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CONTROL CHARTS BASED ON WEIGHTED SUMS[1]

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In a continuous production process, samples of fixed size are taken at regular intervals of time and a statistic X_n is computed from the nth sample, $n = 1, 2, \cdots$. In this paper, we study process inspection schemes which stop the production and take corrective action with $N =$ first $n \geqq 1$ such that $\sum_{i=1}^{n} c_{n-i} X_i \geqq h$, where h is a preassigned constant and $c_0 \geqq c_1 \geqq \cdots \geqq c_{k-1} > 0 = c_k = c_{k+1} = \cdots$ is a suitably chosen sequence of weights. The average run length of such procedures is examined, and in the normal case, numerical comparisons with the average run length of the usual Shewhart Chart are given. In connection with the normal case, the first passage times of more general Gaussian sequences are studied and an asymptotic theorem is obtained. The first passage time N for more general weighted sums, where the sequence (c_n) is not assumed to be eventually zero but is assumed to be at least square summable, is also considered.

1. Introduction. An important industrial application of statistics lies in the area of quality control. In a continuous production process, the quality of the output may be assessed by some characteristic (e.g., the mean life of light bulbs or the fraction of defectives of the output), and we may assign a quality number θ to the characteristic. We are interested in detecting the change in θ once the production process gets out of control. A widely used process inspection scheme is the Shewhart control chart (see [13]), where samples of fixed size are taken at regular intervals of time and a statistic of the sample (e.g., the mean or the number of defectives) is plotted on the chart. If the sample point falls outside the control limits drawn on the chart, rectifying action is taken. This scheme is sometimes referred to as a *single-sample* scheme, since the decision whether or not to take action is based on a single point on the chart. Although the results of previous samples are recorded on the chart, none is used by this single-sample scheme.

It is conceivable that by a judicious use of observations in the immediate past, we may be able to achieve greater efficiency in the process inspection scheme. One method which uses previous observations is suggested by Dudding and Jennett [4] and has been occasionally used. Warning lines within the control limits are drawn on the chart and rectifying action is taken if any point falls outside the control limits or if k out of a sequence of n consecutive points fall outside the warning lines, where k and n are preassigned integers. Such schemes are called *control charts with warning lines.* Performance characteristics of these

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schemes have been studied by Page [12] in the case $k = 2$ and any n and also in the case $k = n$.

In utilizing previous observations for the detection of the lack of control, we have to ensure that the "good old days" of the machine would not outweigh its present misery. To illustrate this, suppose we want to control the mean θ of a normal population $N(\theta, \sigma^2)$, where θ may be changing over time, and production is out of control if $\theta > \theta_0$. Suppose one uses cumulative sum $S_n = Z_1 + \cdots + Z_n$ based on all the observations $Z_1, \cdots, Z_n$ collected so far (since action was last taken) and takes corrective action if $\bar{Z}_n = S_n/n$ falls beyond $\theta_0 + K\sigma/n^{\frac{1}{2}}$. Suppose Z_i has mean $\theta_0 - 1$ for $i \leqq \nu$ and then the mean leaps above θ_0. If ν is large, it may take very long to detect the change in θ. To remedy this, Page [11] proposed to consider $S_n' - \min_{0 \leqq i \leqq n} S_i'$ instead and to take rectifying action as soon as $S_n' - \min_{0 \leqq i \leqq n} S_i' \geqq h$, where $S_n' = \sum_1^n (Z_i - \theta_0)$ and h is a suitably chosen positive constant. By subtracting $\min_{0 \leqq i \leqq n} S_i'$ from S_n', the effect of the good old days is somehow eliminated.

Another way to avoid the shortcoming of the S_n procedure referred to in the preceding paragraph is to sum a certain preassigned segment of the past instead of the entire past, or more generally to take a weighted sum of the entire past putting most weight on the immediate past and zero weight on the remote past. Page ([11], page 100) has pointed out that the performance of such rules based on moving averages is generally difficult to evaluate. In this paper we shall study the performance of this class of rules. Section 2 deals with bounds on the average run length and Section 3 treats the normal case in detail. Section 4 examines the asymptotic behavior of the first passage times for a class of Gaussian sequences related to Section 3, while Section 5 considers the problem of first passage times for more general weighted sums.

2. The average run length. Either in the Shewhart control charts, or in the Page cusum charts, or in the moving average inspection schemes of the preceding section, there is probability one that some point will eventually fall outside the control limit and action will be taken even though θ has constantly remained in control. The S_n procedure referred to in the previous section does not have this property when θ remains $< \theta_0$, but the detection of the deterioration of θ into the out-of-control state becomes slow, particularly when the deterioration occurs late. To achieve quick detection of the deterioration of θ irrespective of when such a deterioration occurs we have to tolerate the property of probability one of false alarms.

In evaluating the performance of process inspection schemes Aroian and Levene [1] propose the use of the *average efficiency number*, now more commonly known as the *average run length* (abbreviated ARL), which is the expected number $E_\theta T$ of articles sampled before action is taken when the quality level has remained constantly at θ. When the quality level θ is satisfactory we want $E_\theta T$ to be large. When the quality level θ is poor we want $E_\theta T$ to be small. For procedures whose performance is not or little affected by when (i.e., at what time point) θ becomes out of control, the ARL is a good measure to evaluate these procedures. In this connection it should be pointed out that for procedures whose speed of detection depends heavily on when the change in θ occurs, such as the S_n procedure referred to in the preceding section, the ARL does not reflect this dependence, since in the ARL we assume that θ stays constantly at a fixed quality level.

We now study the ARL of the moving average inspection schemes. Suppose production is out of control when the quality level θ exceeds θ_0. Samples $B_1, B_2, \cdots$ of size m are taken successively at regular intervals of time since the previous rectifying action, and a statistic $X_i = f(B_i)$ is computed from B_i. If θ is the quality level when the sample B_i is taken, then X_i has distribution function F_θ, and we assume that the family $\{F_\theta, -\infty < \theta < \infty\}$ is stochastically increasing. Choose a sequence of weights $c_0 \geqq c_1 \geqq \cdots \geqq c_{k-1} > 0 = c_k = c_{k+1} = \cdots$. Stop the production and take corrective action with $N =$ first $n \geqq 1$ such that $\sum_{i=1}^n c_{n-i} X_i \geqq h$, where h is a suitably chosen constant.

Since the family $\{F_\theta\}$ is stochastically increasing, the average run length $mE_\theta N$ is a decreasing function in θ. We want $E_\theta N$ to be large for $\theta \leqq \theta_0$, say $E_\theta N \geqq K$ if $\theta \leqq \theta_0$, where K is a preassigned number. Subject to this constraint, we desire $E_\theta N$ to decrease rapidly for $\theta > \theta_0$.

Let $Y_n = \sum_{i=1}^n c_{n-i} X_i$, where $X_1, X_2, \cdots$ are i.i.d. with a common distribution function F_θ. Unlike the sequence $\sum_{i=1}^n c_i X_i$, the sequence Y_n is not a Markov chain, nor does it admit a Wald-type argument to study first passage times. We have shown in [2] that in a limiting sense, the behavior of the sequence Y_n resembles that of a sequence of independent random variables. Such limiting behavior manifests itself in the asymptotic behavior of the ARL, as we shall see in the later sections.

With the above remark in mind, we now find a lower bound for $E_\theta N$. The random variable Y_n can be written as a function of $(X_1, \cdots, X_n)$ which is nondecreasing in each argument. Therefore it follows from Theorem 5.1 of [5] that

$$P_\theta[Y_1 < h, \cdots, Y_n < h] \geqq \prod_{i=1}^n P_\theta[Y_i < h]\,. \tag{1}$$

Let $p_i = P_\theta[Y_i < h]$ for $i = 1, \cdots, k$. We note that for $j > k$, $P_\theta[Y_j < h] = p_k$. Hence (1) implies that

$$\begin{aligned} E_\theta N &= \sum_{n=0}^\infty P_\theta[N > n] \\ &\geqq 1 + p_1 + p_1 p_2 + \cdots + (1 - p_k)^{-1} p_1 \cdots p_{k-1} \qquad (k > 1)\,. \end{aligned} \tag{2}$$

An upper bound for $E_\theta N$ can be easily obtained by noting that for $n > k$,

$$\begin{aligned} P_\theta[Y_1 < h, \cdots, Y_n < h] &\leqq P_\theta[Y_1 < h, \cdots, Y_{n-k} < h, Y_n < h] \\ &= p_k P_\theta[Y_1 < h, \cdots, Y_{n-k} < h]\,. \end{aligned}$$

From this we obtain that

$$E_\theta N \leqq (1 - p_k)^{-1}\{1 + \textstyle\sum_{i=1}^{k-1} P_\theta[Y_1 < h, \cdots, Y_i < h]\}\,. \tag{3}$$

Setting $p_0 = 1$, we have therefore shown that

$$p_0 \cdots p_{k-1}(1 - p_k)^{-1} \leqq E_\theta N \leqq k(1 - p_k)^{-1} . \tag{4}$$

The above upper and lower bounds of $E_\theta N$ are thus multiples of $(1 - p_k)^{-1}$, and they are equal when $k = 1$. In the following section, by sharpening these arguments in the normal case, we shall improve inequalities (2) and (3) and obtain upper and lower bounds which differ little from each other and which are asymptotically equivalent.

3. The normal case. Suppose $X_1, X_2, \cdots$ are i.i.d. $N(\theta, \sigma^2)$. For simplicity, instead of considering the stopping rule N, let us consider

$$N_k = \inf\{n \geqq k : c_{k-1}X_{n-k+1} + \cdots + c_0 X_n \geqq h\} .$$

Let $h_\theta = h - \sum_{i=0}^{k-1} c_i \theta$, $Y_n' = \sum_{i=0}^{k-1} c_i(X_{n-i} - \theta)$. Then $Y_k', Y_{k+1}', \cdots$ is a stationary Gaussian sequence with means 0 and $\operatorname{Cov}(Y_n', Y_{n+\alpha}') = 0$ if $\alpha \geqq k$, and $= \sigma^2 \sum_{i=0}^{k-1-\alpha} c_i c_{i+\alpha}$ if $0 \leqq \alpha < k$ and $n \geqq k$. It is well known (cf. [3], [14]) that if $\varphi(x_1, \cdots, x_n)$ is the density function of the multivariate normal distribution with means 0, variances 1 and correlation matrix (r_{ij}), then

$$\frac{\partial\varphi}{\partial r_{ij}} = \frac{\partial^2\varphi}{\partial x_i\, \partial x_j}, \qquad i \neq j . \tag{5}$$

Let $Q_n = P[Y_k' < h_\theta, \cdots, Y_{n-1}' < h_\theta, Y_n' \geqq h_\theta]$, $n \geqq 2k$. It follows easily from (5) that Q_n is a non-decreasing function of λ_{ij} for $k \leqq i < j \leqq n - 1$, where λ_{ij} is the correlation coefficient between Y_i' and Y_j'. Since $\operatorname{Cov}(Y_i', Y_j') \geqq 0$, we obtain that

$$Q_n \geqq P[Y_k' < h_\theta, \cdots, Y_{n-k}' < h_\theta]P[Y_{n-k+1}' < h_\theta, \cdots, Y_n' \geqq h_\theta] . \tag{6}$$

Let $\lambda_0(x) = 1$, $\lambda_i(x) = P[Y_{k+1}' < x, \cdots, Y_{k+i}' < x]$ $(i \geqq 1)$ and $\rho_i(x) = \lambda_{i-1}(x) - \lambda_i(x)$. We now proceed to prove

$$k(1 - \lambda_k(h_\theta))^{-1} \leqq E_\theta N_k \leqq k + (\lambda_k(h_\theta)/\rho_k(h_\theta)) . \tag{7}$$

To obtain the upper bound in (7), we write $a_n = P[Y_k' < h_\theta, \cdots, Y_n' < h_\theta]$, $n \geqq k$. Then for $n \geqq 2k$, it follows from (6) that

$$a_n = a_{n-1} - Q_n \leqq a_{n-1} - a_{n-k}\rho_k(h_\theta) .$$

Therefore $\sum_{n=2k}^\infty a_n \leqq \sum_{n=2k-1}^\infty a_n - \sum_{n=k}^\infty a_n \rho_k(h_\theta)$, implying that

$$E_\theta N_k = k + \sum_{n=k}^\infty a_n \leqq k + (a_{2k-1}/\rho_k(h_\theta)) = k + (\lambda_k(h_\theta)/\rho_k(h_\theta)) .$$

To obtain the lower bound in (7), for $n = 1, 2, \cdots$, we define the events $A_n = \{Y_j' < h_\theta \text{ for all } nk \leqq j < (n + 1)k\}$, and let A_n^c denote the complement of A_n. Define $M = \inf\{n \geqq 1 : A_n^c \text{ occurs}\}$. Since $P[Y_k' < h_\theta, \cdots, Y_n' < h_\theta]$ is a non-decreasing function of the correlation coefficient λ_{ij} between Y_i' and Y_j' for $k \leqq i < j < n$, it follows that

$$P(\textstyle\bigcap_{i=1}^n A_i) \geqq \prod_{i=1}^n PA_i = (\lambda_k(h_\theta))^n ,$$

and so $EM = 1 + \sum_{n=1}^{\infty} P(\bigcap_{i=1}^{n} A_i) \geqq (1 - \lambda_k(h_\theta))^{-1}$. The desired conclusion then follows since $N_k \geqq kM$.

The upper and lower bounds in (7) are equal when $k = 1$. As $h_\theta \to -\infty$, obviously $E_\theta N_k \to k$ and it is easy to see that both bounds in (7) also tend to k. As $h_\theta \to \infty$, both bounds in (7) are of the order $\{1 - \Phi(h_\theta(\sum_{i=0}^{k-1} c_i^2\sigma^2)^{-\frac{1}{2}})\}^{-1}$, where Φ is the distribution function of the standard normal distribution, and therefore $E_\theta N$ is of the same order. Hence the bounds in (7) are asymptotically sharp.

When $k = 2$, by using tables of the bivariate normal distribution function (see [8]), the upper and lower bounds in (7) can be readily computed. The following table gives some numerical values in the case $\sigma = 1$, $c_0 = c_1 = 1$:

TABLE 1

$h_\theta/2^{\frac{1}{2}}$	Upper bound of $E_\theta N_k$	Lower bound of $E_\theta N_k$
3.5	4425	4357
3	787.4	763.4
2.5	180.3	170.2
2	53.26	48.25
1	9.75	7.84
0	4	3
−1	2.65	2.13

We note that the upper and lower bounds in Table 1 differ little from each other.

In controlling the mean of the normal population $N(\theta, \tau^2)$ where production is out of control when $\theta > \theta_0$, samples $B_1 = \{Z_1^{(1)}, \cdots, Z_m^{(1)}\}$, $B_2 = \{Z_1^{(2)}, \cdots, Z_m^{(2)}\}$, $\cdots$ are taken successively at regular intervals of time and we compute the sample mean $X_i = \sum_{j=1}^{m} Z_j^{(i)}/m$ at each stage. A widely used Shewhart chart is to take rectifying action at stage $N^* = \inf\{n \geqq 1 : X_n - \theta_0 \geqq 3\tau/m^{\frac{1}{2}}\}$. For $\theta \leqq \theta_0$, we have $E_\theta N^* \geqq 740.7$. Defining

$$N_k = \inf\{n \geqq 2 : X_{n-1} + X_n - 2\theta_0 \geqq 3\tau(2/m)^{\frac{1}{2}}\},$$

we find from Table 1 that $E_\theta N_k \geqq 763.4$ for $\theta \leqq \theta_0$. The following table gives a numerical comparison between $E_\theta N_k$ and $E_\theta N^*$ for $\theta > \theta_0$:

TABLE 2

$m^{\frac{1}{2}}(\theta - \theta_0)/\tau$	$E_\theta N^*$	Upper bound of $E_\theta N_k$
$2^{\frac{1}{2}}/4$	248.5	180.3
$2^{\frac{1}{2}}/2$	91.5	53.26
$2^{\frac{1}{2}}$	17.7	9.75
$3(2)^{\frac{1}{2}}/2$	5.28	4
$2(2)^{\frac{1}{2}}$	2.31	2.65

In [15], Weiler discussed the choice of the sample size m when the control limit of the Shewhart chart is specified beforehand in order that a deterioration of given size from θ_0 can be detected as soon as possible. He pointed out that larger samples than usually taken in industry should be used when it is important to detect small changes in the mean, although small frequent samples of the sort usually taken in industry may be satisfactory for detecting large changes. In [10], Page considers the problem of choosing both the sample size m and the control limit in the Shewhart chart, and his results support Weiler's advocacy of large m and show further that it is often advantageous to set the control limit lower than is usual in industrial practice. The following table shows that the ARL of our procedure N_k compares rather closely with that of the Shewhart chart taking samples of size $2m$ at each stage and stopping the production at stage $N' = \inf\{n \geqq 1 : X_n' - \theta_0 \geqq 2.79\tau/(2m)^{\frac{1}{2}}\}$, where X_n' is the mean of the $2m$ observations taken at the nth stage. We note that while $mE_\theta N_k$ is the ARL of the N_k procedure, the ARL of the above Shewhart chart is $2mE_\theta N'$. Now $2E_\theta N' \geqq 763.4$ for $\theta \leqq \theta_0$ and the values of $2E_\theta N'$ for $\theta > \theta_0$ are compared with those of $E_\theta N_k$ below:

TABLE 3

$m^{\frac{1}{2}}(\theta - \theta_0)/\tau$	$2E_\theta N'$	Upper bound of $E_\theta N_k$
$2^{\frac{1}{2}}/4$	181.7	180.3
$2^{\frac{1}{2}}/2$	54.5	53.26
$2^{\frac{1}{2}}$	9.31	9.75
$3(2)^{\frac{1}{2}}/2$	3.43	4
$2(2)^{\frac{1}{2}}$	2.26	2.65

It should be noted as pointed out by Page [12], that although the theoretically better sample size for the Shewhart chart is large, it may not be practically convenient to take such samples; furthermore, because of the cost in man-hours for quality inspection, there are economic advantages in using small samples taken frequently. The same view is also shared by Weiler ([15], page 254) who points out that in practice, other factors than the ARL would favor smaller samples.

The choice of k in the moving average schemes depends on the sample size m at each stage and how frequently the samples are collected, as well as on the variance σ^2. When $k = 3$, one can use tables of the trivariate normal distribution function ([9], pages 207–224) to compute upper and lower bounds of $E_\theta N_k$. For example, when $\sigma = 1$, $c_0 = c_1 = c_2 = 1$, $h_\theta = 0$, we find from (7) that $4.1 \leqq E_\theta N_k \leqq 5.84$. For higher values of k, though tables for our purpose are not readily available, methods for the numerical evaluation of the multivariate normal integral have been extensively studied, and Gupta's bibliography [6] contains a detailed account of the work along these lines up to 1963. In a recent paper, Milton [16] has given an algorithm for the computer evaluation of the multivariate normal integral for arbitrary mean vector, covariance matrix and region of integration. His method is based on a modification of a multi-dimensional adaptive Simpson's quadrature with error control, applied to the iterated integral. For the ease of computation, however, the following cruder bounds can be easily calculated from the available tables and can be used to give us some feeling of what $E_\theta N_k$ looks like:

$$k - 1 + (1 - \psi(h_\theta))^{-1} \leqq E_\theta N_k \leqq \min\{k(1 - \psi(h_\theta))^{-1}, k + \eta_k \gamma(h_\theta)\} \tag{8}$$

where η_k is the smallest integer $\geqq k/2$,

$$\psi(x) = P[V_1 < x],\ \gamma(x) = P[V_1 < x, V_2 < x]/P[V_1 < x, V_2 \geqq x],$$

and (V_1, V_2) have a bivariate normal distribution with means 0,

$$\operatorname{Var} V_1 = \operatorname{Var} V_2 = \sigma^2 \textstyle\sum_{i=0}^{k-1} c_i^2,$$
$$\operatorname{Cov}(V_1, V_2) = \sigma^2 \textstyle\sum_{i=0}^{k-1-\eta_k} c_i c_{i+\eta_k}.$$

To prove (8), we define $M_1 = \inf\{j \geqq 1 : Y'_{jk} \geqq h_\theta\}$, $M_2 = \inf\{i \geqq 0 : Y'_{k+i\eta_k} \geqq h_\theta\}$. Then $N_k \leqq kM_1$ and $N_k \leqq k + \eta_k M_2$. Obviously $EM_1 = (1 - \psi(h_\theta))^{-1}$, and similar arguments as in the proof of the upper bound of (7) give us that $EM_2 \leqq 1 + \gamma(h_\theta)$. To obtain the lower bound in (8), we note that

$$\begin{aligned} E_\theta N_k &= k + \textstyle\sum_{n=k}^\infty P[N_k > n] \\ &\geqq k + \textstyle\sum_{n=k}^\infty P[Y_k' < h_\theta] \cdots P[Y_n' < h_\theta] \\ &= k - 1 + (1 - \psi(h_\theta))^{-1}. \end{aligned}$$

We note that the lower bound in (8) is asymptotic to $E_\theta N_k$ as $h_\theta \to \infty$ or $h_\theta \to -\infty$.

4. An asymptotic theorem. In the preceding section, we have seen from (7) that

$$E_\theta N_k \sim \{1 - \Phi(h_\theta(\textstyle\sum_{i=0}^{k-1} c_i^2\sigma^2)^{-\frac{1}{2}})\}^{-1} \qquad \text{as } h_\theta \to \infty.$$

Actually this is a special case of the theorem below on the asymptotic behavior of first passage times for Gaussian sequences. A function $R(n)$ on $\{0, 1, 2, \cdots\}$ is said to be nonnegative definite if for any finite subset $\{x_1, \cdots, x_k\}$ of real numbers, $\sum_{i=1}^k \sum_{j=1}^k x_i x_j R(|i - j|) \geqq 0$. As is well known, such a function corresponds to the covariance of a stationary Gaussian sequence. Obviously, if R is nonnegative definite, then so is the function R_ε for any $0 \leqq \varepsilon < 1$, where $R_\varepsilon(0) = R(0)$ and $R_\varepsilon(i) = \varepsilon R(i)$ for $i \geqq 1$.

THEOREM. *Let $Y_1, Y_2, \cdots$ be a Gaussian sequence with $EY_i = 0$, $EY_iY_j = r_{ij}$ and $\lim_{i\to\infty} r_{ii} = \sigma^2 > 0$. For any real number c, define*

$$N(c) = \inf\{n \geqq 1 : Y_n \geqq c\}. \tag{9}$$

(i) *If $\lim_{n\to\infty} \sup_{j-i\geqq n; i\geqq i_0} r_{ij} \leqq 0$, then for $\nu = 1, 2, \cdots$, $EN^\nu(c) < \infty$ and*

$$\eta > 1/(2\sigma^2) \Rightarrow EN^\nu(c) = o(\exp(\nu\eta c^2)) \qquad \textit{as } c \to \infty. \tag{10}$$

(ii) *If* $r_{ij} \geqq 0$ *for all* $j \geqq i \geqq i_0$, *then for* $\nu = 1, 2, \cdots$,

$$\eta < 1/(2\sigma^2) \Rightarrow \lim_{c\to\infty} (\exp(\nu\eta c^2))/EN^\nu(c) = 0 . \tag{11}$$

(iii) *Suppose there exist* $\varepsilon < 1$ *and a nonnegative definite function* $R(n)$ *such that* $\sum_0^\infty R(n) < \infty$, $\sigma^2 = R(0) > R(1) \geqq R(2) \geqq \cdots \geqq 0$ *and* $0 \leqq r_{ij} \leqq \varepsilon R(j-i)$ *for* $j > i \geqq i_0$. *If* $r_{ii} = \sigma^2$ *for* $i \geqq i_0$, *then for* $\nu = 1, 2, \cdots$,

$$EN^\nu(c) \sim \nu!\, (2\pi)^{\nu/2}(c/\sigma)^\nu \exp(\nu c^2/(2\sigma^2)) \qquad \text{as } c \to \infty . \tag{12}$$

LEMMA 1. *Let* $(X_1, \cdots, X_n)$, $(Y_1, \cdots, Y_n)$ *have multivariate normal distributions with* $EX_i = EY_i$ *and* Var $X_i \leqq$ Var Y_i *for all* i, *and* Cov $(X_i, X_j) \geqq$ Cov (Y_i, Y_j) *for all* $i \neq j$. *Then given any real numbers* $c_1, \cdots, c_n$,

$$P[X_1 < c_1, \cdots, X_n < c_n] \geqq P[Y_1 < c_1, \cdots, Y_n < c_n] .$$

PROOF OF THEOREM. Without loss of generality, we shall assume throughout the proof that $\sigma = 1$. Also we can assume without loss of generality that $i_0 = 1$, for otherwise we can consider $\tilde{N}(c)$ instead, where $\tilde{N}(c) = \inf\{n \geqq i_0 : Y_n \geqq c\}$. To prove (i), given $\gamma\varepsilon(0, \frac{1}{3})$, we can choose $n_0 \geqq 1$ such that $\sup_{j-i\geqq n_0} r_{ij} < \gamma$. Let $\{U, \tilde{U}_1, \tilde{U}_2, \cdots\}$ be a sequence of independent normal random variables with $EU = E\tilde{U}_i = 0$, $EU^2 = \gamma$ and $E\tilde{U}_i^2 = 1 - 2\gamma$. Set $U_i = U + \tilde{U}_i$. Then $EU_i^2 = 1 - \gamma$ and $EU_iU_j = \gamma$ for $i \neq j$. Let $r_{ii} > 1 - \gamma$ for $i \geqq i_1$, and by our earlier remark, we can assume $i_1 = 1$. Define $Y_n^* = Y_{nn_0}$, $n = 1, 2, \cdots$. Then Var $Y_n^* > 1 - \gamma =$ Var U_n and Cov $(Y_i^*, Y_j^*) < \gamma =$ Cov (U_i, U_j) for $i \neq j$. Hence defining $N^*(c) = \inf\{n \geqq 1 : Y_n^* \geqq c\}$, we have by Lemma 1 that

$$\begin{aligned} P[N^*(c) > n] &= P[Y_1^* < c, \cdots, Y_n^* < c] \\ &\leqq P[U_1 < c, \cdots, U_n < c] \\ &\leqq \textstyle\int_{-\infty}^\infty \Phi^n((1-2\gamma)^{-\frac{1}{2}}(c - \gamma^{\frac{1}{2}}x))\varphi(x)\, dx , \end{aligned} \tag{13}$$

where φ is the standard normal density.

We note that $N(c) \leqq n_0 N^*(c)$. From (13), it follows that

$$\begin{aligned} EN^*(c) &= \textstyle\sum_0^\infty P[N^*(c) > n] \\ &\leqq \textstyle\int_{-\infty}^\infty \{1 - \Phi((1-2\gamma)^{-\frac{1}{2}}(c - \gamma^{\frac{1}{2}}x))\}^{-1}\varphi(x)\, dx \\ &\leqq (1 - \Phi(2))^{-1} \textstyle\int_{(c-1)/\gamma^{\frac{1}{2}}}^\infty \varphi(x)\, dx \\ &\quad + 2(1-2\gamma)^{-\frac{1}{2}} \textstyle\int_{-\infty}^{(c-1)/\gamma^{\frac{1}{2}}} (c - \gamma^{\frac{1}{2}}x) \exp\left\{-\frac{x^2}{2} + \frac{(c-\gamma^{\frac{1}{2}}x)^2}{2(1-2\gamma)}\right\} dx . \end{aligned} \tag{14}$$

Since $\gamma < 1 - 2\gamma$, it is easy to see from (14) that $EN^*(c) < \infty$ and therefore $EN(c) < \infty$. Also given $\eta > \frac{1}{2}$, we can choose γ sufficiently small so that $2\eta > (1-2\gamma)^{-1} + \gamma(1-3\gamma)^{-2}$.

Then (14) implies that $EN(c) = o(e^{\eta c^2})$ as $c \to \infty$.

For $\nu = 2, 3, \cdots$, to prove $EN^\nu(c) < \infty$, we note that

$$\begin{aligned} EN^\nu(c) &= \textstyle\sum_1^\infty n^\nu P[N(c) = n] \\ &= 1 + \textstyle\sum_1^\infty \{\binom{\nu}{1}n^{\nu-1} + \cdots + \binom{\nu}{\nu-1}n + 1\}P[N(c) > n] . \end{aligned} \tag{15}$$

Also it follows from (13) that

$$\begin{aligned}
&\textstyle\sum_{n=\nu-1}^{\infty} n(n-1)\cdots(n-\nu+2)P[N^*(c)>n] \\
(16)\qquad &\leqq \textstyle\int_{-\infty}^{\infty}\sum_{n=\nu-1}^{\infty} n(n-1)\cdots(n-\nu+2) \\
&\qquad \times \Phi^{n-\nu+1}((1-2\gamma)^{-\frac12}(c-\gamma^{\frac12}x))\varphi(x)\,dx \\
&= (\nu-1)!\textstyle\int_{-\infty}^{\infty}\{1-\Phi((1-2\gamma)^{-\frac12}(c-\gamma^{\frac12}x))\}^{-\nu}\varphi(x)\,dx \\
&\equiv I(c;\gamma)\,, \quad \text{say.}
\end{aligned}$$

By a similar argument as before, $I(c;\gamma)<\infty$ if $\nu\gamma<1-2\gamma$. Also given $\eta>\frac12$, we can choose γ sufficiently small so that $I(c;\gamma)=o(\exp(\nu\eta c^2))$ as $c\to\infty$. Hence by induction, we obtain from (15) and (16) that for $\nu=1,2,\cdots, EN^\nu(c)<\infty$ and (10) holds.

Now assume that $r_{ij}\geqq 0$ for all i, j. By Lemma 1, we have

$$(17)\qquad P[Y_1<c,\cdots,Y_n<c]\geqq \textstyle\prod_{i=1}^n P[Y_i<c]\,.$$

First consider the case where $r_{ii}=1$ for $i\geqq 1$. In this case, (17) reduces to $P[Y_1<c,\cdots,Y_n<c]\geqq\Phi^n(c)$. Hence as $c\to\infty$,

$$(18)\qquad EN^\nu(c)\geqq(1+o(1))\nu!\,(2\pi)^{\nu/2}c^\nu\exp(\nu c^2/2)\,.$$

In the more general case where $\lim_{i\to\infty} r_{ii}=1$, (17) leads to the relation (11).

Now assume the conditions of (iii) with $i_0=1$, $\sigma=1$. We shall prove that as $c\to\infty$,

$$(19)\qquad EN^\nu(c)\leqq(1+o(1))\nu!\,(2\pi)^{\nu/2}c^\nu\exp(\nu c^2/2)\,.$$

Let $\varepsilon<\delta<1$ and let $X_1, X_2,\cdots$ be the stationary Gaussian sequence with means 0, variances 1 and Cov $(X_i,X_j)=\delta R(|i-j|)$ for $i\neq j$. Define

$$\begin{aligned}
&f_0(c)=g_0(c)=1 \\
(20)\qquad &f_n(c)=P[Y_1<c,\cdots,Y_n<c] \\
&g_n(c)=P[X_1<c,\cdots,X_n<c]\,, \qquad n\geqq 1 \\
&M(c)=\inf\{n\geqq 1: X_n\geqq c\}\,.
\end{aligned}$$

The conditional distribution of $(X_1,\cdots,X_{n-1})$ given $X_n=x$ is the same as the unconditional distribution of $(Z_1+\delta xR(n-1),\cdots,Z_{n-1}+\delta xR(1))$, where $(Z_1,\cdots,Z_{n-1})$ has a multivariate normal distribution with means 0, Var $Z_i=\lambda_{ii}=1-\delta^2R^2(n-i)$ and Cov $(Z_i,Z_j)=\lambda_{ij}=\delta R(j-i)-\delta^2R(n-i)R(n-j)$ for $j>i$. We obtain

$$\begin{aligned}
g_n(c)&\leqq g_{n-1}(c)-P[X_1<c,\cdots,X_{n-1}<c, c\leqq X_n\leqq c+1] \\
&=g_{n-1}(c) \\
(21)\qquad &\quad -\textstyle\int_c^{c+1}P[Z_i<c-\delta xR(n-i)\ \text{ for }\ i=1,\cdots,n-1]\varphi(x)\,dx \\
&\leqq g_{n-1}(c)-P[Z_i<c-\delta(c+1)R(n-i)\ \text{ for }\ i=1,\cdots,n-1] \\
&\quad \times P[c\leqq X_n\leqq c+1]\,.
\end{aligned}$$

Let $\tilde{Z}_i = \lambda_{ii}^{-\frac{1}{2}} Z_i$. Then $\operatorname{Var} \tilde{Z}_i = 1 = r_{ii}$. Since $\lim_{n\to\infty} R(n) = 0$, $r_{ij} \leqq \varepsilon R(j-i)$ and $R(j-i) \geqq R(n-i)$ for $n \geqq j > i$, it follows that we can choose $k \geqq 2$ such that

$$\text{(22a)} \qquad \operatorname{Cov}(\tilde{Z}_i, \tilde{Z}_j) \geqq r_{ij} \quad \text{whenever} \quad n-j \geqq k \quad \text{and} \quad j > i\,,$$

$$\text{(22b)} \qquad \delta R(i) \leqq \tfrac{1}{8} \quad \text{for} \quad i \geqq k\,.$$

By Lemma 1, we have for $n > k$

$$\begin{aligned} &P[Z_i < c - \delta(c+1)R(n-i) \text{ for } i = 1, \cdots, n-1] \\ &\qquad \geqq P[Z_i < c - \delta(c+1)R(n-i) \text{ for } i = 1, \cdots, n-k] \\ &\qquad\qquad \times \Phi^{k-1}(c - \delta(c+1)R(1)) \\ \text{(23)} \qquad &\qquad = P[\tilde{Z}_i < \lambda_{ii}^{-\frac{1}{2}}\{c - \delta(c+1)R(n-i)\} \text{ for } i = 1, \cdots, n-k] \\ &\qquad\qquad \times \Phi^{k-1}(c - \delta(c+1)R(1)) \\ &\qquad \geqq P[Y_i < \lambda_{ii}^{-\frac{1}{2}}\{c - \delta(c+1)R(n-i)\} \text{ for } i = 1, \cdots, n-k] \\ &\qquad\qquad \times \Phi^{k-1}(c - \delta(c+1)R(1))\,. \end{aligned}$$

From (21) and (23), we have for $n > k$

$$\begin{aligned} g_n(c) &\leqq g_{n-1}(c) - (\Phi(c+1) - \Phi(c))\Phi^{k-1}(c - \delta(c+1)R(1)) \\ \text{(24)} \qquad &\quad \times P[Y_i < (1 - \delta^2 R^2(n-i))^{-\frac{1}{2}} \\ &\quad \times \{c - \delta(c+1)R(n-i)\} \text{ for } i = 1, \cdots, n-k]\,. \end{aligned}$$

Given any integer $N \geqq 1$, take $\gamma_N > 0$ such that $(1+\gamma_N)^2 < (N+\frac{1}{2})/N$. Choose an integer $m \geqq 2$ such that $(1 - \varepsilon^2 R^2(m))^{-1} < \min(\delta/\varepsilon, 1+\gamma_N)$. Denoting

$$\zeta(c, j) = (1 - \delta^2 R^2(j))^{-\frac{1}{2}}\{c - \delta(c+1)R(j)\}\,,$$

we note that

$$\begin{aligned} &P[Y_i < \zeta(c, n-i) \text{ for } i = 1, \cdots, n-k] \\ \text{(25)} \qquad &\quad \geqq P[Y_i < c \text{ for } i = 1, \cdots, n-k] \\ &\qquad - \textstyle\sum_{j=1}^{n-k} P[\zeta(c, n-j) \leqq Y_j < c, \text{ and } Y_i < c \\ &\qquad\quad \text{for all } i = 1, \cdots, n-k \text{ with } |i-j| \geqq m]\,. \end{aligned}$$

For fixed $j = 1, \cdots, n-k$, the conditional distribuion of $\{Y_i : |i-j| \geqq m, i = 1, \cdots, n-k\}$ given $Y_j = y$ is the same as the unconditional distribution of $\{V_i + y r_{ij} : |i-j| \geqq m, i = 1, \cdots, n-k\}$ which has a normal distribution with $EV_i = 0$ and $\operatorname{Cov}(V_\alpha, V_\beta) = r_{\alpha\beta} - r_{\alpha j} r_{\beta j}$. Setting $\tilde{V}_i = (1 - r_{ij}^2)^{-\frac{1}{2}} V_i$, we have $\operatorname{Var} \tilde{V}_i = 1$ and for $\alpha \neq \beta$, $|\alpha - j| \geqq m$, $|\beta - j| \geqq m$,

$$\begin{aligned} \operatorname{Cov}(\tilde{V}_\alpha, \tilde{V}_\beta) &\leqq (1 - r_{\alpha j}^2)^{-\frac{1}{2}}(1 - r_{\beta j}^2)^{-\frac{1}{2}} r_{\alpha\beta} \\ &\leqq (1 - \varepsilon^2 R^2(m))^{-1} r_{\alpha\beta} < \delta R(|\alpha - \beta|)\,. \end{aligned}$$

Also $(1 - r_{\alpha j}^2)^{-\frac{1}{2}} \leqq (1 - \varepsilon^2 R^2(m))^{-\frac{1}{2}} < (1+\gamma_N)^{\frac{1}{2}} < 1 + \gamma_N$. Therefore using

Lemma 1, we obtain for $n > k + 2m$

$$
\begin{aligned}
&P[\zeta(c, n-j) \leqq Y_j < c, \text{ and } Y_i < c \\
&\qquad \text{for } i = 1, \cdots, n-k \text{ with } |i-j| \geqq m] \\
&\qquad = \int_{\zeta(c,n-j)}^{c} P[V_i < c - yr_{ij} \\
&\qquad\qquad \text{for } i = 1, \cdots, n-k \text{ with } |i-j| \geqq m]\varphi(y)\,dy \\
&\qquad \leqq \{\Phi(c) - \Phi(\zeta(c, n-j))\}P[\tilde{V}_i < (1-r_{ij}^2)^{-\frac{1}{2}}\{c - r_{ij}\zeta(c, n-i)\} \\
&\qquad\qquad \text{for } i = 1, \cdots, n-k \text{ with } |i-j| \geqq m] \\
&\qquad \leqq \{\Phi(c) - \Phi(\zeta(c, n-j))\}P[\tilde{V}_i < c(1+\gamma_N) \\
&\qquad\qquad \text{for } i = 1, \cdots, n-k \text{ with } |i-j| \geqq m] \\
&\qquad \leqq \{\Phi(c) - \Phi(\zeta(c, n-j))\}P[X_i < c(1+\gamma_N) \\
&\qquad\qquad \text{for } i = 1, \cdots, n-k \text{ with } |i-j| \geqq m] \\
&\qquad \leqq \{\Phi(c) - \Phi(\zeta(c, n-j))\} \\
&\qquad\qquad \times P[X_1 < c(1+\gamma_N), \cdots, X_{n-k-2m} < c(1+\gamma_N)]\,.
\end{aligned} \tag{26}
$$

The last inequality above follows from Lemma 1 and the fact that $R(n)$ is non-increasing in n. From (24), (25) and (26), we have for $n > k + 2m$

$$
\begin{aligned}
g_n(c) \leqq g_{n-1}(c) &- (\Phi(c+1) - \Phi(c))\Phi^{k-1}(c - \delta(c+1)R(1)) \\
&\times \{f_{n-k}(c) - \textstyle\sum_{j=1}^{n-k} (\Phi(c) - \Phi(\zeta(c, n-j))) \\
&\times g_{n-k-2m}(c(1+\gamma_N))\}\,.
\end{aligned} \tag{27}
$$

Using the inequality $\Phi(x) - \Phi(y) \leqq (\varphi(y) - \varphi(x))/y$ for $x \geqq y > 0$, together with the inequality $\zeta(c, i) \geqq c/2$ for $c \geqq 1$ and $i \geqq k$, we obtain for $c \geqq 1$,

$$
\begin{aligned}
&\textstyle\sum_{j=1}^{n-k} \{\Phi(c) - \Phi(\zeta(c, n-j))\} \\
&\qquad \leqq \textstyle\sum_{i=k}^{n-1} \{\varphi(\zeta(c, i)) - \varphi(c)\}/\zeta(c, i) \\
&\qquad = \varphi(c) \textstyle\sum_{i=k}^{n-1} \Big\{\exp\Big(\dfrac{\delta R(i)c^2}{1+\delta R(i)} - \dfrac{\delta^2 R^2(i)}{2(1-\delta^2 R^2(i))} \\
&\qquad\qquad + \dfrac{\delta R(i)c}{1+\delta R(i)}\Big) - 1\Big\}\Big/\zeta(c, i) \\
&\qquad \leqq \frac{2}{c}\varphi(c) \textstyle\sum_{i=k}^{n-1} \{\exp(2\delta R(i)c^2/(1+\delta R(i))) - 1\} \\
&\qquad \leqq \frac{2}{c}\varphi(c)e^{c^2/4} \textstyle\sum_{i=k}^{n-1} \{1 - \exp(-2\delta R(i)c^2/(1+\delta R(i)))\} \\
&\qquad \leqq (2/\pi)^{\frac{1}{2}} c e^{-c^2/4} \textstyle\sum_{i=k}^{n-1} 2\delta R(i)/(1+\delta R(i)) \leqq \rho c e^{-c^2/4}\,,
\end{aligned} \tag{28}
$$

where $\rho = (2/\pi)^{\frac{1}{2}} \sum_{i=k}^{\infty} 2\delta R(i)/(1+\delta R(i))$. Hence it follows from (27) and (28) that for $c \geqq 1$ and $n > k + 2m$,

$$
\begin{aligned}
g_n(c) \leqq g_{n-1}(c) &- (\Phi(c+1) - \Phi(c))\Phi^{k-1}(c - \delta(c+1)R(1))f_{n-k}(c) \\
&+ \rho c e^{-c^2/4}(\Phi(c+1) - \Phi(c))\Phi^{k-1}(c - \delta(c+1)R(1)) \\
&\times g_{n-k-2m}((1+\gamma_N)c)\,.
\end{aligned} \tag{29}
$$

Summing (29) over $n > k + 2m$, we obtain that as $c \to \infty$,

$$(30) \qquad (\Phi(c+1) - \Phi(c))\Phi^{k-1}(c - \delta(c+1)R(1)) \times \{EN(c) - \rho c e^{-c^2/4} EM((1+\gamma_N)c) + O(1)\} \leqq g_{k+2m}(c) .$$

We have proved that $EN(c) \geqq (1 + o(1))(2\pi)^{\frac{1}{2}} c \exp(c^2/2)$. By (10),

$$\rho c e^{-c^2/4} EM((1+\gamma_N)c) = o(ce^{c^2/2}) , \qquad \text{since} \quad \tfrac{1}{2} > (1+\gamma_N)^2/2 - \tfrac{1}{4} .$$

Therefore (30) implies that $(\Phi(c+1) - \Phi(c))EN(c) \leqq 1 + o(1)$, and so we have established (19) for $\nu = 1$. Proceeding inductively with similar arguments, it is easy to see from (29) together with (15), (18) and our result in (i) for $M((1+\gamma_N)c)$ that (19) holds for $\nu = 1, 2, \cdots, N$. Since N is arbitrary, (19) holds for all ν. □

As an application of the preceding theorem, let us consider the stationary Ornstein–Uhlenbeck process $U(t)$, $t \geqq 0$, which is stationary Gaussian with $EU(t) = 0$ and $\text{Cov}(U(t), U(s)) = \rho \exp(-\alpha|s - t|)$, $\rho > 0$, $\alpha > 0$. Let $Y_n = U(n)$, $n = 1, 2, \cdots$. Take $\beta \in (0, \alpha)$. Then for $n = 1, 2, \cdots$

$$\text{Cov}(Y_i, Y_{i+n}) = \rho e^{-\alpha n} \leqq e^{-(\alpha-\beta)} R(n)$$

where $R(j) = \rho e^{-\beta j}$ $(j = 0, 1, 2, \cdots)$. Hence the conditions of the preceding theorem are satisfied and so defining $N(c)$ by (9), we have

$$(31) \qquad EN(c) \sim (2\pi)^{\frac{1}{2}}(c/\rho^{\frac{1}{2}}) \exp(c^2/(2\rho)) \qquad \text{as } c \to \infty .$$

It should be pointed out that if we define $T(c) = \inf\{t \geqq 0 \colon U(t) \geqq c\}$, then

$$(32) \qquad ET(c) \sim \alpha^{-1}(2\pi)^{\frac{1}{2}}(c/\rho^{\frac{1}{2}})^{-1} \exp(c^2/(2\rho)) \qquad \text{as } c \to \infty$$

(cf. [7]). This shows that the mean first passage time of the discrete-time Gaussian sequence Y_n is not well approximated by that of the corresponding continuous-time process $U(t)$.

5. First passage times for more general weighted sums. For the quality control procedures in Sections 2 and 3, we have taken a weighted sum of the sample scores over a certain preassigned segment of the past, i.e., we have chosen a sequence of weights $a_0 \geqq a_1 \geqq \cdots \geqq a_{k-1} > 0 = a_k = a_{k+1} = \cdots$ and have considered

$$(33) \qquad N = \inf\{n \geqq 1 \colon \textstyle\sum_{i=1}^n a_{n-i} X_i \geqq c\} .$$

In this section, instead of summing over a preassigned segment of the past, we shall consider weighted sums of the entire past which put very little weight on the remote past. In other words, we shall drop the assumption that the sequence (a_n) of weights has to be eventually zero, but we shall assume that it is at least square summable.

First consider the case where $X_1, X_2, \cdots$ are i.i.d. $N(0, \tau^2)$. Let $(a_n, n \geqq 0)$ be a sequence of nonnegative numbers such that $0 < \sum_0^\infty a_n^2 < \infty$. Let $\sigma^2 = \tau^2 \sum_0^\infty a_n^2$, $Y_n = \sum_{i=1}^n a_{n-i} X_i$, and define N by (33). Then $\lim_{n\to\infty} EY_n^2 = \sigma^2$, and using the theorem in the preceding section, we obtain that $EN^\nu < \infty$ for $\nu = 1, 2, \cdots$, and that the asymptotic relations (10) and (11) hold.

Some interesting weighting sequences are:

$$a_n = \rho e^{-\alpha n}, \qquad \rho > 0, \quad \alpha > 0, \qquad n = 0, 1, 2, \cdots \tag{34}$$

$$a_n = (n+1)^{-\beta}, \quad \beta > \tfrac{1}{2}, \qquad n = 0, 1, 2, \cdots. \tag{35}$$

The sequence (34) satisfies $\sum a_n < \infty$, so that if $X_1, X_2, \cdots$ are i.i.d. $N(\mu, \tau^2)$ and the first passage time N is defined by (33), then $EN^\nu < \infty$ for any real number μ. On the other hand, for $\beta \leqq 1$, the sequence (35) has the property $\sum a_n = \infty$. In this case, if $X_1, X_2, \cdots$ are i.i.d. $N(\mu, \tau^2)$ and N is defined by (33), then $EN < \infty$ if $\mu \geqq 0$ and $EN = \infty$ if $\mu < 0$. In fact, for $\mu < 0$, $P[N = \infty] > 0$ (cf. [7]). For the purpose of quality control, where production is out of control if $\mu \geqq 0$, the sequence (35) with $\beta \leqq 1$ has the shortcoming of putting too much weight on the remote past. Like the S_n procedure discussed in Section 1, if X_i has a negative mean for $i \leqq m - 1$, then it may take very long after stage m to detect the change in θ if production is out of control at stage m and m is large.

Suppose $X_1, X_2, \cdots$ are i.i.d. but not necessarily normal random variables. Take the weighting sequence (34) and define N by (33). It is interesting to ask if EN is still finite. This is in general not true, but the following lemma gives a sufficient condition for it to hold.

LEMMA 2. *Suppose $X_1, X_2, \cdots$ are independent random variables such that $X_i \geqq -B$ a.e. for all i. Let $(a_n, n \geqq 0)$ be a sequence of nonnegative numbers such that $\infty > \sum_0^\infty a_i \geqq 1$. Suppose $\liminf_{n\to\infty} P[X_n \geqq d] > 0$ and $c < d$. Define $N = \inf\{n \geqq 1 : \sum_{i=1}^n a_{n-i} X_i > c\}$. Then there exist $\alpha \in (0, 1)$ and $\lambda > 0$ such that $P[N > n] \leqq \lambda \alpha^n$ for all n. Consequently $Ee^{\theta N} < \infty$ for all sufficiently small θ.*

PROOF. Let $c < d_1 < d$. Choose $k \geqq 1$ such that $c + B \sum_k^\infty a_i < d_1$ and $\sum_0^{k-1} a_i > d_1/d$. Suppose $P[X_n \geqq d] \geqq \beta > 0$ for all $n \geqq m$. Then for $n \geqq m + k$,

$$\begin{aligned} P[N > n] &\leqq P[\textstyle\sum_{i=1}^j a_{j-i} X_i \leqq c \ \text{ for } \ j = 1, \cdots, n-k, \\ &\qquad \text{and } \ \textstyle\sum_{i=n-k+1}^n a_{n-i} X_i \leqq c - \sum_{i=1}^{n-k} a_{n-i} X_i] \\ &\leqq P[\textstyle\sum_{i=1}^j a_{j-i} X_i \leqq c \ \text{ for } \ j = 1, \cdots, n-k] \\ &\qquad \times P[\textstyle\sum_{i=n-k+1}^n a_{n-i} X_i \leqq c + B \sum_k^{n-1} a_i] \\ &\leqq P[N > n-k] P[\textstyle\sum_{i=n-k+1}^n a_{n-i} X_i \leqq d_1]. \end{aligned} \tag{36}$$

Noting that $\sum_{i=n-k+1}^n a_{n-i} = \sum_0^{k-1} a_i > d_1/d$, we have for $n \geqq m + k$,

$$\begin{aligned} \beta^k &\leqq P[X_{n-k+1} \geqq d, \cdots, X_n \geqq d] \\ &\leqq P[\textstyle\sum_{i=n-k+1}^n a_{n-i} X_i > d_1]. \end{aligned} \tag{37}$$

From (36) and (37), we obtain for $n \geqq m + k$,

$$P[N > n] \leqq (1 - \beta^k) P[N > n-k]. \tag{38}$$

The desired conclusion then follows easily from (38). □

Suppose in Lemma 2 we drop the assumption that the random variables are uniformly bounded below. Then EN may fail to be finite, as shown by the following example. Define $n_1 = 2$ and for $k \geqq 2$, $n_k = 2^{n_{k-1}}$. Let $X_1, X_2, \cdots$ be i.i.d. random variables such that $P[X_1 = 1] = \frac{1}{2}$, $P[X_1 = -n_k] = 3/(\pi^2 k^2)$, $k = 1, 2, \cdots$. Consider the weighting sequence (34) with $\rho = 1$, $\alpha = \log 2$, i.e., $a_n = 2^{-n}$. Let $N = \inf\{n \geqq 1 : \sum_{i=1}^n a_{n-i} X_i \geqq 0\}$. Then

$$(39) \qquad P[N \geqq n_k] \geqq P[X_1 = -n_2, X_{n_1} = -n_3, X_{n_2} = -n_4, \cdots, X_{n_{k-1}} = -n_{k+1}] = (3/\pi^2)^k((k+1)!)^{-2}.$$

From (39) it is easy to see that $\sum P[N \geqq n] = \infty$ and so $EN = \infty$.

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