \begin{matrix} 2 & 0 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 3 & 8 & 6 & 7 & 1 & 2 & 0 \\ 8 & 6 & 1 & 2 & 0 & 5 & 3 & 4 \end{matrix} \\ \\ G_2 = \begin{matrix} 0 & 1 & 4 & 5 & 3 & 7 & 8 & 6 \\ 3 & 4 & 6 & 7 & 8 & 2 & 0 & 1 \\ 6 & 7 & 2 & 0 & 1 & 3 & 4 & 5 \end{matrix} \end{matrix}$$

$$X^* = \begin{matrix} 0 & 5 & 7 \\ 1 & 3 & 8 \\ 2 & 4 & 6 \end{matrix}$$

$$G_0^* = \begin{matrix} 0 & 5 & 7 & 1 & 3 & 8 & 4 & 2 \\ 1 & 3 & 8 & 2 & 4 & 6 & 7 & 5 \\ 2 & 4 & 6 & 0 & 5 & 7 & 1 & 8 \end{matrix}$$

$$G_1^* = \begin{matrix} 0 & 5 & 7 & 1 & 3 & 8 & 6 & 2 \\ 1 & 3 & 8 & 2 & 4 & 6 & 0 & 5 \\ 2 & 4 & 6 & 0 & 5 & 7 & 3 & 8 \end{matrix}$$

$$G_2^* = \begin{matrix} 0 & 5 & 7 & 1 & 3 & 8 & 6 & 4 \\ 1 & 3 & 8 & 2 & 4 & 6 & 0 & 7 \\ 2 & 4 & 6 & 0 & 5 & 7 & 3 & 1 \end{matrix}$$

The desired design

$$D_1^* = \begin{bmatrix} G_0 & \tau^2 G_1^* & G_2 \\ L_0 & L_1 & L_2 \end{bmatrix}$$

becomes

$$\begin{matrix}
1&2&5&3&4&8&6&7 && 2&4&6&0&5&7&3&8 && 0&1&4&5&3&7&8&6 \\
4&5&7&8&6&0&1&2 && 0&5&7&1&3&8&6&2 && 3&4&6&7&8&2&0&1 \\
7&8&0&1&2&4&5&3 && 1&3&8&2&4&6&0&5 && 6&7&2&0&1&3&4&5 \\
1&2&5&3&4&8&6&7 && 4&5&7&8&6&0&1&2 && 7&8&0&1&2&4&5&3 \\
2&0&3&4&5&6&7&8 && 5&3&8&6&7&1&2&0 && 8&6&1&2&0&5&3&4 \\
0&1&4&5&3&7&8&6 && 3&4&6&7&8&2&0&1 && 6&7&2&0&1&3&4&5 \\
3&4&6&7&8&2&0&1 && 8&6&1&2&0&5&3&4 && 1&2&5&3&4&8&6&7 \\
4&5&7&8&6&0&1&2 && 6&7&2&0&1&3&4&5 && 2&0&3&4&5&6&7&8 \\
5&3&8&6&7&1&2&0 && 7&8&0&1&2&4&5&3 && 0&1&4&5&3&7&8&6 \\
6&7&2&0&1&3&4&5 && 1&2&5&3&4&8&6&7 && 5&3&8&6&7&1&2&0 \\
7&8&0&1&2&4&5&3 && 2&0&3&4&5&6&7&8 && 3&4&6&7&8&2&0&1 \\
8&6&1&2&0&5&3&4 && 0&1&4&5&3&7&8&6 && 4&5&7&8&6&0&1&2 .
\end{matrix}$$

THEOREM 3.4. *There exist* GYD *with parameters* $v = s^2$, $b = k = s(s^2 - 1)$,

$$r = (s^2 - 1)^2 \qquad m = n = s - 1$$
$$r_{(b)} = r_{(k)} = (s^2 - 1)(s - 1) \qquad b_{(v)} = s(s - 1)$$
$$\lambda_1 = \lambda_2 = s^5 - 3s^3 + 3s - 1$$
$$\alpha = \beta = s(s - 1) \qquad \alpha_1 = \beta_1 = s^3 - 2s^2 + 1 \qquad \alpha_0 = \beta_0 = s - 1 .$$

PROOF. Let us permute cyclically the lines within the same parallel pencil in $\tau^{j+1}G_j^*$; this can be accomplished by matrix multiplication as follows:

$$\tau^{j+1} G_j^* \begin{bmatrix} \zeta_s & & & & \\ & \zeta_s & & & \\ & & \ddots & & \\ & & & \zeta_s & \\ & & & & \zeta_{s-1} \end{bmatrix}^j = G_j^{**} ,$$

where there are $s - 1$ matrices ζ_s and all the off diagonal matrices are zero.

The $s(s^2-1) \times s(s^2-1)$ square matrix

$$D^{**} = \begin{bmatrix} G_0 & G_1 & \cdots G_{s-1}^{**} \\ L_0 & L_1 & \cdots L_{s-1} \\ G_1 & G_2 & \cdots G_0^{**} \\ L_0 & L_2 & \cdots L_0 \\ \cdot & \cdot & \cdot \\ G_{s-2} & G_{s-1} & \cdots G_{s-3}^{**} \\ L_{s-2} & L_{s-1} & \cdots L_{s-3} \end{bmatrix}$$

is a GYD.

Using the same argument as in the previous theorem we will prove that the row conditions for GYD are satisfied.

Any given column of D^{**} is made out of $s-1$ rows of the Latin square L, corresponding to the matrices L_i, plus $s-1$ different parallel lines, corresponding to either the matrices G_i or to the matrices G_i^{**} as the case may be. Therefore a point occurs in each column either $s+1$ or s times; it will occur s times if and only if it belongs to one of the $s-1$ parallel lines. Since these parallel lines contain $s(s-1)$ points, the number of points repeated in the column $s = n+1$ times is $s(s-1) = s^2 - s = k_{(v)}$. Furthermore, the missing lines from each column of D^{**} are the columns of the missing G_j, (G_j^{**}), matrix in each block of s^2-1 columns; these matrices are

$$G_{s-1}, G_0, \cdots, G_{s-2}^{**}$$

and they constitute, as we have seen in the previous theorem, the full geometry EG $(2, s)$ replicated $s-1$ times. Therefore each member of a pair of points will appear $s-1$ times in the same column if and only if both points belong to the line missing from that column, and $\beta_0 = s-1$. This concludes the proof that D^{**} is a GYD.

EXAMPLE. For $s = 3$ we have

$$v = 9\,, \quad b = k = 24\,, \quad r = 64\,, \quad \lambda_1 = \lambda_2 = 170\,,$$
$$\alpha = \beta = 6\,, \quad \alpha_0 = \beta_0 = 2\,, \quad \alpha_1 = \beta_1 = 10\,.$$

We add to the previous computations G_0^{**}, G_1^{**} and G_2^{**}; they are:

$$G_0^{**} = \begin{matrix} 1 & 3 & 8 & 2 & 4 & 0 & 7 & 5 \\ 2 & 4 & 6 & 0 & 5 & 7 & 1 & 8 \\ 0 & 5 & 7 & 1 & 3 & 8 & 4 & 2 \end{matrix} \qquad G_1^{**} = \begin{matrix} 4 & 6 & 2 & 5 & 7 & 0 & 8 & 3 \\ 5 & 7 & 0 & 3 & 8 & 1 & 2 & 6 \\ 3 & 8 & 1 & 4 & 6 & 2 & 5 & 0 \end{matrix}$$

$$G_2^{**} = \begin{matrix} 7 & 0 & 5 & 8 & 1 & 3 & 6 & 4 \\ 8 & 1 & 3 & 6 & 2 & 4 & 0 & 7 \\ 6 & 2 & 4 & 7 & 0 & 5 & 3 & 1\,. \end{matrix}$$

The desired design becomes:

$$D^{**} = \begin{bmatrix} G_0 & G_1 & G_2^{**} \\ L_0 & L_1 & L_2 \\ G_1 & G_2 & G_0^{**} \\ L_1 & L_2 & L_0 \end{bmatrix}.$$

Using trial and error with EG (2, 3), a GYD with the same parameter values as D^{**}, but not isomorphic to it, was constructed. Details are available from the authors.

As stated in Section 1, J. Kiefer (1958) proved that GYD are D-optimum when either $b_{(v)} = 0$ or $k_{(v)} = 0$. He also gave an example of a GYD with $v = 4$, $b = k = 6$ which is not optimal, raising the question whether the divisibility condition is necessary for D-optimality. Kiefer (1971) also indicated that his example could be generalized to $v = t$, $b = k = 6t$, t odd. We shall present presently a class of GYD with these parameters and show that they are in fact not D-optimal. However, it seems that our method of construction is not along the lines suggested by Kiefer.

THEOREM 3.5. *There exists* GYD *with* $v = 4$, $b = k = 6t$ *for any odd integer* t.

PROOF. The other parameters are

$$r = 9t^2 \qquad b_{(b)} = k_{(v)} = 2 \qquad r_{(b)} = r_{(k)} = 3t$$

$$m = n = \frac{3t-1}{2} \qquad \lambda_1 = \lambda_2 = \frac{27t^3 - t}{2}$$

$$\alpha = \beta = 2t \qquad \alpha_0 = \beta_0 = t \qquad \alpha_1 = \beta_1 = t\,.$$

Let the set of varieties of $V = \{A, 1, 2, 3\}$ and let ζ be a permutation on v^b defined as follows:

$$\zeta(a_1, \cdots, a_b) = (a_b, a_1, \cdots, a_{b-1})\,, \quad \forall (a_1, \cdots, a_b) \in V^b\,.$$

Let τ be a transformation on V which leaves exactly one variety fixed; by renaming the varieties if necessary we may assume without loss of generality that

$$\tau(A) = A\,, \qquad \tau(1) = 2\,, \qquad \tau(2) = 3\,, \qquad \tau(3) = 1\,.$$

Finally, let $p \in V^b$ be

$$p = (A \overset{m}{\cdots} A, 1 \overset{m+1}{\cdots} 1, 2 \overset{m+1}{\cdots} 2, 3 \overset{m}{\cdots} 3)$$

and let D be a $k \times b$ matrix whose first row is p and such that every row and column is the transform of the preceding one by $\zeta \circ \tau$.

Since τ leaves A fixed, A will occur m times in each row and column of D; since τ^3 is the identity every variety other then A will appear $m + 1$ times in two of every three consecutive rows or columns.

Let d_{ij}, $i = 1, 2, \cdots, k$, $j = 1, 2, \cdots, b$ be the (i, j) entry of the matrix D. We claim that if we make $d_{i,3t+i} = A$, $i = 1, 2, \cdots, 3t$, the resulting matrix D^* is a GYD.

Variety A appears $m + 1$ times in each of the first $3t = r_{(k)}$ rows; any other variety $x \neq A$ appears $m + 1$ times in one out of every three consecutive rows for the first $3t$ rows, and in two out of every three consecutive rows for the last $3t$ rows, that is in a total of $(3t/3) + (3t/3)2 = 3t = r_{(k)}$ rows. Moreover, the pair of distinct varieties A, x $(x \neq A)$ appear $m+1$ times each in the same row $t = \alpha_1$ times.

A pair of distinct varieties other than A can occur $m+1$ times each in the same row only in the last $3t$ rows and in exactly one out of every three consecutive rows, that is in $t = \alpha_1$ rows.

The same arguments applied to the columns would allow us to conclude that D^* is a GYD.

EXAMPLE. For $t = 1$, we have

$$v = 4, \quad b = k = 6, \quad r = 9, \quad \lambda_1 = \lambda_2 = 13, \quad m = n = 1,$$
$$r_{(b)} = r_{(k)} = 3, \quad b_{(v)} = k_{(v)} = 2, \quad \alpha_0 = \beta_0 = 3. \quad \alpha_1 = \beta_1 = 1.$$

$$D = \begin{matrix} A & 1 & 1 & 2 & 2 & 3 \\ 1 & A & 2 & 2 & 3 & 3 \\ 1 & 2 & A & 3 & 3 & 1 \\ 2 & 2 & 3 & A & 1 & 1 \\ 2 & 3 & 3 & 1 & A & 2 \\ 3 & 3 & 1 & 1 & 2 & A \end{matrix} \qquad D^* = \begin{matrix} A & 1 & 1 & A & 2 & 3 \\ 1 & A & 2 & 2 & A & 3 \\ 1 & 2 & A & 3 & 3 & A \\ 2 & 2 & 3 & A & 1 & 1 \\ 2 & 3 & 3 & 1 & A & 2 \\ 3 & 3 & 1 & 1 & 2 & A \end{matrix}.$$

We will show now that the GYD D^* is not D-optimum, by comparing it with the non-symmetrical design D.

The hypothesis to be tested is that variety has no effect on yield, that is

$$\gamma_A = \gamma_1 = \gamma_2 = \gamma_3 .$$

In the two-way heterogeneity setting where we have v varieties and a $k \times b$ array of plots, the covariance matrix is given by

$$c_{ij} = \delta_{ij} r_i - \frac{\lambda_{ij}^{(1)}}{b} - \frac{\lambda_{ij}^{(2)}}{k} + \frac{r_i r_j}{kb}$$

where δ_{ij} is the Kronecker delta, r_i is the number of replications of the ith variety and

$$\lambda_{ij}^{(1)} = \sum_{l=1}^{k} n_{il}^{(1)} n_{jl}^{(1)}$$
$$\lambda_{ij}^{(2)} = \sum_{l=1}^{b} n_{il}^{(2)} n_{jl}^{(2)}$$

with $n_{il}^{(q)}$ equal to the number of occurrences of the ith variety in the lth row $(q = 1)$ or the lth column $(q = 2)$.

It is a straightforward but long computation to obtain in the case of D^*

$$c_{ii}^* = \frac{27t^2 - 2}{4} \qquad c_{ij}^* = \frac{2 - 27t^2}{12}$$

for $i \neq j$, $i, j = A, 1, 2, 3$.

For the design D one would obtain

$$c_{AA} = \frac{27t^2 - 6t - 1}{4} \qquad c_{Ai} = \frac{27t^2 - 6t - 1}{12}$$

$$c_{ii} = \frac{243t^2 + 18t - 17}{36} \qquad c_{ij} = \frac{81t^2 - 18t + 7}{36}$$

$$i \neq j, i, j = 1, 2, 3$$

and for the corresponding determinants Δ^* and Δ,

$$\Delta^* = \frac{27t^2 - 2}{3}$$

$$\Delta = \frac{[27t^2 + 3t - 2]^2[27t^2 - 6t - 1]}{3^3}.$$

The difference $\Delta - \Delta^* = (108t^3 - 45t^2 - 12t + 4)/3^3$ is positive for any positive t; therefore D^* is not D-optimum.

Note, however, that for the eigenvalues we still have $(27t^2 - 2)/3 > (27t^2 - 6t - 1)/3$, that is the smallest eigenvalue of D^* is larger than the smallest eigenvalue of D, as it should be.

For $t = 1$

$$\Delta^* = \frac{25^3}{3^3} = \frac{15625}{27}$$

$$\Delta = \frac{28^2}{3^2}\frac{20}{3} = \frac{15680}{27}$$

$$\Delta - \Delta^* = \frac{55}{27} > 0.$$

4. Modification of construction of GYD's when divisibility conditions are satisfied. In this section we shall describe mappings of the sets of elements of the GYD's constructed previously onto new sets aimed at the construction of optimal GYD's with suitably chosen parameter sets.

Case 1. $b = k = tv$. Take any Latin square of order tv and divide its elements arbitrarily into b/t sets of v elements each. Identify all the elements of each set with exactly one element of a set of v distinct elements.

We may specify this method to the case $v = 4$, $b = 6t$, t even and obtain an optimal GYD.

Case 2. Consider the class of GYD's following from Theorem 3.1. Modifying the set of parameters to $v = s$, $b = k = s(s + 1)$ we may construct optimal GYD's essentially in two ways. We may apply the method of Case 1. On the other hand we may apply the construction of case 1 only to the inner Latin square of size s^2 and replace the bordering $2s + 1$ squares of size s by arbitrary (the same or different) Latin squares of order s.

Case 3. We shall modify now the parameter set of Theorem 3.2 using $v = s$, $b = s(s^2 - 1)$, $k = s(s + 1)$. Notice that we may divide the original design into

$s + 1$ groups of s rows each of the following structure. The s rows consist of s, $s \times (s^2 - 1)$ matrices which contain $s - 1$ squares of size s in which each of the s^2 elements appears, and $s \times (s - 1)$ matrix of all distinct elements. We may replace all the squares of size s by arbitrary Latin squares of order s and the remaining $s - 1$ columns of any of the s matrices of size $s \times (s^2 - 1)$ by $s - 1$ columns of a Latin square. To complete the construction we shall replace the $s - 1$ columns of the remaining $s - 1$ matrices of size $s \times (s^2 - 1)$ by $s - 1$ columns whose elements are cyclic permutations of the first chosen $s - 1$ columns. We shall illustrate this construction by specifying the example of Theorem 3.2.

$$\begin{array}{cccccccc@{\qquad}cccccccc@{\qquad}cccccccc}
A & B & A & B & C & A & B & C & A & B & C & A & B & C & B & C & A & B & C & A & B & C & C & A \\
B & C & B & C & A & B & C & A & B & C & A & B & C & A & C & A & B & C & A & B & C & A & A & B \\
C & A & C & A & B & C & A & B & C & A & B & C & A & B & A & B & C & A & B & C & A & B & B & C \\
A & B & A & B & C & A & B & C & B & C & A & B & C & A & B & C & C & A & B & B & C & A & C & A \\
B & C & B & C & A & B & C & A & B & A & B & C & A & B & C & A & A & B & C & C & A & B & A & B \\
C & A & C & A & B & C & A & B & A & B & C & A & B & C & A & B & B & C & A & A & B & C & B & C \\
A & B & A & B & C & A & B & C & B & C & A & B & C & A & B & C & C & A & B & B & C & A & C & A \\
B & C & B & C & A & B & C & A & C & A & B & C & A & B & C & A & A & B & C & C & A & B & A & B \\
C & A & C & A & B & C & A & B & A & B & C & A & B & C & A & B & B & C & A & A & B & C & B & C \\
A & B & A & B & C & A & B & C & B & C & A & B & C & A & B & C & C & A & B & B & C & A & C & A \\
B & C & B & C & A & B & C & A & C & A & B & C & A & B & C & A & A & B & C & C & A & B & A & B \\
C & A & C & A & B & C & A & B & A & B & C & A & B & C & A & B & B & C & A & A & B & C & B & C \, .
\end{array}$$

Notice that using different Latin squares we could obtain a non-isomorphic design. We could also describe many other mappings yielding a design with the same parameters.

Case 4. We shall conclude this discussion modifying the parameter set of Theorem 3.3 to $v = s$, $b = k = s(s^2 - 1)$. Here again we may apply the method of Case 1 or of Case 3. In the latter case we may apply juxtaposition of either the same or different $s - 1$ designs of the same structure.

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