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ALMOST SURE BEHAVIOUR OF *U*-STATISTICS AND VON MISES' DIFFERENTIABLE STATISTICAL FUNCTIONS[1]

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For *U*-Statistics and von Mises' differentiable statistical functions, when the regular functional is stationary of order zero, almost sure convergence to appropriate Wiener processes is studied. A second almost sure invariance principle, particularly useful in the context of the law of iterated logarithm and the probability of moderate deviations, is also established.

1. Introduction. Let $\{X_i, i \geqq 1\}$ be a sequence of independent and identically distributed random vectors (i.i.d. rv) defined on a probability space $(\Omega, \mathscr{A}, P)$, with each X_i having a distribution function (df) $F(x)$, $X \in R^p$, the $p(\geqq 1)$-dimensional Euclidean space. Let $g(X_1, \cdots, X_m)$, symmetric in its $m(\geqq 1)$ arguments, be a Borel measurable *kernel* of *degree m*, and consider the *regular functional*

$$\theta(F) = \int_{R^{pm}} \cdots \int g(x_1, \cdots, x_m)\, dF(x_1) \cdots dF(x_m)\,; \qquad F \in \mathscr{F}, \tag{1.1}$$

where $\mathscr{F} = \{F: |\theta(F)| < \infty\}$. The minimum variance unbiased estimator of $\theta(F)$ based on a sample $X_1, \cdots, X_n$ of size n is (*the U-statistic*)

$$U_n = \binom{n}{m}^{-1} \sum_{C_{n,m}} g(X_{i_1}, \cdots, X_{i_m})\,; \qquad C_{n,m} = \{1 \leqq i_1 < \cdots < i_m \leqq n\}. \tag{1.2}$$

If we let $c(u)$ be equal to 1 if all the p components of u are nonnegative and otherwise let $c(u) = 0$, then on defining the empirical df

$$F_n(x) = n^{-1} \sum_{i=1}^{n} c(x - X_i)\,, \qquad x \in R^p\,,\ n \geqq 1\,, \tag{1.3}$$

the corresponding functional

$$\begin{aligned}\theta(F_n) &= \int_{R^{pm}} \int g(x_1, \cdots, x_m)\, dF_n(x_1) \cdots dF_n(x_m) \\ &= n^{-m} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} g(X_{i_1}, \cdots, X_{i_m})\end{aligned} \tag{1.4}$$

is termed a von Mises' (1947) *differentiable statistical function*.

Asymptotic normality of $n^{\frac{1}{2}}[U_n - \theta(F)]$ and $n^{\frac{1}{2}}[\theta(F_n) - \theta(F)]$ are studied in von Mises (1947) and Hoeffding (1948). Under the same set of regularity conditions, Loynes (1970) has shown that a process obtained by linear interpolation from $\{n^{\frac{1}{2}}[U_k - \theta(F)];\ k \geqq n\}$ weakly converges to a Wiener process, as $n \to \infty$. Also, Miller and Sen (1972) have shown that under the same set of conditions, processes obtained by linear interpolation from $\{n^{-\frac{1}{2}}k[U_k - \theta(F)],\ m \leqq k \leqq n\}$

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and $\{n^{-\frac{1}{2}}k[\theta(F_k) - \theta(F)],\ 1 \leqq k \leqq n\}$ converge in distribution in the uniform topology on the $C[0, 1]$ space to a common Wiener process, as $n \to \infty$. In the present paper, among other results, these weak convergence results are strengthened to almost sure (a.s.) convergence results.

For sums of independent random variables and martingales, Strassen (1967) studied three a.s. invariance principles. His first invaraince principle, namely, the a.s. convergence of sample partial cumulative sums to Wiener processes is extended here (see Theorem 2.1) to a broad class of $\{U_n\}$ and $\{\theta(F_n)\}$. In Theorem 2.2, we consider a result analogous to his third a.s. invariance principle, and this is particularly useful in the context of the law of iterated logarithm and the probability of moderate deviations for U_n and $\theta(F_n)$, for which we may refer to Rubin and Sethuraman (1965), Ghosh and Sen (1970) and Serfling (1971), among others.

The basic results along with the regularity conditions are stated in Section 2. The proofs of the theorems are presented in Section 3. Some applications are sketched in the last section.

2. Statement of the main results. Define for every $h (0 \leqq h \leqq m)$,

$$g_h(x_1, \cdots, x_h) = Eg(x_1, \cdots, x_h, X_{h+1}, \cdots, X_m)\,, \qquad g_0 = \theta(F)\,; \tag{2.1}$$

$$\zeta_h(F) = Eg_h^2(X_1, \cdots, X_h) - \theta^2(F)\,, \qquad \zeta_0(F) = 0\,; \tag{2.2}$$

$$\zeta^*(F) = \max_{1 \leqq i_1 \leqq \cdots \leqq i_m \leqq m} Eg^2(X_{i_1}, \cdots, X_{i_m})\,. \tag{2.3}$$

We term that $\theta(F)$ is *stationary of order zero* (cf. Hoeffding (1948)), if

$$0 < \zeta_1(F) < \infty\,. \tag{2.4}$$

Let $\mathbf{S} = \{S(t) \colon 0 \leqq t < \infty\}$ be a random process, where

$$\begin{aligned} S(k) = S_k &= 0\,, & 0 \leqq k \leqq m-1\,, \\ &= k[U_k - \theta(F)]\,, & k \geqq m\,, \end{aligned} \tag{2.5}$$

and $S(t) = S_k$ for $k \leqq t < k+1$, $k \geqq 0$. Similarly, let $\mathbf{S}^* = \{S^*(t) \colon 0 \leqq t < \infty\}$ be a random process, where

$$\begin{aligned} S^*(k) = S_k^* &= k[\theta(F_k) - \theta(F)]\,, & k \geqq 1\,, \\ &= 0\,, & k = 0\,, \end{aligned} \tag{2.6}$$

and $S^*(t) = S_k^*$ for $k \leqq t < k+1$, $k \geqq 0$. Alternatively, $S(t)$ (and $S^*(t)$) can also be defined by linear interpolation between (S_k, S_{k+1}) (and (S_k^*, S_{k+1}^*)) when $t \in [k, k+1]$, $k \geqq 0$. Consider now a positive and real-valued function $f(t)$, $t \in [0, \infty)$, such that

$$f(t) \text{ is } \uparrow \text{ but } t^{-1}f(t) \text{ is } \downarrow \text{ in } t \colon 0 \leqq t < \infty\,; \tag{2.7}$$

$$\sum_{n \geqq 1} [f(cn)]^{-1} E\{[g_1^*(X_n)]^2 I([g_1^*(X_n)]^2 > f(cn))\} < \infty\,, \tag{2.8}$$

for every $c > 0$, where $g_1^*(x) = g_1(x) - \theta(F)$ and $I(A)$ denotes the indicator function of a set A. (2.8) is framed by analogy with the basic condition of

Theorem 4.4 of Strassen (1967) (see his (138)). Since $g_1^*(x) = E([g(X_1, \cdots, X_m) - \theta(F)] | X_1 = x)$, and the X_i are i.i.d., (2.8) can also be restated in terms of the kernel $g(X_1, \cdots, X_m)$. In applications, this does not make much difference, though the current form of (2.8) is a little less restrictive. For example, when the X_i are univariate rv's with mean μ and variance σ^2, the U-statistic corresponding to σ^2 has the kernel $g(x, y) = \frac{1}{2}(x-y)^2$, so that $g_1^*(x) = [(x-\mu)^2 - \sigma^2]/2$, and hence (2.8) is equally easily verifiable. A similar case follows with the Wilcoxon signed rank statistic or the Kendall tau which are expressible as U-statistics or von Mises functionals. Also, as in Theorems 4.6 and 4.8 of Strassen (1967), we may strengthen (2.8) to $E|g(X_1, \cdots, X_m)|^r < \infty$, for some $r > 2$, and this improves the approximation in our Theorem 2.1. Such a condition is of course easier to verify. Finally, let $\zeta = \{\zeta(t) \colon 0 \leqq t < \infty\}$ be a standard Wiener (Brownian motion) process, and we let

$$\gamma^2 = m^2\zeta_1(F) \quad (> 0 \text{ by } (2.4)) . \tag{2.9}$$

Then, the following theorem extends Strassen's (1967) first a.s. invariance principle to $\{U_n\}$ and $\{\theta(F_n)\}$.

THEOREM 2.1. *If $\theta(F)$ is stationary of order zero and $\zeta_m(F) < \infty$, then under* (2.7) *and* (2.8), *as $t \to \infty$,*

$$S(t) = \gamma\xi(t) + o((tf(t))^{\frac{1}{2}} \log t) \quad \text{a.s.} \tag{2.10}$$

Also, under (2.4), (2.7) *and* (2.8), *if $\zeta^*(F) < \infty$, then as $t \to \infty$,*

$$S^*(t) = \gamma\xi(t) + o((tf(t))^{\frac{1}{2}} \log t) \quad \text{a.s.}; \tag{2.11}$$

$$S(t) - S^*(t) = o((tf(t))^{\frac{1}{2}} \log t) \quad \text{a.s.} \tag{2.12}$$

Let now $\boldsymbol{\phi} = \{\phi(t) \colon 0 \leqq t < \infty\}$ be a positive function with a continuous derivative $\{\phi'(t)\}$, such that as $t \to \infty$,

$$s/t \to 1 \Rightarrow \phi'(s)/\phi'(t) \to 1 , \tag{2.13}$$

$$t^{-\frac{1}{2}}\phi(t) \text{ is } \uparrow \text{ but } t^{-h}\phi(t) \text{ is } \downarrow \text{ in } t \text{ for some } \tfrac{1}{2} < h < \tfrac{3}{5} ; \tag{2.14}$$

$$\int_1^\infty t^{-\frac{3}{2}}\phi(t) \exp\{-\tfrac{1}{2}t^{-1}\phi^2(t)\}\, dt < \infty . \tag{2.15}$$

Then, as an extension of Theorems 1.4 and 4.9 of Strassen (1967), we have the following theorem where we define $\{S_m\}$ and $\{S_n^*\}$ as in (2.5) and (2.6).

THEOREM 2.2. *If $g(X_1, \cdots, X_m)$ has a finite moment generating function in a neighbourhood of* 0, *then under* (2.4), (2.13), (2.14) *and* (2.15), *as $n \to \infty$,*

$$P\{\sup_{k \geqq n} S_k/\phi(k) \geqq \gamma\} \sim \frac{1}{(2\pi)^{\frac{1}{2}}} \int_n^\infty t^{-\frac{1}{2}}\phi'(t) \exp\{-\tfrac{1}{2}t^{-1}\phi^2(t)\}\, dt , \tag{2.16}$$

while, if in addition, $g(X_{i_1}, \cdots, X_{i_m})$ has a finite moment generating function in a neighbourhood of 0 *for every $1 \leqq i_1 \leqq \cdots \leqq i_m \leqq m$, then as $n \to \infty$,*

$$P\{\sup_{k \geqq n} S_k^*/\phi(k) \geqq \gamma\} \sim \frac{1}{(2\pi)^{\frac{1}{2}}} \int_n^\infty t^{-\frac{1}{2}}\phi'(t) \exp\{-\tfrac{1}{2}t^{-1}\phi^2(t)\}\, dt . \tag{2.17}$$

The proofs of the theorems rest on the reverse martingale property of $\{U_n\}$ (see Berk (1966)), certain related results studied in Miller and Sen (1972) and the basic theorems of Strassen (1967). We shall see in Section 4 that Theorem 2.2 strengthens some earlier results of Rubin and Sethuraman (1965) on U-statistics.

3. Derivation of the main results. Proceeding as in Miller and Sen (1972), we have for every $n \geqq 1$,

$$\theta(F_n) = \theta(F) + \textstyle\sum_{h=1}^{m} \binom{m}{h} V_n^{(h)}\,; \tag{3.1}$$

$$V_n^{(h)} = \int_{R^{ph}} \cdots \int g_h(x_1, \cdots, x_h) \textstyle\prod_{j=1}^{h} d[F_n(x_j) - F(x_j)]\,, \qquad 1 \leqq h \leqq m\,, \tag{3.2}$$

and for every $n \geqq m$,

$$U_n = \theta(F) + \textstyle\sum_{h=1}^{m} \binom{m}{h} U_n^{(h)}\,, \qquad U_n^{(1)} = V_n^{(1)}\,; \tag{3.3}$$

$$U_n^{(h)} = n^{-[h]} \textstyle\sum_{P_{n,h}} \int_{R^{ph}} \cdots \int g_h(x_1, \cdots x_h) \prod_{j=1}^{h} d[c(x_j - X_{i_j}) - F(x_j)]\,, \tag{3.4}$$

where $n^{-[h]} = \{n \cdots (n-h+1)\}^{-1}$ and $P_{n,h} = \{1 \leqq i_1 \neq \cdots \neq i_h \leqq n\}$, $h = 1, \cdots, m$.

We start with the proof of (2.10) in Theorem 2.1. Consider a random process $\mathbf{S}^{(1)} = \{S^{(1)}(t)\colon 0 \leqq t < \infty\}$, where for $k \leqq t < k+1$, $S^{(1)}(t) = S^{(1)}(k) = S_k^{(1)}$, $k \geqq 0$, and

$$S_k^{(1)} = kU_k^{(1)} \quad \text{if} \quad k \geqq m\,, \quad \text{and} \quad 0\,, \quad \text{if} \quad k \leqq m-1\,. \tag{3.5}$$

Further, let

$$U_n^* = \textstyle\sum_{h=2}^{m} \binom{m}{h} U_n^{(h)}\,, \qquad n \geqq m\,. \tag{3.6}$$

Then, to prove (2.10), it suffices, by virtue of (2.5), (3.3), (3.5) and (3.6), to prove that as $t \to \infty$,

$$S^{(1)}(t) = \gamma\xi(t) + o((tf(t))^{\frac{1}{2}} \log t) \quad \text{a.s.}, \tag{3.7}$$

and as $n \to \infty$,

$$\sup_{k \geqq n} \{U_k^*[(kf(k))^{\frac{1}{2}} \log k]^{-1}\} \to_P 0\,. \tag{3.8}$$

Now, since $S_k^{(1)} = kU_k^{(1)} = \sum_{i=1}^{k} [g_1(X_i) - \theta(F)]$, $k \geqq 1$, involve the sequence of i.i.d. rv $\{g_i^*(X_i) = g_1(X_i) - \theta(F);\, i \geqq 1\}$ where $Eg_1^*(X_i) = 0$ and $E[g_1^*(X_i)]^2 = \zeta_1(F)$ (> 0), and (2.7)—(2.8) hold, the proof of (3.7) follows directly from Theorem 4.4 of Strassen (1967). So, we need to prove only (3.8). For this, on writing

$$c_k = k[(kf(k)^{\frac{1}{2}} \log k]^{-1} = [(k^{\frac{1}{2}}/\log k)(k/f(k))^{\frac{1}{2}}]\,, \qquad k \geqq m(\geqq 2)\,, \tag{3.9}$$

we note that $\{c_k\}$ is a sequence of positive numbers such that (by (2.7))

$$c_k \quad \text{is} \uparrow \text{in} \quad k\,, \qquad \text{for} \quad k \geqq 8\,. \tag{3.10}$$

Also, by (3.6), for $k \geqq m$, U_k^* is a U-statistic, and hence, $\{U_k^*, k \geqq m\}$ is a reverse martingale with respect to a non-increasing sequence of σ-fields (cf. Berk

(1966)). Further, as in Hoeffding (1948), it can be shown on using (3.6) that for every $n \geqq m$,

$$
\begin{aligned}
E[U_n^* - U_{n+1}^*]^2 &= E[U_n^*]^2 - E[U_{n+1}^*]^2 \\
&= E[U_n - \theta(F)]^2 - m^2 E[U_n^{(1)}]^2 \\
&\qquad - E[U_{n+1} - \theta(F)]^2 + m^2 E[U_{n+1}^{(1)}]^2 \\
&= \binom{n}{m}^{-1} \sum_{h=1}^{m} \binom{m}{h}\binom{n-m}{m-h} \zeta_h(F) - m^2 n^{-1} \zeta_1(F) \\
&\qquad - \binom{n+1}{m}^{-1} \sum_{h=1}^{m} \binom{m}{h}\binom{n-m+1}{m-h} \zeta_h(F) + m^2 (n+1)^{-1} \zeta_1(F) \\
&\leqq C(F) n^{-3}, \qquad \text{where } C(F) < \infty \text{ when } \zeta_m(F) < \infty .
\end{aligned} \tag{3.11}
$$

Finally, as in (3.11), $E[U_n^*]^2 \leqq C(F) n^{-2}$ for every $n \geqq m$, so that by (2.7) and (3.9),

$$\lim_{n\to\infty} \{c_n^2 E[U_n^*]^2\} = 0 . \tag{3.12}$$

Consequently, by (3.9), (3.11), (3.12) and Theorem 1 of Chow (1960) (i.e., the Hájek–Rényi inequality for sub-martingales), we obtain on noting that $c_n^2 = o(n^{\frac{3}{2}})$ (by (2.7) and (3.10)) that for every $\varepsilon > 0$, (and $n \geqq 8$),

$$
\begin{aligned}
P\{\max_{k\geqq n} c_k |U_k^*| > \varepsilon\} &\leqq \varepsilon^{-2} \{\textstyle\sum_{k=n}^{\infty} c_k^2 E[U_k^* - U_{k+1}^*]^2\} \\
&\geqq C(F) \varepsilon^{-2} \textstyle\sum_{k=n}^{\infty} c_k^2 k^{-3} \\
&= C(F) \varepsilon^{-2} [o(n^{-\frac{1}{2}})] \to 0 \qquad \text{as } n \to \infty .
\end{aligned} \tag{3.13}
$$

Thus, (3.8) holds and the proof of (2.10) is complete. We next consider the proof of (2.12). Proceeding as in the proof of Lemma 2.6 of Miller and Sen (1972), it can be shown that for every $n \geqq m$,

$$E\{n[\theta(F_n) - U_n]\}^2 \leqq C^*(F) n^{-1} , \tag{3.14}$$

where $C^*(F) < \infty$ whenever $\zeta^*(F) < \infty$. Therefore, for every $\varepsilon > 0$,

$$
\begin{aligned}
&P\{\sup_{t\geqq n} [|S(t) - S^*(t)/[(tf(t))^{\frac{1}{2}} \log t]] > \varepsilon\} \\
&\quad \leqq P\{\max_{k\geqq n} [k|\theta(F_k) - U_k|/[(kf(k))^{\frac{1}{2}} \log k]] > \varepsilon\} \\
&\quad \leqq \textstyle\sum_{k=n}^{\infty} P\{k|\theta(F_k) - U_k| > \varepsilon[(kf(k))^{\frac{1}{2}} \log k]\} \\
&\quad \leqq C^*(F) \varepsilon^{-2} \textstyle\sum_{k=n}^{\infty} k^{-1} [(kf(k))^{\frac{1}{2}} \log k]^{-2} \\
&\quad \leqq C^*(F) \varepsilon^{-2} [f(n)]^{-\frac{1}{2}} (\log n)^{-2} \textstyle\sum_{k=n}^{\infty} k^{-\frac{3}{2}} \\
&\quad = C^*(F) \varepsilon^{-2} [f(n)]^{-\frac{1}{2}} (\log n)^{-2} [O(n^{-\frac{1}{2}})] \to 0 \qquad \text{as } n \to \infty .
\end{aligned} \tag{3.15}
$$

Thus, (2.12) follows from (3.15), and (2.11) follows directly from (2.10) and (2.12), and hence, the proof of Theorem 2.1 is complete.

For the proof of Theorem 2.2, we first consider the following.

LEMMA 3.1. *For even* $k_n (2 \leqq k_n < o(n^{\frac{1}{2}}))$, *as* $n \to \infty$, *for every* $1 \leqq h \leqq m$,

$$E[n^{h/2} U_n^{(h)}]^{k_n} \leqq [C_h(F)]^{k_n} \left\{ \frac{(hk_n)!}{2^{\frac{1}{2}hk_n} (\frac{1}{2}hk_n)!} \right\} \{1 + o(1)\} , \tag{3.16}$$

where $E[|g(X_1, \cdots, X_m)|^{k_n}] < \infty \Rightarrow C_h(F) < \infty$ *for* $h = 1, \cdots, m$.

PROOF. We sketch the proof only for the case of $h = 2$; the same proof holds for every $1 \leqq h \leqq m$. By (3.4) and the Fubini theorem,

$$\begin{aligned} E[nU_n^{(2)}]^{k_n} &= E\{(n-1)^{-1} \sum_{\Gamma_{n,2}} \int_{R^{2p}} \cdots \int g_2(x_1, x_2) \\ &\qquad \times \prod_{j=1}^{2} d[c(x_j - X_{i_j}) - F(x_j)]\}^{k_n} \\ &= (n-1)^{-k_n} \sum_n^* \int_{R^{2pk_n}} \cdots \int \{\prod_{l=1}^{k_n} g_2(x_{l1}, x_{l2})\} \\ &\qquad \times E\{\prod_{l=1}^{k_n} \prod_{j=1}^{2} d[c(x_{lj} - X_{i_{lj}}) - F(x_{lj})]\}, \end{aligned} \tag{3.17}$$

where the summation $\sum_n^*$ extends over all $1 \leqq i_{l1} \neq i_{l2} \leqq n$, $l = 1, \cdots, k_n$.

For a given set $\{i_{lj}, j = 1, 2, l = 1, \cdots, k_n\}$ of $2k_n$ integers, suppose that there are s_n distinct integers $j_1, \cdots, j_{s_n}$, $s_n \geqq 1$, where j_k occurs $r_k(\geqq 1)$ times, so that $r_1 + \cdots + r_{s_n} = 2k_n$. Then,

$$\begin{aligned} |E\{\prod_{l=1}^{k_n} \prod_{j=1}^{2} d[c(x_{ij} - X_{i_{lj}}) - F(x_{lj})]\}| & \\ = 0\,, \quad & \text{if at least one of } r_1, \cdots, r_{k_n} = 1\,, \\ \leqq \prod_{l=1}^{k_n} \prod_{j=1}^{2} dF(x_{lj})\,, \quad & \text{otherwise,} \end{aligned} \tag{3.18}$$

for every set of $\{i_{lj}, j = 1, 2, l = 1, \cdots, k_n)$. Thus, the leading terms in (3.17) arise from sets for which $s_n = k_n$, $r_1 = \cdots = r_{k_n} = 2$; there being $[(2k_n)!/2^{k_n} \cdot k_n!]$ such sets, their total contribution in (3.17) is bounded by

$$\begin{aligned} \frac{(2k_n)!}{2^{k_n} k_n!} [\int_{R^{2p}} \cdots \int |g_2(x_1, x_2)|\, dF(x_1)\, dF(x_2)]^{k_n} & \\ = [C_2(F)]^{k_n} \cdot [(2k_n)!/2^{k_n} \cdot k_n!]\,, & \end{aligned} \tag{3.19}$$

where $C_2(F) < [\zeta_m(F) + \theta^2(F)]^{\frac{1}{2}} < \infty$ whenever $\zeta_m(F) < \infty$. The other sets with non-zero contributions in (3.17) correspond to values of $s_n \leqq k_n - 1$ with $r_k \geqq 2$ for $k = 1, \cdots, s_n$. If $s_n = k_n - u$, $u \geqq 1$, and $E|g|^{1+u} < \infty$ (as assumed), then the contribution of the sets to (3.17) is bounded by (3.19) times a coefficient which is

$$O[(k_n^2/n)^u]\,, \qquad \text{for} \quad u = 1, \cdots, k_n - 1\,. \tag{3.20}$$

Thus, the total contribution of these sets (with $s_n \leqq k_n - 1$) is bounded above by (3.19) times a coefficient

$$\begin{aligned} & \sum_{u=1}^{k_n-1} \{O[(n^{-1}k_n^2)^u]\} \\ & \leqq O(n^{-1}k_n^2) + (k_n - 2)\{O(n^{-1}k_n^2)^2\} = O(k_n^2/n) = o(1)\,. \qquad \square \end{aligned} \tag{3.21}$$

In the context of convergence rates for U-statistics and related statistics, results similar to the one in Lemma 3.1 have been considered by Abdalimov and Malevic (1970) and by Grams and Serfling (1973).

A direct consequence of (3.6) and Lemma 3.1 is the following.

LEMMA 3.2. *If* $E|g|^{k_n} < \infty$, *then for every even* $k_n[2 \leqq k_n \leqq o(n^{\frac{1}{3}})]$, *as* $n \to \infty$,

$$E[nU_n^*]^{k_n} \leqq \binom{m}{2}^{k_n} [C_2(F)]^{k_n} \{(2k_n)!/2^{k_n} k_n!\}\{1 + o(1)\}\,. \tag{3.22}$$

(Note that in deriving (3.22), we make use of the fact that for every $h > 2$, $[n^{-(h-2)/2}]^{k_n}\{[(hk_n)!/(2k_n)!][k_n!/(\frac{1}{2}hk_n)!]2^{-(h/2-1)k_n}\} \to 0$ as $n \to \infty$.)

Also, by using (3.1)—(3.3), and proceeding as in Lemma 3.1, we have the following.

LEMMA 3.3. *If* $E|g(X_{i_1}, \cdots, X_{i_m})|^{k_n} < \infty$ *for every* $1 \leqq i_1 \leqq \cdots \leqq i_m \leqq m$, *then for every even* $k_n[2 \leqq k_n \leqq o(n^{\frac{1}{3}})]$, *as* $n \to \infty$

$$E[S_n^* - mS_n^{(1)}]^{k_n} \leqq \binom{m}{2}^{k_n}[C_2(F)]^{k_n}\{(2k_n)!/2^{k_n}k_n!\}\{1 + o(1)\}, \tag{3.23}$$

where S_n^* *and* $S_n^{(1)}$ *are defined by* (2.6) *and* (3.5).

LEMMA 3.4. *If* $E(\exp\{ug(X_1, \cdots, X_m)\}) < \infty$ *for* $|u| < \varepsilon(> 0)$ *and* $\frac{1}{4} < b < \frac{1}{2}$, $0 < 3a < 4b - 1$ $(\Rightarrow 0 < a < \frac{1}{3})$, *then as* $n \to \infty$,

$$P\{k|U_k^*| > \tfrac{1}{2}k^b \text{ for some } k > n\} = o(e^{-n^a}), \tag{3.24}$$

and, if $E(\exp\{ug(X_{i_1}, \cdots, X_{i_m})\}) < \infty$ *for every* $1 \leqq i_1 \leqq \cdots \leqq i_m \leqq m$, $|u| < \varepsilon$, *then, as* $n \to \infty$,

$$P\{|S_k^* - mS_k^{(1)}| > \tfrac{1}{2}k^b \text{ for some } k > n\} = o(e^{-n^a}). \tag{3.25}$$

PROOF. By (3.22) and the Markov inequality, for large n,

$$\begin{aligned} P\{n|U_n^*| > \tfrac{1}{2}n^b\} &\leqq (\tfrac{1}{2}n^b)^{-k_n}E|nU_n^*|^{k_n} \\ &\leqq (\tfrac{1}{2}n^b)^{-k_n}\binom{m}{2}^{k_n}[C_2(F)]^{k_n}\{(2k_n)!/2^{k_n}k_n!\}\{1 + o(1)\}, \end{aligned} \tag{3.26}$$

so that on choosing k_n as the largest even integer contained in n^a, we obtain that the right-hand side of (3.26) is asymptotically (as $n \to \infty$) equal to

$$\begin{aligned} &[2\tbinom{m}{2}C_2(F)/n^b]^{k_n}\{(2k_n)^{2k_n+\frac{1}{2}}e^{-2k_n}/2^{k_n}k_n^{k_n+\frac{1}{2}}e^{-k_n}\}\{1 + o(1)\} \\ &\quad \sim [m(m-1)C_2(F)/n^b]^{n^a}2^{n^a}e^{-n^a}(n^a)^{n^a}\{1 + o(1)\} \\ &\quad = [2m(m-1)C_2(F)/n^{b-a}]^{n^a}e^{-n^a}\{1 + o(1)\}. \end{aligned} \tag{3.27}$$

Now, $3a < 4b - 1 \Rightarrow 4(b-a) > 1 - a > \frac{2}{3} \Rightarrow b - a > \frac{1}{6}$. Therefore the first factor on the right hand-side of (3.27) is bounded above by $[2m(m-1)C_2(F)n^{-\frac{1}{6}}]^{n^a}$. Consequently, by (3.26), (3.27) and the Bonferroni inequality, as $n \to \infty$,

$$\begin{aligned} &P\{k|U_k^*| > \tfrac{1}{2}k^b \text{ for some } k \geqq n\} \\ &\quad \leqq \textstyle\sum_{k=n}^{\infty} P\{k|U_k^*| > \tfrac{1}{2}k^b\} \\ &\quad \leqq \textstyle\sum_{k=n}^{\infty} [2m(m-1)C_2(F)k^{-\frac{1}{6}}]^{k^a}e^{-k^a}[1 + o(1)] \\ &\quad \leqq e^{-n^a}[1 + o(1)] \textstyle\sum_{k=n}^{\infty} [2m(m-1)C_2(F)k^{-\frac{1}{6}}]^{k^a} \\ &\quad = o(e^{-n^a}), \end{aligned} \tag{3.28}$$

as $\sum_{k=n}^{\infty} [2m(m-1)C_2(F)k^{-\frac{1}{6}}]^{k^a} \to 0$ as $n \to \infty$. The proof of (3.25) follows similarly by using Lemma 3.3. □

THEOREM 3.5. *If* $\theta(F)$ *is stationary of order* 0 *and* $E(\exp\{ug(X_1, \cdots, X_m)\}) < \infty$ *for* $|u| < \varepsilon(> 0)$, *then there is a standard Brownian motion* $\boldsymbol{\xi} = \{\xi(t) \colon 0 \leqq t < \infty\}$

such that if $\frac{1}{4} < b < \frac{1}{2}$ *and* $0 < 3a < 4b - 1$, *then as* $s \to \infty$,

$$P\{|S(t) - \gamma\xi(t)| > t^b \text{ for some } t > s\} = o(e^{-s^a}), \tag{3.29}$$

and, if further, $E(\exp\{ug(X_{i_1}, \cdots, X_{i_m})\}) < \infty$ *for* $|u| < \varepsilon(> 0)$, *uniformly in* $1 \leqq i_1 \leqq \cdots \leqq i_m \leqq m$, *then as* $s \to \infty$,

$$P\{|S^*(t) - \gamma\xi(t)| > t^b \text{ for some } t > s\} = o(e^{-s^a}), \tag{3.30}$$

where $S(t)$ *and* $S^*(t)$ *are defined as in Section* 2.

PROOF. We only prove (3.29) as (3.30) follows on parallel lines. By virtue of (3.3) and (3.5), $S_k = mS_k^{(1)} + kU_k^*$, so that the event $[|S(t) - \gamma\zeta(t)| > t^b$ for some $t > s]$ is contained in the union of the two events $[|mS^{(1)}(t) - \gamma\zeta(t)| > \frac{1}{2}t^b$ for some $t > s]$ and $[|kU_k^*| > \frac{1}{2}k^b$ for some $k > s]$. Thus, the left-hand side of (3.29) is bounded above by

$$P\{|mS^{(1)}(t) - \gamma\zeta(t)| > \tfrac{1}{2}t^b \text{ for some } t > s\} + P\{|kU_k^*| > \tfrac{1}{2}k^b \text{ for some } k > s\}. \tag{3.31}$$

Since $S^{(1)}(t)$ involves the i.i.d. rv $\{g_1(X_i) - \theta(F), i \geqq 1\}$, by Theorem 4.8 of Strassen (1967), it can be shown that the first term in (3.31) is $o(e^{-s^a})$ as $s \to \infty$, while by Lemma 3.4, it follows that the second term is also $o(e^{-s^a})$ as $s \to \infty$. Hence the theorem follows.

Returning now to the proof of Theorem 2.2, we observe that the proof follows along the same line as in Corollary 4.9 of Strassen (1967) where in his (204) and (206), we need to use our Theorem 3.5, instead of his Theorem 4.8. For brevity, the details are therefore omitted.

4. Some applications. For $\{U_n\}$ and $\{\theta(F_n)\}$, the law of iterated logarithm has been studied by Sproule (1969), Ghosh and Sen (1970) and Serfling (1971). The same result follows from Theorem 2.2 by letting

$$\phi(n) = [2n(1 + \varepsilon)\log\log n]^{\frac{1}{2}}, \qquad \varepsilon > 0, \tag{4.1}$$

and noting that the right-hand side of (2.16) or (2.17) is then asymptotically equal to $[(4\pi)^{\frac{1}{2}}\varepsilon(\log n)^{\varepsilon}]^{-1}$, and hence, converges to o as $n \to \infty$ (for every $\varepsilon > 0$).

Rubin and Sethuraman (1965) have shown that as $n \to \infty$,

$$(\log n)^{-1}\log P\{n^{\frac{1}{2}}|U_n - \theta(F)| > \gamma c(\log n)^{\frac{1}{2}}\} \to -\tfrac{1}{2}c^2, \qquad c > 0. \tag{4.2}$$

On substituting $\phi(n) = c[n\log n]^{\frac{1}{2}}$, $c > 0$, we obtain from Theorem 2.2 that as $n \to \infty$,

$$\begin{aligned} &P\{k^{\frac{1}{2}}|U_k - \theta(F)| > \gamma c(\log k)^{\frac{1}{2}} \text{ for some } k \geqq n\} \\ &\quad \sim (c/2(2\pi)^{\frac{1}{2}}) \textstyle\int_{\log n}^{\infty} u^{-\frac{1}{2}}e^{-\frac{1}{2}c^2u}\,du \\ &\quad = [c(2\pi)^{\frac{1}{2}}]^{-1}\{n^{-\frac{1}{2}c^2}(\log n)^{-\frac{1}{2}}[1 + O((\log n)^{-1})]\}. \end{aligned} \tag{4.3}$$

Thus, not only (4.3) specifies a better order in asymptotic expression, but also strengthens (4.2) to the entire tail of $\{U_k, k \geqq n\}$. The same result holds for $\{\theta(F_k); k \geqq n\}$.

Theorem 2.1 is of great help in the developing area of sequential procedures based on U-statistics and $\{\theta(F_n)\}$, where the derived Wiener process approximation simplifies the ASN and the OC functions in a certain asymptotic sense. These will be considered in a separate paper.

REFERENCES

[1] ABDALIMOV, B. and MALEVIC, T. L. (1970). Sharpening the limit theorem for *U*-statistics. *Izv. Acad. Nauk Uzbek. SSR* **14** 6–12.

[2] BERK, R. H. (1966). Limiting behavior or posterior distributions when the model is incorrect. *Ann. Math. Statist.* **37** 51–58.

[3] CHOW, Y. S. (1960). A martingale inequality and the law of large numbers. *Proc. Amer. Math. Soc.* **11** 107–111.

[4] GHOSH, M. and SEN, P. K. (1970). On the almost sure convergence of von Mises' differentiable statistical functions. *Calcutta Statist. Assoc. Bull.* **19** 41–44.

[5] GRAMS, W. F. and SERFLING, R. J. (1973). Convergence rates for *U*-statistics and related statistics. *Ann. Statist.* **1** 153–160.

[6] HOEFFDING, W. (1948). A class of statistics with asymptotically normal distribution. *Ann Math. Statist.* **19** 293–325.

[7] LOYNES, R. M. (1970). An invariance principle for reverse martingales. *Proc. Amer. Math. Soc.* **25** 56–64.

[8] MILLER, R. G. JR. and SEN, P. K. (1972). Weak convergence of *U*-statistics and von Mises' differentiable statistical functions. *Ann. Math. Statist.* **43** 31–41.

[9] MISES, R. VON (1947). On the asymptotic distribution of differentiable statistical functions. *Ann. Math. Statist.* **18** 309–348.

[10] RUBIN, H. and SETHURAMAN, J. (1965.) Probabilities of moderate deviations. *Sankhyā Ser A* **27** 325–346.

[11] SERFLING, R. J. (1971). The law of iterated logarithm for *U*-statistics and related von Mises' statistics (abstract). *Ann. Math. Statist.* **42** 1794.

[12] SPROULE, R. N. (1969). A sequential fixed-width confidence interval for the mean of a *U*-statistic. Ph. D. dissertation, Univ. N. Carolina, Chapel Hill.

[13] STRASSEN, V. (1967). Almost sure behavior of sums of independent random variables and martingales. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **2** 315–343.

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