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NECESSARY AND SUFFICIENT CONDITIONS FOR INEQUALITIES OF CRAMÉR-RAO TYPE[1]

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For a random variable X with possible distributions indexed by a parameter θ, and for real-valued $T = T(X)$ and $V = V(X, \theta)$ with Var $T < \infty$ and $0 <$ Var $V < \infty$, Schwarz's inequality gives Var $T \geqq \{\text{Cov}(T, V)\}^2/\text{Var } V$. Necessary and sufficient conditions are given for this inequality to be of Cramér-Rao type: Var $T \geqq \{a_m(\theta)\}^2/\text{Var } V$ where $m(\theta)$ is a notation for ET and $a_m(\theta)$ is a notation for Cov (T, V). Specialized to $V = \{\partial p_\theta(X)/\partial\theta\}/p_\theta(X)$, where p_θ is a probability density function for X, these conditions are necessary and sufficient for validity of the Cramér-Rao inequality. The use of these inequalities in proving an estimator minimum variance unbiased is shown to be superfluous. The use of these inequalities in proving admissibility is discussed, with examples.

1. Introduction and summary. Let X be a random variable with possible probability measures P_θ, $\theta \in \Omega$ on a σ-field of subsets of space $\mathfrak{X}$. This is the general probability model of statistics; any restrictions on the space $\mathfrak{X}$ or on the family of measures will be stated where needed. For

$$T = T(X) \quad \text{any real-valued statistic,}$$
$$V = V(X, \theta) \quad \text{any real-valued random variable,}$$
$$\text{with} \qquad \text{Var } T < \infty \quad \text{and} \quad 0 < \text{Var } V < \infty,$$

Schwarz's inequality with optimal centering gives

$$\text{Var } T \geqq \frac{\{\text{Cov}(T, V)\}^2}{\text{Var } V}.$$

These conditions on T and V are assumed throughout, for all of the T's and V's that appear.

When a particular V is such that

$$\begin{aligned} &\text{Cov}(T, V) \quad \text{depends on} \quad T \quad \text{only through} \quad ET, \\ &\text{i.e.} \quad ET_1 \equiv ET_2 \quad \text{implies} \quad \text{Cov}(T_1, V) \equiv \text{Cov}(T_2, V), \\ &\text{i.e.} \quad ET = m(\theta) \quad \text{implies} \quad \text{Cov}(T, V) = a_m(\theta) \qquad \text{for all } m \end{aligned} \tag{1}$$

then the Schwarz inequality takes the useful form of a lower bound on Var T in terms of ET, and is referred to in [1] as an *inequality of Cramér–Rao type*:

$$\begin{aligned} &\text{Var } T \geqq \{a_m(\theta)\}^2/\text{Var } V, \quad \text{where} \\ &m(\theta) \quad \text{is a notation for} \quad ET, \quad \text{and} \\ &a_m(\theta) \quad \text{is a notation for} \quad \text{Cov}(T, V). \end{aligned} \tag{2}$$

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Notice that for the Schwarz inequality to be of Cramér–Rao type (2), it is enough that V satisfy the apparently weaker condition

$$\begin{aligned} &ET_1 \equiv ET_2 \quad \text{implies} \quad \{\mathrm{Cov}\,(T_1, V)\}^2 \equiv \{\mathrm{Cov}\,(T_2, V)\}^2\,, \\ (3)\qquad &\text{i.e.}\quad ET_1 \equiv ET_2 \quad \text{implies} \quad \mathrm{Cov}\,(T_1, V) = S(\theta)\cdot \mathrm{Cov}\,(T_2, V) \\ &\text{where}\quad S(\theta) \quad \text{takes on only the values} \quad \pm 1\,. \end{aligned}$$

But this apparently weaker condition (3) is actually equivalent to condition (1), because if $ET_1 \equiv ET_2$ then $(T_1 + T_2)/2$ has this same expectation, and the condition (3) implies

$$\left\{\mathrm{Cov}\left(\frac{T_1 + T_2}{2}\,, V\right)\right\}^2 \equiv \{\mathrm{Cov}\,(T_1, V)\}^2 \equiv \{\mathrm{Cov}\,(T_2, V)\}^2\,,$$

i.e.

$$\{\tfrac{1}{2}\,\mathrm{Cov}\,(T_1, V) + \tfrac{1}{2}\,\mathrm{Cov}\,(T_2, V)\}^2 \equiv \{\mathrm{Cov}\,(T_1, V)\}^2 \equiv \{\mathrm{Cov}\,(T_2, V)\}^2\,.$$

At a value of θ for which $\mathrm{Cov}\,(T_1, V) = -\mathrm{Cov}\,(T_2, V)$ this becomes

$$0 = \{\mathrm{Cov}\,(T_1, V)\}^2 = \{\mathrm{Cov}\,(T_2, V)\}^2\,,$$

giving

$$0 = \mathrm{Cov}\,(T_1, V) = \mathrm{Cov}\,(T_2, V)\,,$$

and showing that condition (1) must hold for all θ.

In Section 2, necessary and sufficient conditions are given for V to give an inequality of Cramér–Rao type. Specialized to the particular $V = V_1 = \{\partial p_\theta(X)/\partial\theta\}/p_\theta(X)$, where p_θ is a probability density function for X, these conditions are necessary and sufficient for validity of the Cramér–Rao inequality.

In Section 3, it is shown that the use of Cramér–Rao type inequalities in proving an estimator minimum variance unbiased, either locally at $\theta = \theta_0$ or uniformly, is made superfluous by a theorem of Lehmann and Scheffé. The use of Cramér–Rao type inequalities in proving an estimator quadratic-loss admissible is also discussed, with examples.

The purpose of this paper is to do, in general, what is done in [1] for a model having a minimal sufficient statistic with a complete family of possible distributions; and to relate the results of [1] to those of Lehmann and Scheffé [3].

2. Necessary and sufficient conditions.

THEOREM 1. *$V = V(X, \theta)$, with $0 < \mathrm{Var}\, V < \infty$, gives an inequality of Cramér–Rao type if and only if V has 0 covariance with every finite-variance unbiased estimator of 0.*

Necessity. If V gives an inequality of Cramér–Rao type, i.e. satisfies condition (1), then for arbitrary T and every finite-variance unbiased estimator T_0 of 0, since T and $T + T_0$ have the same expectation they have the same covariance with V, i.e. $\mathrm{Cov}\,(T + T_0, V) \equiv \mathrm{Cov}\,(T, V)$, i.e. $\mathrm{Cov}\,(T, V) + \mathrm{Cov}\,(T_0, V) \equiv \mathrm{Cov}\,(T, V)$, showing that $\mathrm{Cov}\,(T_0, V) \equiv 0$ and proving the necessity.

Sufficiency. If V has 0 covariance with every finite-variance unbiased estimator of 0, then, for T_1 and T_2 with finite variances, $ET_1 \equiv ET_2$, i.e. $E(T_1 - T_2) \equiv 0$, implies $\mathrm{Cov}(T_1 - T_2, V) \equiv 0$, i.e. $\mathrm{Cov}(T_1, V) - \mathrm{Cov}(T_2, V) \equiv 0$, showing that V gives an inequality of Cramér–Rao type.

Note 1. (Theorem 2 of [1]). Every V satisfying the conditions of Theorem 1 must, with probability 1 for all θ, depend on X only through a minimal sufficient statistic. (This is proved in [1] by applying the Rao–Blackwell theorem at $\theta = \theta_0$ to $V(X, \theta_0)$ considered as an unbiased estimator of $EV(X, \theta_0)$.) If the minimal sufficient statistic has a complete family of possible distributions, then every V depending on X only through this statistic gives an inequality of Cramér–Rao type.

Note 2. If V is such that the third form of condition (1) holds non-vacuously for even one specified m, then the full condition (1) holds. To see this, notice that the conditions of Theorem 1 do not involve m; or notice that the necessity proof needs only one T for one specified m.

Note 3. A necessary and sufficient condition for the existence of an achievable Cramér–Rao type lower bound (2) for the variance of an estimator T having a specified m for expectation is that $m(\theta)$ possess a uniformly minimum variance unbiased estimator with positive variance. Because, for $T = T^*$ to achieve equality in (2), the V of that inequality must be $V = c(\theta) + d(\theta)T^*$ or equivalently, because of invariance under linear transformations, $V = T^*$, this with probability 1 for all θ. Therefore T^* must have 0 covariance with every finite-variance unbiased estimator of 0; but this condition is necessary and sufficient for T^* to be a uniformly minimum variance unbiased estimator of ET^* (Lehmann and Scheffé [3], Theorem 5.3).

Now, and for the rest of this section, we take the model to be a dominated family: Let p_θ be a probability density function for X, with respect to some fixed measure μ on subsets of $\mathfrak{X}$, and consider this particular V:

$$V_1 = \frac{1}{p_\theta(X)} \frac{\partial}{\partial\theta} \{p_\theta(X)\} .$$

To consider this V_1 we must further have θ real-valued (easily extended to vector-valued θ), and the parameter space Ω must be a union of non-degenerate intervals, with the derivative understood to mean the appropriate one-sided derivative at any end-points that are included in Ω.

Let us refer to the Cramér–Rao type inequality (2) with $V = V_1$ as the *Cramér–Rao inequality*:

$$\begin{aligned} &\mathrm{Var}\, T \geqq \{a_m(\theta)\}^2/\mathrm{Var}\, V_1\,, \quad \text{where} \\ &m(\theta) \quad \text{is a notation for} \quad ET, \\ &a_m(\theta) \quad \text{is a notation for} \quad \mathrm{Cov}(T, V_1), \\ &\text{and} \qquad V_1 = \{\partial p_\theta(X)/\partial\theta\}/p_\theta(X)\,. \end{aligned} \tag{4}$$

This differs from the usual *regular Cramér–Rao inequality*:

$$ET = m(\theta) \quad \text{implies} \quad \mathrm{Var}\, T \geqq \{m'(\theta)\}^2/\mathrm{Var}\, V_1 \tag{5}$$

in that (5) is a statement for the specified m only, and in that the $\{a_m(\theta)\}^2$ of (4) is replaced by $\{m'(\theta)\}^2$.

COROLLARY TO THEOREM 1. *The Cramér–Rao inequality* (4) *is valid if and only if* $0 < \operatorname{Var} V_1 < \infty$ *and* ET_0 *is differentiable under the expectation sign for every finite-variance unbiased estimator* T_0 *of* 0.

PROOF. Here the assertion $0 < \operatorname{Var} V_1 < \infty$ includes the assertion that $\operatorname{Var} V_1$ exists, which includes the assertion that V_1 exists with probability 1, for all θ. This corollary is just Theorem 1 with $V = V_1$. The condition that V_1 have 0 covariance with every finite-variance unbiased estimator of 0 is that $ET_0 \equiv 0$ should imply $ET_0 V_1 \equiv 0$. Now, the result of differentiating

$$0 = ET_0 = \int_{\mathfrak{X}} T_0(x) p_\theta(x)\, d\mu(x)$$

under the expectation sign is

$$\int_{\mathfrak{X}} T_0(x) \frac{\partial}{\partial\theta} p_\theta(x)\, d\mu(x) = \int_{\mathfrak{X}} T_0(x) \left\{\frac{\partial}{\partial\theta} p_\theta(x)\right\} \cdot \frac{p_\theta(x)}{p_\theta(x)}\, d\mu(x) = ET_0 V_1 .$$

This must be identically 0, making it coincide with the derivative of $ET_0 \equiv 0$.

Note 1. (As for Theorem 1). When the minimal sufficient statistic has a complete family of possible distributions, the differentiability condition of the corollary must be automatically satisfied, since V_1 depends on X only through a minimal sufficient statistic, from the factorization theorem.

Note 2. (As for Theorem 1). If the Cramér–Rao inequality (4) is true non-vacuously for even one specified m, then (4) is true (for all m).

Note 3. The usual sufficient conditions for the regular Cramér–Rao inequality (5), due to Wolfowitz [5], are

(i) $0 < \operatorname{Var} V_1 < \infty$,
(ii) $EV_1 \equiv 0$,
(iii) $ET = m(\theta)$ implies $ETV_1 = m'(\theta)$ for every finite-variance T with the specified expectation.

Condition (ii) is equivalent to differentiability of $E1$ under the expectation sign, and condition (iii) is equivalent to differentiability of ET under the expectation sign for every finite variance T with the specified expectation m. Notice that non-vacuous truth of (iii) for even one specified m implies the differentiability condition of the corollary: if ET and $E(T + T_0)$ are both differentiable under the expectation sign, then the same is true of their difference $E\{T + T_0 - T\} = ET_0$. So under the Wolfowitz conditions, the Cramér–Rao inequality (4) is valid (for every m), with $a_m(\theta) = m'(\theta)$ for the m specified in (iii). The conditions of the corollary together with the condition $\operatorname{Cov}(T, V_1) = m'(\theta)$ are necessary and sufficient for the Cramér–Rao inequality (4) with the same bound as in (5). No simple necessary and sufficient conditions are available for the inequality (5), which might conceivably be true for particular cases in which differentiability under the integral sign fails and $m'(\theta) \neq \text{Cov}(T, V_1)$.

3. Applications and examples. One use of Cramér–Rao type inequalities has been in proving that T^*, as an estimator of $g(\theta)$, has uniformly minimum quadratic-loss risk for a given expectation. Such a proof, for a T^* with $ET^* = m(\theta)$, goes like this:

> If (i) $ET = m(\theta)$ implies $\text{Var}\, T \geqq \{a_m(\theta)\}^2/\text{Var}\, V$, and (ii) $\text{Var}\, T^* = \{a_m(\theta)\}^2/\text{Var}\, V$, then T^* is the uniformly minimum variance unbiased estimator of $m(\theta)$.

Now, for T^* to achieve the equality (ii), the inequality (i) must be the one given by $V = T^*$: See Theorem 1, Note 3. So before we can write down the required inequality (i) we must first prove that T^* satisfies the 0 covariance condition of Theorem 1. But that condition already implies the uniformly minimum variance property of T^* that we are trying to prove—by the Lehmann–Scheffé theorem [3, Theorem 5.3]. So nothing would be gained by proceeding to write down the inequality (i).

For example, with X Binomial (n, p), to prove that X/n is the uniformly minimum variance unbiased estimator of p, we would like to write down the Cramér–Rao inequality $ET(X) \equiv p$ implies $\text{Var}\, T(X) \geqq p(1-p)/n$, and observe that $T(X) = X/n$ achieves equality in this. But before we can write down this inequality we must first verify the Wolfowitz conditions (easily done) and these imply the 0 covariance condition of Theorem 1 for the V involved, namely $X - np$ or equivalently X/n. So we now know (from Lehmann–Scheffé) that X/n is the uniformly minimum variance unbiased estimator of p, before we have had a chance to write down the inequality, and the reason for writing it down no longer exists.

The corresponding local at $\theta = \theta_0$ use of Cramér–Rao type inequalities in proving that T^* is the unbiased estimator of ET^* with locally minimum variance at $\theta = \theta_0$, is also superfluous. This can be seen in exactly the same way, using the local at $\theta = \theta_0$ version of the Lehmann–Scheffé theorem (T is the unbiased estimator of ET with locally minimum variance at $\theta = \theta_0$ if and only if T has 0 covariance at $\theta = \theta_0$ with every finite-variance unbiased estimator of 0) and the corresponding local at $\theta = \theta_0$ version of Theorem 1 and its corollary.

The use of Cramér–Rao type inequalities in Hodges–Lehmann type admissibility proofs is discussed in [1] as follows. An estimator T^* with $ET^* = m^*(\theta)$ is inadmissible for estimating $g(\theta)$, for every quadratic loss, if and only if there is an estimator T with $ET = m(\theta)$ that is a non-trivial solution of the inadmissibility inequality

$$\text{Var}\, T + \{m(\theta) - g(\theta)\}^2 \leqq E\{T^* - g(\theta)\}^2 \tag{6}$$

or equivalently

$$E\{T - T^*\}^2 + 2E\{T - T^*\}\{T^* - g(\theta)\}^2 \leqq 0\,. \tag{6'}$$

A non-trivial solution (there is always the trivial solution $T = T^*$) is one for which the inequality is strict for at least one value of θ. The Hodges–Lehmann method [2] of proving T^* admissible consists of showing that there are no non-trivial solutions m (there is always the trivial solution $m = m^*$) of the relaxation of (6) obtained by replacing Var T by its lower bound from the Cramér–Rao type inequality with $V = T^*$.

$$\frac{\{\mathrm{Cov}\,(T, T^*)\}^2}{\mathrm{Var}\,T^*} + \{m(\theta) - g(\theta)\}^2 \leqq E\{T^* - g(\theta)\}^2 \tag{7}$$

or equivalently

$$\frac{\{\mathrm{Cov}\,(T - T^*, T^*)\}^2}{\mathrm{Var}\,T^*} + \{m(\theta) - m^*(\theta)\}^2 + 2E\{T - T^*\}\{T^* - g(\theta)\} \leqq 0\,. \tag{7'}$$

Alternatively, admissibility of T^* can be proved by showing that there are no non-trivial solutions (there is always the trivial solution $m = m^*$) of the further relaxation of (7):

$$\{m(\theta) - m^*(\theta)\}^2 + 2E\{T - T^*\}\{T^* - g(\theta)\} \leqq 0 \tag{8}$$

or equivalently

$$2\,\mathrm{Cov}\,(T, T^*) + \{m(\theta) - g(\theta)\}^2 + \{m^*(\theta) - g(\theta)\}^2 \leqq 2E\{T^* - g(\theta)\}^2 \tag{8'}$$

which amounts to using the convexity inequality instead of the Cramér–Rao type inequality on $E(T - T^*)^2$ in (6′). Instead of proving that (7), or (8), has no non-trivial solution m, it is enough to prove that this inequality has no non-trivial solution m such that $ET = m(\theta)$ for some T; this is what is done in Example 1 of [1].

The essential idea of the Hodges–Lehmann method is to replace the integral inequality (6) in T by an inequality in m that is easier to solve. The inequalities (7) and (8) are always relaxations of (6), but for (7) or (8) to be an inequality in the function m, it is necessary and sufficient that Cov (T, T^*) depend on T only through $m(\theta) = ET$; and this is so if and only if T^* has 0 covariance with every finite-variance unbiased estimator of 0. So the method can be used for proving T^* admissible only if T^* has 0 covariance with every finite-variance unbiased estimator of 0. For using the inequality (7) there is the additional requirement that Var $T^* > 0$ for all θ, except that points θ at which Var $T^* = 0$ and $E\{T^* - g(\theta)\}^2 = 0$ cause no difficulty.

The method fails to prove the admissibility of $T^* \equiv a$ as an estimator of any $g(\theta)$ that possesses an unbiased estimator (with the trivial exception $g(\theta) \equiv a$) even though T^* may be admissible. Here, since Var $T^* \equiv 0$, the further relaxed inequality (8) would have to be used:

$$\{m(\theta) - a\}^2 + 2\{m(\theta) - a\}\{a - g(\theta)\} \leqq 0\,.$$

Setting $m(\theta) = g(\theta)$ this becomes

$$-\{g(\theta) - a\}^2 \leqq 0\,,$$

showing that $m(\theta) = g(\theta)$ is a non-trivial solution of the inequality unless $g(\theta) \equiv a$. Example 4 of [1] is a special case of this.

When there is a minimal sufficient statistic with a complete family of possible distributions, the method is always available (though it may fail) because the only T^*'s that can be admissible are functions of the minimal sufficient statistic, and these all satisfy the 0 covariance condition.

But when the minimal sufficient statistic has a non-complete family of possible distributions, the 0 covariance condition is a very restrictive one: the class of T^*'s satisfying it can be very small as in Example 1 or empty (except for constants) as in Example 2. For all other T^*'s the method is unavailable because (7) and (8) are not inequalities in m only, and so offer no advantage over (6).

EXAMPLE 1. (This is an example where the Hodges–Lehmann method fails, in that the relaxed inequality (7) has non-trivial solutions even though the inadmissibility inequality (6) does not.) The number X of independent tosses made with a coin having $P(\text{Head}) = p$, when we stop after the first toss if it is a head and otherwise stop after the second tail, has possible probability distributions

$$P(X = 1) = p$$
$$P(X = x) = (1 - p)^2 p^{x-2} \qquad \text{for } x = 2, 3, \cdots \text{ with } 0 \leqq p \leqq 1 .$$

Lehmann and Scheffé [3, Examples 3.1 and 5.2] give this as an example of a family that is boundedly complete but not complete. They give the general unbiased estimator T_0 of 0:

$$T_0(x) = -a(x - 2) \qquad \text{for } x = 1, 2, \cdots$$

and the general estimator T_1 that has 0 covariance with every such T_0:

$$T_1(x) = c_1 \qquad \text{for } x = 1, 3, 4, \cdots$$
$$T_1(2) = c_1 + c_2 .$$

Only when T^* is one of these T_1's, which is a very small class indeed, is the Hodges–Lehmann method available; and even then the method using (7) can fail to work, as the following special case shows.

One of the above T_1's is T^*, given by

$$T^*(x) = 0 \qquad \text{for } x = 1, 3, 4, \cdots$$
$$T^*(2) = 1 .$$

Is this T^*, with $ET^* = m^*(p) = (1 - p)^2$ and $\text{Var}\, T^* = (1 - p)^2 p(2 - p)$, a quadratic-loss admissible estimator for its expectation $(1 - p)^2$? From the inadmissibility inequality (6) which here is

$$\text{Var}\, T + \{m(p) - (1 - p)^2\}^2 \leqq (1 - p)^2 p(2 - p)$$

it can be shown that the answer is yes, because this inequality at $p = 0$ and $p = 1$ requires $T(1) = 0$ and $T(2) = 1$; and as $p \to 1$ then requires $0 = T(3) = T(4) = \cdots$.

Can the Hodges–Lehmann method be used to prove this admissibility? For $0 < p < 1$ the relaxed inequality (7) is

$$(1-p)^2\{m(p)-m(0)\}^2 + \{m(p)-(1-p)^2\}^2 p(2-p) - (1-p)^2 p^2 (2-p)^2 \leqq 0\,,$$

since $ETT^* = T(2)(1-p)^2 = m(0)\cdot(1-p)^2$. Writing in the expression for $m(p) = ET$, this inequality is

$$(1-p)^2\{pT(1) - p(2-p)T(2) + (1-p)^2 pT(3) + (1-p)^2 p^2 T(4) + \cdots\}^2 + p(2-p)\{pT(1) + (1-p)^2[T(2)-1] + (1-p)^2 pT(3) + (1-p)^2 p^2 T(4) + \cdots\}^2 - (1-p)^2 p^2 (2-p)^2 \leqq 0\,.$$

As $p \to 0$ this inequality requires $T(1) = 0$, and as $p \to 1$ it requires $T(2) = 1$; but then it does not restrict $T(3)$ to a unique value. And $T(1) = 0$, $T(2) = 1$, $T(3) = 1$, $T(4) = 0 = T(5) = \cdots$ is easily seen to be a non-trivial solution; so the method fails. And of course the further relaxed inequality (8) would all the more have non-trivial solutions.

EXAMPLE 2. For $X_1, \cdots, X_n$ independent, each with Rectangular $(\theta - \frac{1}{2}, \theta + \frac{1}{2})$ distribution, $-\infty < \theta < \infty$, Lehmann and Scheffé show [3, Examples 4.1, 5.3 and 6.4] that the minimal sufficient statistic $U, V = \min X_\iota, \max X_\iota$ has a family of possible distributions that is not boundedly complete, and that constants are the only functions of U, V that have 0 covariance with every finite-variance unbiased estimator of 0. Therefore, although we can write down the inequality (7), or (8), for any T^*, this is not an inequality in the function m only, so the method is not available for proving T^* admissible.

EXAMPLE 3. (This is an example of a complete discrete-parameter family in which the Hodges–Lehmann method using (8), and so also using (7), can be used to prove an estimator admissible.) For

$$P(X = x) = \frac{1}{\theta + 1} \qquad \text{for } x = 0, 1, \cdots, \theta \text{ with } \theta = 0, 1, 2, \cdots$$

the method using (8) can be used to show that X is a quadratic-loss admissible estimator of θ. This is done (the details are omitted here) by showing successively for $\theta = 0$, $\theta = 1, \cdots$ that the inequality (8) requires $T(0) = 0$, $T(1) = 1, \cdots$.

EXAMPLE 4. (This is an example of a complete exponential family with vector-valued parameter $\theta = \theta_1, \cdots, \theta_k$ in which the Hodges–Lehmann method using (8), and so also using (7), can be used to prove quadratic-loss admissibility of a real-valued function of θ.) The family X_1, X_2 independent with Binomial (n_1, p_1), Binomial (n_2, p_2) distributions respectively, $0 \leqq p_1 \leqq 1$ and $0 \leqq p_2 \leqq 1$, is complete. For $T^* = X_1/n_1 + X_2/n_2$ as an estimator of $p_1 + p_2$, the inequality (8) in $m(p_1, p_2) = ET(X_1, X_2)$ is

$$(m - p_1 - p_2)^2 + 2\left\{\frac{p_1(1-p_1)}{n_1}\left[\frac{\partial m}{\partial p_1} - 1\right] + \frac{p_2(1-p_2)}{n_2}\left[\frac{\partial m}{\partial p_2} - 1\right]\right\} \leqq 0\,.$$

In terms of $r(p_1, p_2) = m(p_1, p_2) - p_1 - p_2$, this inequality is

$$\{r(p_1, p_2)\}^2 + 2\left\{\frac{p_1(1-p_1)}{n_1}\frac{\partial r}{\partial p_1} + \frac{p_2(1-p_2)}{n_2}\frac{\partial r}{\partial p_2}\right\} \leqq 0 .$$

Admissibility of T^* is established by showing that the trivial solution $r(p_1, p_2) \equiv 0$ is the only solution of this inequality. For $p_2 = 0$, it reduces to

$$\{r(p_1, 0)\}^2 + 2\,\frac{p_1(1-p_1)}{n_1}\frac{dr(p_1, 0)}{dp_1} \leqq 0 .$$

This gives $r(0, 0) = 0$ and $r(1, 0) = 0$ and $dr(p_1, 0)/dp_1 \leqq 0$ which together imply $r(p_1, 0) \equiv 0$. And in the same way we see that $r(p_1, p_2)$ must be 0 along all edges of the unit square. Now consider the curves in the unit square along which

$$\frac{n_1\, dp_1}{p_1(1-p_1)} = \frac{n_2\, dp_2}{p_2(1-p_2)}$$

which integrates to

$$n_1 \log \frac{p_1}{1-p_1} = n_2 \log \frac{p_2}{1-p_2} + \text{const.},$$

that is,

$$\left(\frac{p_1}{1-p_1}\right)^{n_1} = \text{const.}\left(\frac{p_2}{1-p_2}\right)^{n_2} .$$

These curves all pass through the points (0, 0) and (1, 1), and there is one of them passing through an arbitrary interior point of the unit square. In terms of distance s along this curve,

$$\begin{aligned}\frac{dr}{ds} &= \frac{\partial r}{\partial p_1}\cdot\frac{dp_1}{ds} + \frac{\partial r}{\partial p_2}\cdot\frac{dp_2}{ds} \\ &= \frac{(\partial r/\partial p_1)p_1(1-p_1)/n_1 + (\partial r/\partial p_2)p_2(1-p_2)/n_2}{\{[p_1(1-p_1)/n_1]^2 + [p_2(1-p_2)/n_2]^2\}^{\frac{1}{2}}} .\end{aligned}$$

And along this curve the inequality is

$$\{r(s)\}^2 + 2\left\{\left[\frac{p_1(1-p_1)}{n_1}\right]^2 + \left[\frac{p_2(1-p_2)}{n_2}\right]^2\right\}^{\frac{1}{2}} \cdot \frac{dr(s)}{ds} \leqq 0 .$$

This shows that $dr/ds \leqq 0$. And $r(s) = 0$ at the two s values corresponding to $(p_1, p_2) = (0, 0)$ and $(1, 1)$, so it follows that $r(s) \equiv 0$ along the curve, which includes an arbitrary interior point of the unit square. We therefore have $r(p_1, p_2) \equiv 0$, proving the admissibility of T^*.

(For X_i Binomial (n_i, p_i), $i = 1, \cdots, k$, all independent the same method shows that $\sum a_i X_i/n_i$ is an admissible estimator of $\sum a_i p_i$. And for Y_j Normal $(\theta_j, 1)$, $j = 1, \cdots, m$ all independent and independent of the X_i's the same method shows that $\sum a_i X_i/n_i + \sum b_j Y_j$ is an admissible estimator of $\sum a_i p_i + \sum b_j \theta_j$.)

The associate editor points out that Stein's admissibility proof in [4] provides an example in which the Hodges–Lehmann method is used to prove admissibility of an estimator of a vector-valued function of a vector-valued parameter, with the sum of the squares of errors in the components as loss. The associate editor has also shown, using an adaptation of the argument in [4], that for X Normal $(\theta, 1)$ the method using (8) can be used to prove, for $0 \leqq k \leqq 1$, the quadratic-loss admissibility of $T^* = X$ as an estimator of $g(\theta) = \theta\{1 + k/(1 + \theta^2)\}$; and he comments that the same reasoning can be applied to certain functions other than this g.

I still do not know of an example in which the inequality (8) has non-trivial solutions and the inequality (7) is available and does not have non-trivial solutions.

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