The Annals of Statistics
1974, Vol. 2, No. 1, 182–189

THE INTERSECTION OF RANDOM SPHERES AND THE NONCENTRAL RADIAL ERROR DISTRIBUTION FOR SPHERICAL MODELS[1]

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The paper gives an expression suitable for numerical computation and recursive relationships for the expected volume of the intersection of an n dimensional sure sphere and an n dimensional random sphere with a different radius, whose center follows a spherical distribution, the center of which does not coincide with that of the sure sphere. Several expressions are given for the distribution of the squared noncentral radial error when the vector observation follows a spherical distribution—a generalization of the noncentral chi-square distribution. Applications to specific spherical models are presented.

1. Introduction and summary. Smith and Stone (1961) obtained the "expected coverage" of a fixed circle by a random circle of different radius, whose center followed a circular normal distribution. This result was generalized in 1962 to the case of two n-dimensional random spheres by Laurent [4], who also gave several expressions suitable for computation and recursive relationships for the noncentral chi-square cumulative distribution. The present paper generalizes the preceding results (that were presented in an earlier version of this paper and included in [5]) to the case when the center of the random sphere follows an n-dimensional spherical distribution. It gives an expression, suitable for numerical computations, for the expected value of the intersection of the fixed and the random sphere, together with some recursive relationships. It gives also several expressions (and corresponding recursive relationships) for the cumulative distribution of the noncentral squared radial error when the vector observation is spherically distributed—a generalization of the noncentral chi-square distribution. Expressions for the pdf of the noncentral radial error have been given, by different methods and under stricter conditions, by Thomas [8].

The motivation for this research stemmed from the consideration of problems pertaining to bombing, firing, and search theories. In those as in other fields of application the interest in studying the class of spherical models rather than the spherical normal distribution rests on the plausibility that, in many instances, normality is postulated when, actually, only the hypothesis of sphericity, that is, essentially, of rotational invariance, is intended—this due, likely, to some

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Received October 1972; revised April 1973.

[1] Partially supported by Goodyear Aircraft Corporation, Dayton, Ohio, and by a summer grant from Wayne State University.

AMS 1970 *subject classifications.* Primary 62H10, 60D05; Secondary 44A30.

Key words and phrases. Random sets, spherical distributions, noncentral radial error, multivariate transforms.

implicit assumption made about the independence of the components of the vector observation. In bombing theory and, more generally, in the theory of errors of observations, as has been often suggested by Maurice Fréchet [3], it is plausible that a Laplace I type (double exponential) model is more adequate than the classical Gaussian model (Lord in [6] has given a spherical multivariate generalization of the double exponential model; Laurent in several contributed papers has considered ellipsoidal generalizations of the same model).

2. General formulation and notations. Let S_0 be a fixed n-dimensional sphere with radius r centered at the origin. Let the center C, with coordinates ξ, of an n-dimensional sphere S, with radius R, follow a spherical distribution about a point A with coordinates ζ, that is, a distribution that is invariant under the group of orthogonal transformations applied to $x = \xi - \zeta$, and whose probability density function (pdf) will be denoted $f_n(\xi; \zeta, \tau) = g_n(x'x/\tau^2)$. There will be no loss of generality in assuming $\tau = 1$. One wants to obtain the expected value $\mu_n = E[V(S_0 \cap S)]$ of the volume $V[S_0 \cap S]$ of the intersection of S_0 and S.

Let m be the coordinates of a point M. The indicator of the set $S_0 \cap S$ is the product of the indicators $I_{S_0}(m; r)$, $I_S(m; \xi, R)$ of the sets S_0 and S, the volume of the intersection is

$$V(S_0 \cap S) = \int I_{S_0} I_S \, dm = \int_{S_0} I_S \, dm \tag{1}$$

and its expected value, if it exists, is

$$\mu_n = \int [\int I_{S_0} I_S \, dm] f_n(\xi; \zeta, 1) \, d\xi \, . \tag{2}$$

These formulas admit another interpretation: if M is uniformly distributed in S_0, then the joint pdf of ξ and m is

$$w(\xi, m) = f_n(\xi; \zeta, 1) I_{S_0} / V(S_0) \, ; \tag{3}$$

$V(S_0 \cap S)/V(S_0)$ is the conditional probability, given ξ, that S covers M; μ_n is the marginal probability that S covers M. Also the integral between brackets in (2) is the conditional expected value of $V(S_0 \cap S)$ given ξ and the event $\{M \in S_0 \cap S\}$; hence, one can change in (2) the order of integration and write

$$\mu_n = \int [\int I_S f_n(\xi; \zeta, 1) \, d\xi] I_{S_0} \, dm \, , \tag{4}$$

where the integral between brackets is the probability $P((\xi - m)'(\xi - m) \leqq R^2)$ that S covers M when M is fixed or, in another terminology, that the noncentral radial error $\rho = |x|$ does not exceed R. When ξ is normally distributed, it is well known that ρ^2 follows the noncentral chi-square distribution with n degrees of freedom and noncentrality parameter $\delta^2 = (\zeta - m)'(\zeta - m)$. In all cases, this probability depends on M through δ^2 and, therefore, is invariant under the group of orthogonal transformations about A; it is also the probability that S covers a point that is uniformly distributed on a sphere of center A and radius δ. Smith and Stone have used (2), when ξ is normally distributed, to obtain

μ_2; we will derive μ_n from (4) in the general case, as this will offer an opportunity to discuss the problem of the noncentral squared radial error distribution. The latter will be denoted by $Q_n(R^2; \delta^2)$ and its pdf by $q_n(\rho^2; \delta^2)$; hence,

(5) $$\mu_n = \int Q_n(R^2; \delta^2) I_{S0}\, dm .$$

3. The squared noncentral radial error distribution. We will first obtain different expressions for the conditional distribution of the squared noncentral radial error given $x'x$, then use these results to obtain the marginal distribution, and discuss specific examples.

(a) As x is spherical the statistic $x'x$ is sufficient for the distribution of x; the conditional distribution of x, given $x'x = |x|^2$, is uniform on the sphere of radius $|x|$ (and center C); and the conditional and marginal distribution of $x/|x|$ is uniform on the unit sphere of center C. The conditional probability $P(A; |x|^2)$, given $x'x = |x|^2$, of an event A, provides an unbiased estimator $P(A; x'x)$ of the probability of A, and if $x'x$ is complete, this estimator is the minimum variance unbiased estimator of $P(A)$.

As $\overline{MC} = \overline{AC} - \overline{AM}$, $\overline{MC}^2 = \rho^2$, $\overline{AC}^2 = |x|^2$, $\overline{AM}^2 = \delta^2$, $\rho^2 = |x|^2 + \delta^2 - 2|x|\delta \cos \Phi$. The conditional distribution $Q_n(R^2; \delta^2; |x|^2)$ of the noncentral squared radial error, given $x'x = |x|^2$, which is also the MVUE of $Q_n(R^2; \delta^2)$ if $x'x$ is complete, is the area of the polar cap determined on the unit sphere by the angle $\Phi = (AM, AC)$, namely,

(6) $$Q_n(R^2; \delta^2; |x|^2) = \int_0^\Phi \sin^{n-2} d\varphi \Big/ B\left(\frac{n-1}{2}, \frac{1}{2}\right),$$

with

$$\begin{aligned} \Phi = 0\,, &\quad \text{if} \quad R^2 < (|x| - \delta)^2 \\ = \arc\cos\,(x^2 + \delta^2 - R^2)/2x\delta\,, &\quad \text{if} \quad (|x| - \delta)^2 < R^2 < (|x| + \delta)^2 \\ = \pi\,, &\quad \text{if} \quad R^2 > (|x| + \delta)^2\,, \end{aligned}$$

as can be shown by using polar coordinates and integrating out all angles except φ.

It is known (see e.g. Erdélyi [2]) that (6) can also be written as

(7) $$Q_n(R^2; \delta^2; |x|^2) = \Gamma(n/2)(2\pi)^{-\frac{1}{2}} 2^{n/2-1} \delta^{\frac{1}{2}} (\sin \phi)^{(n-1)/2} P_{(n-3)/2}^{-(n-1)/2}(\cos \phi)\,,$$

where P denotes a Legendre function, and, according to Watson ([9] page 411) as

(8) $$Q_n(R^2; \delta^2; |x|^2) = \Gamma(n/2)(2R/|x|\delta)^{n/2-1} R \int_0^\infty u^{-(n-2)/2} J_{n/2-1}(|x|u) J_{n/2}(Ru) J_{n/2-1}(\delta u)\, du\,,$$

where $J_v(t)$ denotes a Bessel function of first kind of order v. Further, differentiating with respect to R^2, one obtains the conditional pdf of ρ^2, given $x'x = |x|^2$, as

(9) $$q_n(\rho^2; \delta^2; |x|^2) = (\tfrac{1}{2})\Gamma(n/2)(2\rho/|x|\delta)^{n/2-1} \int_0^\infty u^{-(n-4)/2} J_{n/2}(|x|u) J_{n/2-1}(\delta u) J_{n/2-1}(\rho u)\, du\,;$$

an alternative from being (Watson [9] page 411)

$$(10)\qquad q_n(\rho^2;\,\delta^2;\,|x|^2) = \frac{(2|x|\delta)^{-(n-2)}[\rho^2-(\delta-|x|)^2]^{(n-3)/2}[(\delta+|x|)^2-\rho^2]^{(n-3)/2}}{B\left(\frac{n-1}{2},\frac{1}{2}\right)},$$

a formula which can be directly established by writing that the projection $x_1 = -\delta + (x^2 - \rho^2 - \delta^2)/2\delta$ of x on $\overline{AM}$ follows a Thompson distribution (that is, that $x_1{}^2/|x|^2$ follows an incomplete beta distribution with parameters $\frac{1}{2}$ and $(n-1)/2$). Equations (8) and (9) can be expressed as Hankel transforms and can be obtained also by the direct use of discontinuous factors.

(b) To obtain the marginal distribution $Q_n(R^2;\,\delta^2)$ of R^2 it suffices to take the expected value of the estimator $Q_n(R^2;\,\delta^2;\,|x'x|)$ with respect to the distribution of $x'x$ or, equivalently, of x, thus defining an integral transform $\mathscr{C}(f;\,R^2,\,\delta^2)$ and a symbolic calculus relating spherical distributions and the distributions of their squared noncentral radial errors. As transforms, the latter may be considered functions of R^2 with parameter δ^2 or functions of δ^2 with parameter R^2. Therefore,

$$(11)\qquad Q_n(R^2;\,\delta^2) = \int_0^\infty Q_n(R^2;\,\delta^2;\,|x|^2)h_n(|x|^2)d|x|^2\,,$$

where

$$h_n(|x|^2)\,dx^2 = 2\pi^{n/2}|x^2|^{(n-2)/2}g_n(|x|^2)\,,$$

and $Q_n(R^2;\,\delta^2;\,|x|^2)$ is given by (8).

Assuming that the conditions under which the change of the order of integration are met (11) reads

$$(12)\qquad \begin{aligned} Q_n(R^2;\,\delta^2) &= (R/\delta)^{(n-2)/2}R\int[(2\pi)^{n/2}|x|^{n/2}u^{1-n/2}J_{n/2-1}(|x|u)g_n(|x|^2)d|x|] \\ &\qquad\times J_{n/2-1}(\delta u)J_{n/2}(Ru)\,du \\ &= (R/\delta)^{(n-2)/2}\,R\int\phi(u)J_{n/2-1}(\delta u)J_{n/2}(Ru)\,du\,, \end{aligned}$$

where $\phi(|t|)$ denotes the characteristic function $E[\exp(it'x)]$ of the pdf of x, as a function of $|t| = (t't)^{\frac{1}{2}}$. It is well known that the characteristic function ϕ of a spherical distribution, as a function of its argument, does not depend on n, that is, is invariant under projection (and, therefore, the same for a projective family $f_n(x)$, $n \leqq n_0$ for some n_0 not necessarily finite). In terms of Hankel transforms one has

$$(13)\qquad R^{-n/2}Q_n(R^2;\,\delta) = \delta^{(1-n)/2}H_{n/2-1}[u^{-\frac{1}{2}}\phi(u)J_{n/2}(Ru);\,\delta]$$

and

$$(14)\qquad R^{-(n-1)/2}Q_n(R^2;\,\delta^2) = \delta^{1-n/2}H_{n-2}[u^{-\frac{1}{2}}\phi(u)J_{n/2-1}(\delta u);\,R]\,.$$

Q_n may be found with the help of tables of the Hankel transform or is amenable to numerical integration.

Equation (13) can also be written

(15 i) $$R^{-n/2}Q_n(R^2;\delta^2) = F_{n,u}[(2\pi u)^{-n/2}\phi(u)J_{n/2}(Ru);\delta]\,,$$

where

$$F_{n,u}\{h_n(u);v\} = (2\pi)^{n/2}v^{1-n/2}\int_0^\infty u^{n/2}J_{n/2-1}(uv)h_n(u)\,du = \mathbb{F}_n h_n$$

is the Fourier transform of a spherical function $f = h_n(u)$, with $u = (u'u)^{\frac{1}{2}}$.

It is essential to note that, up to a factor in π, the operator $\mathbb{F}$ is its own inverse, i.e., $\mathbb{F}_n\mathbb{F}_n = I$ and, as $\mathbb{F}_n f_n = \Phi$ and $\mathbb{F}_k\Phi = f_k$, where f_k is the projection of f_n on the k dimensional space, the $\mathbb{F}_k\mathbb{F}_n$ is a projection operator from the n dimensional space to the k dimensional space.

Equation (15 i) shows $R^{-n/2}Q_n(R^2;\delta^2)$ as (up to factor in π)

(15 ii) $$R^{-n/2}Q_n(R^2;\delta^2) = \mathbb{F}_n u^{-n/2}J_{n/2}(R(u))\mathbb{F}_n f_n\,,$$

that is, $Q_n(R^2;\delta^2)$ is obtained from f_n by the application of the operator $R^{n/2}\mathbb{F}_n u^{-n/2}J_{n/2}(Ru)\mathbb{F}_n$.

Similarly, one has the dual formula

(15 iii) $$F_{n,\delta}[(2\pi R)^{-n/2}Q_n(R^2;\delta^2);s] = s^{-n/2}\phi(s)J_{n/2}(Rs)\,,$$

that is,

(15 iv) $$\mathbb{F}_n Q_n = (R/s)^{n/2}J_{n/2}(Rs)\mathbb{F}_n f_n = (R/s)^{n/2}J_{n/2}/Rs)\mathbb{F}_k f_k$$

(Q_n is spherical as a function of δ; therefore, its projection on the k dimensional space is $\mathbb{F}_k(R/s)^{n/2}J_{n/2}(Rs)\phi$ and different from Q_k, which is $\mathbb{F}_k(R/s)^{k/2}J_{k/2}(Rs)\phi)$.

$$Q_n = \mathbb{F}_n(R/u)^{(n-k)/2}J_{n/2}/J_{k/2}\mathbb{F}_k Q_k$$

may be considered a recursive formula for Q_n.

Similarly, (14) can be written

(16 i) $$R^{-2}Q_n(R^2;\delta^2) = F_{n+2,u}[(2\pi u\delta)^{-n/2+1}\phi(u)J_{n/2-1}(\delta u);R]$$

with dual

(16 ii) $$F_{n+2,R}[(2\pi)^{-n/2+1}R^{-n}Q_n(R^2;\delta^2);s] = \phi(s)J_{n/2-1}(\delta s)(s\delta)^{-n/2+1}\,.$$

Formula (16 ii) shows that the proper operator, applied to Q_n, factors it into one factor accounting for noncentrality and one factor accounting for the nature of the original spherical distribution.

If x follows a multivariate spherical Laplace distribution, then $\phi(u) = (1+u^2)^{-1}$ and

(17) $$\begin{aligned} Q_n(R^2;\delta^2) &= (R/\delta)^{n/2}I_{n/2}(R)K_{n/2-1}(\delta)\,, && R<\delta\,,\\ &= (R/\delta)^{(n-1)/2}\delta I_{n/2}(\delta)K_{n/2-1}(R)\,, && R>\delta\,,\end{aligned}$$

where I_v and K_v denote the modified Bessel functions of first and third kind of order v.

4. The squared noncentral radial error pdf. By taking the derivatives with respect to R^2 of the expressions given above for $Q_n(R^2; \delta^2)$ one obtains the pdf $q_n(\rho^2; \delta^2)$, namely

$$q_n(\rho^2; \delta^2) = 2^{-1}(\rho/\delta)^{n/2-1} \int_0^\infty uJ_{n/2-1}(\delta u)J_{n/2-1}(\rho u)\phi(u)\, du\,. \tag{18}$$

This formula has also been obtained by Thomas by different methods and under more stringent conditions. In terms of Hankel and F transforms one has

$$q_n(\rho^2; \delta^2) = 2^{-1}(\rho/\delta)^{n/2}\delta^{\frac{1}{2}}H_{n/2-1}[u^{\frac{1}{2}}\phi(u)J_{n/2-1}(\rho u); \delta] \tag{19}$$

$$= 2^{-1}(\rho/\delta)^{(n-1)/2}\rho^{-1}H_{n/2-1}[u^{\frac{1}{2}}\phi(u)J_{n/2-1}(\delta u); \rho]\,, \tag{20}$$

$$2\rho^{1-n/2}q_n(\rho^2; \delta^2) = F_{n,u}[(2\pi)^{-n/2}\phi(u)u^{1-n/2}J_{n/2-1}(\rho u); \delta]\,, \tag{21}$$

$$2\rho^{2-n}q_n(\rho^2; \delta^2) = F_{n,u}[(2\pi)^{-n/2}\phi(u)(u\delta)^{1-n/2}J_{n/2-1}(\delta u); \rho]\,, \tag{22 i}$$

the latter with dual

$$F_{n,\rho}[(2\pi)^{-n/2}2\rho^{2-n}q_n(\rho^2, \delta^2); s] = \phi(s)(s\delta)^{1-n/2}J_{n/2-1}(s\delta)\,, \tag{22 ii}$$

showing again the factorization into a factor of noncentrality and a factor resulting from the application of the F transform. In case x is spherical Laplace distributed, then

$$\begin{aligned} 2q_n(\rho^2; \delta^2) &= (\rho/\delta)^{n/2-1}I_{n/2-1}(\rho)K_{n/2-1}(\delta)\,, \qquad && 0 < \rho < \delta\,, \\ &= (\rho/\delta)^{n/2-1}I_{n/2-1}(\delta)K_{n/2-1}(\rho)\,, && \rho \geqq \delta\,, \end{aligned}$$

a result already given by Thomas [8].

5. Recursive relationships. Taking the derivative of (12) with respect to R^2 yields (19). If x is normally distributed, integrating (19) by parts yields

$$\begin{aligned} 2p_n(\rho^2; \delta^2) &= (\rho/\delta)^{n/2-1} \int_0^\infty \exp(-u^2/2)d[J_{n/2-1}/\delta u]J_{n/2-1}(\rho u)]\, du \\ &= Q_{n-2}(\rho^2; \delta^2) - Q_n(\rho^2; \delta^2)\,, \end{aligned} \tag{23}$$

which establishes a recursive relationship between the Q_n; this relationship, however, involves q_n. If x is not normally distributed, integration by part does not lead to simplifications.

In (12), $\phi(u)$ does not depend on the dimension n and it is possible to establish recursive relationships between integrals of type $\int \phi(u)u^{-k}J_r(\delta u)J_s(Ru)\, du$, namely, one may express the Bessel's functions $J_{n/2}(Ru)$, $J_{n/2-1}(\delta u)$, as well as their product, as functions of Lommel polynomials in u^{-1} and Bessel functions of order 0 and 1 if n is even, and of order $\frac{1}{2}$ and $-\frac{1}{2}$ if n is odd. Hence, $Q_n(R^2; \delta^2)$ can be expressed as a sum of integrals of type $\int \phi(u)u^{-m} \sin(\delta u) \sin(Ru)\, du$, $\int \phi(u)u^{-m} \operatorname{trig}(\delta u) \operatorname{trig}(Ru)\, du$, (where trig z stands for $\sin z$ and/or $\cos z$) if n is odd, and $\int \phi(u)u^{-m}J_i(\delta u)J_j(Ru)\, du$, $i, j = 0$ or 1, if n is even.

Other recursive relationships involving differentiation can be based on the relationship

$$J_p(uv)u^{-p} = (-2/v)^m d^m[J_{p-m}(uv)/u^{(p-m)}]/d(u^2)^m\,.$$

If n is even ($n = 2m + 2$) these lead to

$$\rho^{-n/2}Q_n(\rho^2; \delta^2) = (-2d/d\delta^2)^{n/2-1} \int_0^\infty s^{1-n/2}\phi(s)J_{n/2}(\rho s)J_0(\delta s)\, ds\,,$$

$$\rho^{-n}Q_n(\rho^2; \delta^2) = 2^{n-1}(d/d\delta^2)^{n/2-1}(d/d\rho^2)^{n/2} \int_0^\infty s^{1-n}\phi(s)J_0(\delta s)J_0(\rho s)\, ds\,.$$

If n is odd ($n = 2m + 1$) they lead to

$$\rho^{-n/2}Q_n(\rho^2; \delta^2) = (2/\pi)^{\frac{1}{2}}(-2d/d\delta^2)^{(n-1)/2} \int_0^\infty s^{-n/2}\phi(s)J_{n/2}(\rho s)\cos(s\delta)\, ds\,,$$

$$\rho^{-n}Q_n(\rho^2; \delta^2) = -(2/\pi)2^n(d/d\delta^2)^{(n-1)/2}(d/d\rho^2)^{(n+1)/2} \int_0^\infty s^{-(n+1)}\phi(s)\cos(\rho s)\cos(s\delta)\, ds\,.$$

6. The expected value of the intersection. In view of (4) and (12) the expected value of the intersection, if it exists, is

$$\mu_n = \int_0^\infty (R/\delta)^{(n-2)/2}R \int_{S_0} \phi(u)J_{n/2-1}(\delta u)J_{n/2}(Ru)\, du\, dm\,. \tag{24}$$

Using polar coodinates $l, \theta, \varphi_1, \cdots, \varphi_{n-2}$ ($0 \leqq \theta \leqq 2\pi$, $0 \leqq \varphi_i \leqq \pi$) for m and integrating out $\theta, \varphi_2, \cdots, \varphi_{n-2}$ yields

$$\mu_n = 2\pi^{(n-1)/2} \int_0^\pi \int_0^\pi Q_n(R^2; \theta^2)l^{n-1} \sin^{n-2}\varphi_1\, dl\, d\varphi\,,$$

where $\delta^2 = l^2 + |\zeta|^2 - 2l|\zeta|\cos\varphi_1$. Interchanging the order of integration with respect to φ_1 and s, which is permissible, and using the formula

$$\int_0^\pi J_\lambda[(R^2 + r^2 - 2Rr\cos\varphi)^{\frac{1}{2}}](R^2 + r^2 - 2Rr\cos\varphi)^{-d/2}\sin^{2d}\varphi\, d\varphi$$
$$= 2^\lambda\Gamma(\lambda + \tfrac{1}{2})\Gamma(\tfrac{1}{2})J_\lambda(R)J_\lambda(r)R^{-\lambda}r^{-\lambda}$$

gives

$$\mu_n = (2\pi R/|\zeta|)^{n/2}|\zeta| \int_0^\pi \int_0^\infty \phi(s)J_{n/2}(Rs)_{(n/2)-1}(|\zeta|s)J_{(n/2)-1}(ls)s^{1-n/2}\, ds l^{n/2}\, dl$$

and

$$\mu_n = (2\pi R/|\zeta|)^{n/2}|\zeta| \int_0^\infty \phi(s)J_{n/2}(Rs)J_{(n/2)-1}(|\zeta|s)J_{n/2}(rs)s^{-n/2}\, ds\,, \tag{25}$$

which shows μ_n as the result of a linear transform on ϕ or on $f(\xi)$, and is amenable to numerical computations, or may be expanded through the use of Bessel functions recursive formulas. It can be shown directly or by a passage to the limit that the volume $V[S_0 \cap S]$ of the intersection of the two spheres is given by

$$V[S_0 \cap S] = (2\pi Rr/|\xi|)^{n/2}|\xi| \int_0^\infty J_{n/2}(Rs)J_{(n/2)-1}(|\xi|s)J_{n/2}(rs)s^{-n/2}\, ds\,. \tag{26}$$

An alternative approach consists in taking directly the expected value of this volume; it leads readily to (25).

In (26) taking the derivative of V with respect to $|\xi|$ yields

$$\begin{aligned} dV/d|\xi| &= -(2\pi Rr/|\xi|)^{n/2}|\xi|^{\frac{1}{2}}H_{n/2}[J_{n/2}(Rs)J_{n/2}(rs)s^{-(n-1)/2}; |\xi|] \\ &= (2\pi^{\frac{1}{2}}/|\xi|)^{n-1}[|\xi^2| - (R-r)^2]^{(n-1)/2}[(R+r)^2 - |\xi|^2]^{(n-1)/2}\Big/\Gamma\left(\frac{n+1}{2}\right), \end{aligned} \tag{27}$$

which is the area of the polar cap cut out of S by S_0. Recursive relationships may be obtained for $V(S_0 \cap S)$ by integrating $dV/d|\xi|$ as expressed by (27) and, consequently, for the expected value of that volume.

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