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COMPARING RANK TESTS FOR ORDERED ALTERNATIVES IN RANDOMIZED BLOCKS

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Two classes of rank tests are considered for ordered alternatives in a randomized block design with k treatments and n blocks: tests based on among-blocks rankings (A-tests) and tests based on within-blocks rankings (W-tests). Previous efficiency comparisons for fixed k, $n \to \infty$ under the normal distribution have indicated that A-tests are more sensitive. In the present paper it is shown that this behavior is not typical under other distributions. Further, analysis of efficiencies for fixed n, $k \to \infty$ indicates greater sensitivity for W-tests. Considering these results and certain other desirable properties of the W-tests, the latter are recommended for most applications.

0. Summary. Let $X_{ij} = Y_{ij} + d_j + b_i$, $j = 1, \cdots, k$, $i = 1, \cdots, n$ represent a randomized blocks experiment, where $\{Y_{ij}\}$ is a collection of independent random variables with common distribution function F and density f. The constants $\{d_j\}$ are treatment effects, and b_i represents the nuisance effect (fixed or random) of block i. This paper is concerned with procedures for testing the null hypothesis

$$H_0: d_j \equiv 0 \tag{0.1}$$

against the ordered location alternative

$$H_1: d_1 \leqq d_2 \leqq \cdots \leqq d_k , \tag{0.2}$$

where at least one of the inequalities is strict. The procedures discussed are in fact sensitive to the more general stochastic ordered alternatives (cf. Hollander [6]) but for mathematical convenience only (0.2) will be considered. The model and the tests are also appropriate for certain replicated regression problems.

Asymptotic results obtained by Hollander [6] and Puri and Sen [12] for fixed k, $n \to \infty$ convey the impression that tests based on rankings across blocks are generally more sensitive to ordered alternatives than tests based on rankings within blocks, and therefore should be preferred.

The present paper examines these results along with asymptotic properties for fixed n, $k \to \infty$.

In Section 2, asymptotic distributions for fixed n, $k \to \infty$ are obtained for the test statistics, employing a projection lemma due to Hájek [3]. Pitman asymptotic relative efficiencies (ARE) are derived and discussed in Section 3, and a table of ARE values is presented for a specific example of each class of tests.

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From the efficiency results and other considerations, the conclusion is drawn that the within-blocks rank tests would more often be preferred.

1. Introduction and notation. Two basic classes of rank tests are considered for testing (0.1) vs. (0.2):

(i) *Within-blocks rank tests* (W-tests), which reject H_0 for large values of

$$W = \sum_{i=1}^{n} w_i , \tag{1.1}$$

where w_i is a simple linear rank statistic (cf. Hájek [3]), computed on the observations in block i;

$$w_i = \sum_{j=1}^{k} c_j a(R_{ij}) , \tag{1.2}$$

R_{ij} is the rank of X_{ij} among $\{X_{iu}, u = 1, \cdots, k\}$, $a(\cdot)$ is a monotone scores function, and $\{c_j, j = 1, \cdots, k\}$ are so-called "regression" constants, chosen to reflect the alternatives; in particular $c_1 < c_2 < \cdots < c_k$. Tests of the W-class have been proposed by Page [9] and Pirie and Hollander [11].

(ii) *Among-blocks rank tests* (A-tests), which reject H_0 for large values of

$$A = \sum_{u<v} T_{uv} , \tag{1.3}$$

where T_{uv} is a simple linear rank statistic for paired samples, computed on the column vectors for treatments u and v;

$$T_{uv} = \sum_{i=1}^{n} a^*(R_{uv}^{(i)})\eta(X_{uv}^{(i)}) , \tag{1.4}$$

with $X_{uv}^{(i)} = X_{iv} - X_{iu}$, $R_{uv}^{(i)}$ the rank of $|X_{uv}^{(i)}|$ among $\{|X_{uv}^{(t)}|, t = 1, \cdots, n\}$, $a^*(\cdot)$ a monotone scores function and

$$\begin{aligned} \eta(x) &= 1 \qquad \text{if} \quad x \geqq 0 \\ &= 0 \qquad \text{if} \quad x < 0 . \end{aligned}$$

Tests of the A-class have been proposed by Hollander [6] and Puri and Sen [12].

For parametric and other nonparametric tests for this problem see Doksum [2], Hollander [6], Jonckheere [7] and Pirie [10].

For fixed k, $n \to \infty$ (n-asymptotic) Hollander's paper contains ARE values of A with respect to W, $e_{k,\infty}(A, W)$, when F is a normal distribution function and $c_j = a(j) = a^*(j) = j$, $j = 1, \cdots, k$. This represents Hollander's A-test and Page's W-test. The results, reproduced in the lower left column of Table 1, showed that $e_{k,\infty}(A, W) > 1$ for all k. As the rest of Table 1 shows, this behavior is not typical.

2. The k-asymptotic distributions of A and W. In Section 3, ARE of A with respect to W will be discussed for fixed n, $k \to \infty$ (k-asymptotic). In the definition of Pitman efficiency, it is assumed the test statistics being compared are asymptotically normal (AN); (cf. Noether [8]).

Hájek [3] and Hájek and Šidák [4] present proofs that W is AN for $n = 1$, $k \to \infty$. The extension to general n is trivial.

For proof that A is AN, the following multivariate form of a lemma due to Hájek [3] is used.

LEMMA 2.1 (Hájek). *Let* $\mathbf{Y}_1, \mathbf{Y}_2, \cdots, \mathbf{Y}_k$ *be independent random n-vectors, and* $A = A(\mathbf{Y}_1, \cdots, \mathbf{Y}_k)$ *be a statistic such that* $\operatorname{Var} A < \infty$. *Let* $\mathscr{L}$ *denote the class of statistics L, of the form* $L = \sum_{j=1}^k l_j(\mathbf{Y}_j)$, *with* $\operatorname{Var} L < \infty$. *Finally let*

$$\hat{A} = \sum_{j=1}^k E(A \mid \mathbf{Y}_j) - (k-1)E(A) . \tag{2.1}$$

Then $\hat{A}$ *is a member of* $\mathscr{L}$,

$$E(\hat{A}) = E(A) \quad \text{and} \quad E(A - \hat{A})^2 = \operatorname{Var} A - \operatorname{Var} \hat{A} , \tag{2.2}$$

and for any L in $\mathscr{L}$,

$$E(A - \hat{A})^2 \leqq E(A - L)^2 . \tag{2.3}$$

In attempting to prove that a statistic of the form A in the theorem is AN, $\hat{A}$ is the best mean square approximation to A among a class, $\mathscr{L}$, of statistics for which the usual independent central limit theorems can be used to investigate asymptotic normality.

First a bound on the mean squared error of the approximation is obtained, then asymptotic normality of $\hat{A}$ is established. The desired result then follows. The following two easily proven lemmas are used in the proof of Theorem 2.4.

LEMMA 2.2. *Let* $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ *be independent random elements on a space* Ω, *and* $U(\cdot, \cdot)$, $V(\cdot, \cdot)$ *be real-valued measurable functions on* $\Omega \times \Omega$. *Then*

$$\operatorname{Cov} \{E[U(\mathbf{X}, \mathbf{Z}) \mid \mathbf{Z}], E[V(\mathbf{Y}, \mathbf{Z}) \mid \mathbf{Z}]\} = \operatorname{Cov} \{U(\mathbf{X}, \mathbf{Z}), V(\mathbf{Y}, \mathbf{Z})\} . \tag{2.4}$$

PROOF. A straightforward exercise in conditional expectations.

LEMMA 2.3. *If* A *is given by* (1.3) *then a convenient formulation of the approximation* (2.1) *is the following*:

$$\hat{A} - E(A) = \sum_{u=1}^k B_u , \tag{2.5}$$

where $\{B_u\}$ *are independent random variables defined by*

$$\begin{aligned} B_u = {} & \sum_{v=u+1}^k [E(T_{uv} \mid \mathbf{Y}_u) - E(T_{uv})] \\ & - \sum_{v=1}^{u-1} [E(T_{uv} \mid \mathbf{Y}_u) - E(T_{uv})] , \qquad u = 1, \cdots, k . \end{aligned} \tag{2.6}$$

PROOF. This result is easily verified from the definitions of A and $\hat{A}$ and some routine algebraic manipulation.

THEOREM 2.4. *Let* A *and* $\hat{A}$ *be given by* (1.3) *and* (2.1) *respectively, where* $\mathbf{Y}_u = (Y_{1u}, Y_{2u}, \cdots, Y_{nu})'$. *Then there exists a positive constant M, depending on n and* $a^*(\cdot)$, *but not on k, such that*

$$E(A - \hat{A})^2 \leqq Mk^2 . \tag{2.7}$$

PROOF. Noting that $\operatorname{Cov}(T_{uv}, T_{u'v'}) = 0$ if u, v, u', v' are distinct, and

$\operatorname{Cov}(T_{uv}, T_{wu}) = -\operatorname{Cov}(T_{uv}, T_{uw})$ if u, v, w are distinct, then

$$\begin{aligned} \operatorname{Var} A = \textstyle\sum\sum_{u<v} \operatorname{Var} T_{uv} + 2 \sum\sum\sum_{u<v<w} \operatorname{Cov}(T_{uv}, T_{uw}) \\ + 2 \textstyle\sum\sum\sum_{v<w<u} \operatorname{Cov}(T_{uv}, T_{uw}) \\ - 2 \textstyle\sum\sum\sum_{v<u<w} \operatorname{Cov}(T_{uv}, T_{uw}) . \end{aligned} \tag{2.8}$$

From (2.6),

$$\begin{aligned} \operatorname{Var} \hat{A} = \textstyle\sum_{u=1}^{k} \operatorname{Var} B_u = \sum_{u=1}^{k} \{ \sum_{v \neq u} \operatorname{Var} E(T_{uv} \mid \mathbf{Y}_u) \\ + 2 \textstyle\sum\sum_{u<v<w} \operatorname{Cov}[E(T_{uv} \mid \mathbf{Y}_u), E(T_{uw} \mid \mathbf{Y}_u)] \\ + 2 \textstyle\sum\sum_{v<w<u} \operatorname{Cov}[E(T_{uv} \mid \mathbf{Y}_u), E(T_{uw} \mid \mathbf{Y}_u)] \\ - 2 \textstyle\sum\sum_{v<u<w} \operatorname{Cov}[E(T_{uv} \mid \mathbf{Y}_u), E(T_{uw} \mid \mathbf{Y}_u)]\} . \end{aligned} \tag{2.9}$$

Since $T_{uv} = T_{uv}(\mathbf{Y}_u, \mathbf{Y}_v)$ and $\{\mathbf{Y}_u\}$ is a collection of independent random n-vectors, Lemma 2.2 provides $\operatorname{Cov}(T_{uv}, T_{uw}) = \operatorname{Cov}[E(T_{uv} \mid \mathbf{Y}_u), E(T_{uw} \mid \mathbf{Y}_u)]$. Thus, from (2.2), (2.8) and (2.9),

$$\begin{aligned} E(A - \hat{A})^2 = \textstyle\sum\sum_{u<v} \operatorname{Var} T_{uv} - \sum\sum_{u \neq v} \operatorname{Var} E(T_{uv} \mid \mathbf{Y}_u) \\ \leqq \textstyle\sum\sum_{u<v} \operatorname{Var} T_{uv} . \end{aligned} \tag{2.10}$$

From (1.4), $|T_{uv}| \leqq \sum_{i=1}^{n} |a^*(i)| = M'$, say, which, with (2.10), yields

$$E(A - \hat{A})^2 \leqq \tbinom{k}{2}(M')^2 . \tag{2.11}$$

Finally (2.7) follows with $M = (M')^2/2$.

Thus, while A and $\hat{A}$ are of order k^2, the variance of each is of order k^3, and the mean squared difference is of order only k^2. This provides the following theorem.

THEOREM 2.5. *Consider the statistics A and $\hat{A}$ in Theorem* 2.4. *If*

$$\lim_{k \to \infty} \operatorname{Var}(k^{-\frac{3}{2}} \hat{A}) = D^2 , \qquad 0 < D^2 < \infty , \tag{2.12}$$

then $k^{-\frac{3}{2}}[\hat{A} - E(A)]$ is AN $(0, D^2)$.

Further, (2.12) and asymptotic normality will hold, or not hold, simultaneously for A and $\hat{A}$.

PROOF. Clearly $\{B_{uk} = B_u/k, u = 1, \cdots, k, k = 1, 2, \cdots\}$ is a uniformly bounded double sequence of random variables, independent within rows. Thus asymptotic normality of $\hat{A}$ follows from (2.12) by a well-known form of the central limit theorem (cf. Chung [1], page 186). The equivalence for A follows directly from Theorem 2.4.

REMARK. Since F is assumed continuous, (2.9) reveals that (2.12) will hold for sequences of alternatives that converge, in either the Pitman (cf. Noether [8]) or contiguous (cf. Hájek–Šidák [4]) sense, to the null distribution. The usual regularity conditions on the scores function $a^*(\cdot)$ (*ibid.*) are not required here since n is assumed fixed.

3. Asymptotic efficiencies. Emphasis in this section will center on the ARE of A with respect to W for uniform (or Wilcoxon) scores, $a(j) = a^*(j) = j$, and normal scores $a(j) = E_{kj}$, $a^*(j) = E_{nj}$, where E_{mj} is the expected value of the jth order statistic from a normal sample of size m, $m = k$ or n. For convenience $A(U)$, $W(U)$ and $A(N)$, $W(N)$ will denote the test statistics with uniform and normal scores, respectively. In practice these two scores functions are almost exclusively employed. Many of the following results, however, extend readily to general scores.

Hollander [6] showed that for $c_j \equiv j$, and linear ordered alternatives $d_j - d_{j-1} \equiv$ constant,

$$(3.1) \qquad e_{k,\infty}(A(U), W(U)) = 2(k+1)^2\{k[3 + 2(k-2)\rho^*(F)]\}^{-1}[\textstyle\int g^2/\int f^2]^2$$

where g is the density of $Y_2 + Y_1$ when Y_2, Y_1 are independent with density f, and $\rho^*(F) = \lim_{n\to\infty} \operatorname{Cov}(T_{uv}, T_{uw})$ when H_0 holds. This corresponds to Hollander's original A-test and Page's W-test. He also presents a table with which (3.1) can be evaluated when F is a normal distribution function. Those values and some for other distributions comprise the lower half of Table 1. The normal distribution values led Hollander (and others) to conclude that A-tests in general are more sensitive to ordered alternatives than are W-tests. The original motivation for the present paper was the conviction that this could not be true for (relatively) large values of k, and small n, and that k-asymptotic relative efficiencies, $e_{\infty,n}(A, W)$, would provide corroborating evidence. The present section is devoted to this purpose. However, examination of the lower half of Table 1 shows that the earlier conclusion (at least for that choice of scores functions) is not universally valid even for n-asymptotic comparisons.

In the following discussion of k-asymptotic results, a condition of bounded variation will be assumed for the alternatives, namely that there exists a positive constant K such that

$$(3.2) \qquad k^{-1} \textstyle\sum_{j=1}^{k} (d_j - \bar{d})^2 \leqq K\,, \qquad k = 2, 3, \cdots,$$

where $\bar{d} = k^{-1} \sum_j d_j$. For the consideration of ARE we define the Pitman sequence of alternatives to be

$$(3.3) \qquad X_{ij} = Y_{ij} + k^{-\frac{1}{2}}\phi d_j + b_i\,, \qquad j = 1, \cdots, k,\ i = 1, \cdots, n\,.$$

The definition of Pitman k-asymptotic relative efficiency of A with respect to W is

$$(3.4) \qquad e_{\infty,n}(A, W) = \lim_{k\to\infty} \{(d/d\theta)E_\theta(A)\,|_{\theta=0}[(d/d\theta)E_\theta(W)\,|_{\theta=0}]^{-1}\}^2 \times \operatorname{Var}_0 W/\operatorname{Var}_0 A\,,$$

where $\theta \doteq k^{-\frac{1}{2}}\phi$, and the zero subscript refers to moments computed for $\theta = 0$.

THEOREM 3.1. *If* $\rho_0^n(F) = \operatorname{Cov}_0(T_{12}, T_{13})$,

$$(3.5) \qquad \lim_{k\to\infty} k^{-3} \textstyle\sum_j (c_j - \bar{c})^2\{\sum_j (2j - k - 1)d_j/\sum_j (c_j - \bar{c})d_j\}^2 = M_{cd}$$

with $0 < M_{cd} < \infty$, *and* (3.2) *holds, and if the densities f and g are square integrable, the k-asymptotic relative efficiency of Hollander's test with respect to Page's* *test with arbitrary constants c_j is*

$$(3.6)\qquad e_{\infty,n}(A(U), W(U)) = 6M_{cd}[\rho_0^n(F)(n+1)(2n+1)]^{-1}\{[(n-1)\int g^2 + \int f^2]/\int f^2\}^2 .$$

PROOF. The individual rank statistics in $A(U)$ can be represented as

$$(3.7)\qquad \begin{aligned} T_{uv} &= \sum\sum_{i<j} \eta(X_{iv} - X_{iu} + X_{jv} - X_{ju}) + \sum_{i=1}^{n} \eta(X_{iv} - X_{iu}) \\ &= \sum\sum_{i<j} \eta[(Y_{iv} + Y_{jv}) - (Y_{iu} - Y_{ju}) + 2\theta(d_v - d_u)] \\ &\quad + \sum_{i=1}^{n} \eta[Y_{iv} - Y_{iu} + \theta(d_v - d_u)] , \end{aligned}$$

and letting G denote the distribution function corresponding to density g,

$$(3.8)\qquad \begin{aligned} E_\theta T_{uv} &= \sum\sum_{i<v} P[(Y_{iu} + Y_{ju}) \leqq (Y_{iv} + Y_{jv}) + 2\theta(d_v - d_u)] \\ &\quad + \sum_{i=1}^{n} P[Y_{iu} \leqq Y_{iv} + \theta(d_v - d_u)] \\ &= \tfrac{1}{2}n(n-1) \int G(y + 2\theta(d_v - d_u))\, dG(y) \\ &\quad + n \int F(y + \theta(d_v - d_u))\, dF(x) . \end{aligned}$$

Taking the derivative and summing yields

$$(3.9)\qquad \begin{aligned} (d/d\theta)E_\theta A(U)\,|_{\theta=0} &= n \sum\sum_{u<v} (d_v - d_u)\{(n-1)\int g^2 + \int f^2\} \\ &= n\{\sum_j [2j - (k+1)]d_j\}\{(n-1)\int g^2 + \int f^2\} . \end{aligned}$$

The null variance is given by Hollander [6] as

$$(3.10)\qquad \mathrm{Var}_0\, A(U) = n(n+1)(2n+1)k(k-1)[3 + 2(k-2)\rho_0^n(F)]/144 .$$

A similar treatment of $W(U)$ gives

$$(3.11)\qquad \begin{aligned} E_\theta W(U) &= n \sum_{j=1}^{k} c_j E_\theta(R_{1j}) \\ &= n \sum_j c_j E_\theta\{\sum_{u=1}^{k} \eta(X_{1j} - X_{1u})\} \\ &= n \sum_j c_j\{1 + \sum_{u\neq j} \int F(y + \theta(d_j - d_u))\, dF(x)\} , \end{aligned}$$

with the resulting derivative

$$(3.12)\qquad \begin{aligned} (d/d\theta)W(U)\,|_{\theta=0} &= n \sum_j c_j\{\sum_u (d_j - d_u) \int f^2\} \\ &= nk\{\sum_j (c_j - \bar{c})d_j\}/\int f^2 . \end{aligned}$$

The null variance is readily obtained from standard rank distributions as

$$(3.13)\qquad \mathrm{Var}_0\, W(U) = nk(k+1) \sum_j (c_j - \bar{c})^2/12 .$$

Substituting (3.9), (3.10), (3.12) and (3.13) into (3.4), and assuming that (3.5) holds, (3.6) is established.

To compare (3.6) with the n-asymptotic results of Table 1, let $c_j \equiv j$. The appropriate linear ordered alternatives subject to (3.2) are $d_j \equiv j/k$. With these values, $M_{cd} = \frac{1}{3}$ and the upper half of Table 1 gives the results for the indicated distributions. Values of $\rho_0^n(F)$ are given by Hollander [5]. Hollander [5, 6] showed that $\rho_0^n(F)$, and therefore also the null distribution of A, is dependent on F. A method is presented in his 1967 paper for obtaining a consistent estimate of $\lim_{n\to\infty} \mathrm{Var}_0(A)$ which yields an n-asymptotically distribution-free test based on A. The estimate is not k-asymptotically consistent.

TABLE 1
Asymptotic relative efficiencies of test A with respect to test W for linear ordered alternatives and
$a(j) = a^*(j) = c_j = j$
(Hollander's test vs. Page's test)

Distribution	normal	uniform	exponential
	k-asymptotic relative efficiency		
n			
1	1.	1.	1.
2	1.012	.957	.794
5	1.023	.928	.643
10	1.030	.918	.587
20	1.032	.911	.559
50	1.034	.909	.541
∞	1.035	.907	.530
	n-asymptotic relative efficiency		
k			
2	1.5	1.333	.750
5	1.224	1.077	.617
10	1.129	.991	.615
20	1.083	.948	.551
50	1.055	.908	.538
∞	1.035	.907	.530

From Table 1, it is seen that ARE values alone do not provide a definitive choice between $A(U)$ and $W(U)$. Which test performs better apparently depends both on the distribution F, and the values of k and n. In any case the difference in power cannot be expected to be great. For other reasons however, the author recommends using $W(U)$ in most cases. Specifically:

(i) The W-tests are distribution-free, whereas the A-tests are not. In terms of ability to control type I error then, W-tests are clearly favored.

(ii) In ease of computation also, W-tests are favored. A complicated estimate of the variance is necessary for employment of any A-test (cf. Puri and Sen [12]). In addition the proposed variance estimate is not consistent for $k \to \infty$, which limits its appeal. On the other hand, W-tests are easily computed from the regression constants c_j, and the scores $a(\cdot)$. Also, small sample null distribution tables are available for $W(U)$ (Page [9]), and $W(N)$ (Pirie and Hollander [11]).

(iii) Finally, if a specific form of non-linear growth rate of the constants d_j is suspected (say quadratic), the W-tests can take this into account by adjusting the corresponding growth rate of the constants c_j. This is not true of the A-tests, at least in their present form.

For other scores functions $a(\cdot)$ and $a^*(\cdot)$, such as normal scores, the results are less complete. The difficulty lies in evaluation of $\lim_{k\to\infty} \mathrm{Var}_0 A$. Since n is fixed, the scores function $a^*(\cdot)$ does not converge to a "well behaved" function as in Chernoff–Savage theorems (cf. Puri and Sen [12]). Thus $\mathrm{Cov}_0(T_{12}, T_{13})$

would have to be evaluated explicitly for any distribution F and scores function $a^*(\cdot)$. This has been accomplished only for uniform scores (i.e., for $A(U)$).

Nevertheless, some comparisons for $A(N)$ and $W(N)$ can be made. Details are omitted, for brevity, but the following results have been obtained: If $\int g^2 < \infty$, then

$$0 < e_{\infty,n}(A(N), A(U)) < \infty . \tag{3.14}$$

From (3.14), if F represents a distribution for which $e_{\infty,n}(W(N), W(U)) = \infty$ (e.g., uniform or exponential), then $e_{\infty,n}(W(N), A(N)) = \infty$, all n. Similarly, from results in Pirie and Hollander [11], if $\int f^2 < \infty$, then, assuming the same regression constants c_j are used in $W(N)$ and $W(U)$,

$$0 < e_{k,\infty}(W(N), W(U)) < \infty . \tag{3.15}$$

From (3.15), if F represents a distribution for which $e_{k,\infty}(A(N), A(U)) = \infty$ (again uniform or exponential), then $e_{k,\infty}(A(N), W(N)) = \infty$, all k. As with uniform scores, ARE results do not provide a definitive choice between $A(N)$ and $W(N)$, but for the same reasons as previously stated the author recommends use of the W-test in most cases.

Concluding remarks. A comparison of Pitman ARE of among-blocks (A) and within-blocks (W) rank tests for ordered alternatives in the randomized blocks model has been undertaken in two ways; (i) fixed k, $n \to \infty$ and (ii) fixed n, $k \to \infty$. In particular, detailed results are presented comparing Hollander's A-test [6] and Page's W-test [9], and a partial result is given comparing the normal scores versions (cf. Puri and Sen [12] and Pirie and Hollander [11]).

Contrary to previous conclusions the ARE values are at least as likely to favor the W-tests as the A-tests. Because of this, and certain desirable properties of the W-tests, the author recommends W-tests for most uses. A simulation study could be useful to determine if in some cases A-tests perform sufficiently better than W-tests so that some of the undesirable properties of the former could be tolerated.

REMARK. The model $X_{ij} = Y_{ij} + d_j\theta + b_i$ is also appropriate for certain replicated regression problems where $\{d_j\}$ represent known constants and b_i accounts for an inability to accurately reproduce a fixed reference point for the ith replicate. In this case the flexibility of the W-tests allows the choice $c_j \equiv d_j$ with considerable advantage in power.

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REFERENCES

[1] CHUNG, K. L. (1968). *A Course in Probability Theory.* Harcourt, Brace & World, New York.

[2] DOKSUM, K. A. (1967). Robust procedures for some linear models with one observation per cell. *Ann. Math. Statist.* **38** 878–883.

[3] HÁJEK, J. (1968). Asymptotic normality of simple linear rank statistics under alternatives. *Ann. Math. Statist.* **39** 325–346.

[4] HÁJEK, J. and ŠIDÁK, Z. (1967). *Theory of Rank Tests*. Academic Press, New York.

[5] HOLLANDER, M. (1966). An asymptotically distribution-free multiple comparison procedure-treatments vs. control. *Ann. Math. Statist.* **37** 735–738.

[6] HOLLANDER, M. (1967). Rank tests for randomized blocks when the alternatives have an a priori ordering. *Ann. Math. Statist.* **38** 867–877.

[7] JONCKHEERE, A. R. (1954). A test of significance for the relation between m rankings and k categories. *British J. Statist. Psych.* **7** 93–100.

[8] NOETHER, G. E. (1955). On a theorem of Pitman. *Ann. Math. Statist.* **26** 64–68.

[9] PAGE, E. B. (1963). Ordered hypotheses for multiple treatments: a significance test for linear ranks. *J. Amer. Statist. Assoc.* **58** 216–230.

[10] PIRIE, W. R. (1970). Distribution-free tests for ordered alternatives in the randomized block model. Ph.D. dissertation, Florida State Univ.

[11] PIRIE, W. R. and HOLLANDER, M. (1972). A distribution-free normal scores test for ordered alternatives in the randomized block design. *J. Amer. Statist. Assoc.* **67** 855–757.

[12] PURI, M. L. and SEN, P. K. (1968). On Chernoff–Savage tests for ordered alternatives in randomized blocks. *Ann. Math. Statist.* **39** 967–972.

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