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ASYMPTOTIC SUFFICIENCY OF THE VECTOR OF RANKS IN THE BAHADUR SENSE[1]

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We shall consider the hypothesis of randomness under which two samples $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$ have an identical but arbitrary continuous distribution. The vector of ranks $(R_1, \cdots, R_{n+m})$ will be shown to be asymptotically sufficient in the Bahadur sense for testing randomness against a general class of two-sample alternatives, simple ones as well as composite ones. In other words, the best exact slope will be attainable by rank statistics, uniformly throughout the alternative.

1. Introduction. We shall consider the hypothesis of randomness under which two samples $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$ have an identical but arbitrary continuous distribution. The vector of ranks $(R_1, \cdots, R_{n+m})$ will be shown to be asymptotically sufficient in the Bahadur sense for testing randomness against a general class of two-sample alternatives, simple ones as well as composite ones. In other words, the best exact slope will be attainable by rank statistics, uniformly throughout the alternative.

Our basic tool will be a law of large numbers for simple linear statistics and a large deviation theorem for those statistics due to G. Woodworth (1970). Established properties of simple linear statistics will have important consequences for the rank likelihood ratio statistic, without which the composite alternative could not be treated.

We shall also provide a specialized version of the Berk–Savage theorem [3].

The reader not acquainted with Bahadur's theory may consult papers [1] and [2].

2. The two-sample case. Fix two densities f and g defined on R, and a number λ, $0 < \lambda < 1$. Put $F(x) = \int_{-\infty}^{x} f(y)\, dy$, $G(x) = \int_{-\infty}^{x} g(y)\, dy$ and

$$H(x) = \lambda F(x) + (1 - \lambda)G(x) \quad -\infty < x < \infty .$$

Introduce new densities

$$\bar{f}(u) = \frac{d}{du} F(H^{-1}(u)) , \qquad \bar{g}(u) = \frac{d}{du} G(H^{-1}(u)) , \qquad 0 < u < 1 .$$

Obviously, if f and g correspond to X and Y, then $\bar{f}$ and $\bar{g}$ correspond to $H(X)$

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and $H(Y)$, respectively. Since $\lambda F(H^{-1}(u)) + (1-\lambda)(G(H^{-1}(u)) = u$, we have

$$(1) \qquad \lambda \bar{f}(u) + (1-\lambda)\bar{g}(u) = 1\,, \qquad 0 < u < 1\,.$$

Now given a vector of ranks $(R_1, \cdots, R_N)$ and a double sequence of scores $a_N(i)$, $1 \leqq i \leqq N < \infty$, let us consider simple linear rank statistic

$$(2) \qquad S_N = \sum_{i=1}^{n} a_N(R_i)\,.$$

We shall assume that for some integrable function $\phi(u)$, $0 < u < 1$, the following holds:

$$(3) \qquad \lim_{N\to\infty} \int_0^1 |a_N(1 + [uN]) - \phi(u)|\, du = 0\,,$$

where $[uN]$ denotes the integral part of uN.

Statistics S_N satisfy the following law of large numbers:

THEOREM 1. *Assume that the functions $a_N(1 + [uN])$ have uniformly bounded variation on closed subintervals of $(0, 1)$. Let $X_1, X_2, \cdots, Y_1, Y_2, \cdots$ be independent random variables, the first sequence having density f and the other density g. Let $R_1, \cdots, R_N$ be the ranks corresponding to $(X_1, \cdots, X_n, Y_1, \cdots, Y_m)$, where $n + m = N$ and*

$$(4) \qquad \frac{n}{N} \to \lambda\,, \qquad 0 < \lambda < 1\,.$$

Then, under (3),

$$\frac{1}{N} S_N \to \lambda \int_0^1 \phi(u)\bar{f}(u)\, du$$

holds with probability 1, *with $\bar{f}(u)$ defined above.*

PROOF. For every $\delta > 0$ we can find $K > 0$ such that truncated scores

$$\begin{aligned} \bar{a}_N(i) &= a_N(i)\,, \qquad \text{if } |a_N(i)| \leqq K\,,\\ &= 0\,, \qquad \text{otherwise} \end{aligned}$$

satisfy

$$(5) \qquad \frac{1}{N} \sum_{i=1}^{N} |a_N(R_i) - \bar{a}_N(R_i)| < \delta$$

for all $N > N_0$ and $(R_1, \cdots, R_N)$. Similarly

$$\begin{aligned} \bar{\phi}(u) &= \phi(u)\,, \qquad \text{if } |\phi(u)| \leqq K\,,\\ &= 0\,, \qquad \text{otherwise} \end{aligned}$$

will satisfy

$$\int_0^1 |\phi(u) - \bar{\phi}(u)|\, du < \delta\,.$$

Relation (1) entails $\lambda \bar{f}(u) \leqq 1$, so that we also have

$$(6) \qquad |\lambda \int_0^1 \phi(u)\bar{f}(u)\, du - \lambda \int_0^1 \bar{\phi}(u)\bar{f}(u)\, du| < \delta\,.$$

From (5) and (6) it follows that it is sufficient to prove the theorem for the case when ϕ and $a_N(\cdot)$ have bounded variations over all $(0, 1)$.

Denoting by F_n the empirical distribution function corresponding to $(X_1, \cdots, X_n)$ and by H_N the empirical distribution corresponding to $(X_1, \cdots, X_n, Y_1, \cdots, Y_m)$, we can write

$$\begin{aligned} \frac{1}{N} S_N &= \frac{n}{N} \int_0^1 a_N(1 + [uN])\, dF_n(H_N^{-1}(u)) \\ &= \frac{n}{N} \int_0^1 a_N(1 + [uN])\, d\bar{F}_n(u) \qquad (7) \\ &= \frac{n}{N} \int_0^1 \phi \bar{f}\, du + \frac{n}{N} \int_0^1 (a_N - \phi)\bar{f}\, du + \frac{n}{N} \int_0^1 a_N d(\bar{F}_n - \bar{F}) , \end{aligned}$$

where

$$\bar{F}(u) = F(H^{-1}(u)) , \qquad \bar{F}_n(u) = F_N(H_N^{-1}(u)) .$$

Now (3) and (4) entail

$$\left| \frac{n}{N} \int_0^1 (a_N - \phi)\bar{f}\, du \right| \to 0 \qquad \text{as } N \to \infty . \tag{8}$$

Moreover, by the Glivenko–Cantelli theorem

$$H_N(x) \to H(x) , \qquad F_n(y) \to F(y) , \qquad \bar{F}_N(u) \to \bar{F}(u) , \qquad \text{as } N \to \infty ,$$

the convergence being uniform. Thus, if the variations of $a_N(\cdot)$ are bounded by V, we have

$$\begin{aligned} \left| \frac{n}{N} \int_0^1 a_N d(\bar{F}_n - \bar{F}) \right| &= \frac{n}{N} \left| \int (\bar{F}_n - \bar{F}) da_N(\cdot) \right| \qquad (9) \\ &= \max_{0<u<1} |\bar{F}_n(u) - \bar{F}(u)| V \to 0 . \end{aligned}$$

Substituting relations (8) and (9) into (7), we can see that the theorem is proved.

Our next aim is to establish the Bahadur exact slope for S_N-tests used for discriminating the hypothesis of randomness from the alternative considered in Theorem 1. The necessary (extra) large deviation result is contained in the paper by G. Woodworth (1970). For any ρ satisfying

$$\lambda \int_0^1 \phi\, du < \rho < \sup_A \{\int_A \phi\, du : \int_A du = \lambda\}$$

we have, under the hypothesis of randomness (the distribution of $(R_1, \cdots, R_N)$ is uniform over the space of all permutations), that

$$\lim_{N\to\infty} \frac{1}{N} \log P\left(\frac{1}{N} S_N > \rho\right) = -b(\lambda, \rho)$$

where

$$\begin{aligned} b(\lambda, \rho) = \rho H + (1 - \lambda) \log R - \int_0^1 \log (e^{H\phi(u)} + R)\, du \\ - \lambda \log \lambda - (1 - \lambda) \log (1 - \lambda) \end{aligned} \tag{10}$$

with (H, R) being the unique solution of the following two equations:

$$\int_0^1 [1 + R \exp(-H\phi(u))]^{-1}\, du = \lambda , \tag{11}$$

$$\int_0^1 \phi(u)[1 + R \exp(-H\phi(u))]^{-1}\, du = \rho . \tag{12}$$

It is obvious that $-b(\lambda, \rho)$ is continuous in ρ.

REMARK 1. G. Woodworth, in applying his Theorem 1 to the two-sample case, assumes that $a_N(\cdot) \to \phi$ in L_2. But it is possible to show that convergence in L_1 is also sufficient for establishing Property A of his article.

The above rather complicated result simplifies tremendously, if applied to ϕ and ρ of our specific interest. We shall put

$$\phi(u) = \log \frac{\bar{f}(u)}{\bar{g}(u)}, \tag{13}$$

and

$$\rho = \lambda \int_0^1 \bar{f}(u) \log \frac{\bar{f}(u)}{\bar{g}(u)}\, du = \lambda K(\bar{f}, \bar{g}), \tag{14}$$

$K(\cdot, \cdot)$ denoting the Kullback–Leibler information number. Note that $(1/N)S_N \to \rho$ under the alternative (f, g), and also $(\bar{f}, \bar{g})$, according to Theorem 1, if ϕ is given by (13). Introduce

$$\bar{J}(f, g, \lambda) = \lambda \int_0^1 \bar{f} \log \bar{f}\, du + (1 - \lambda) \int_0^1 \bar{g} \log \bar{g}\, du. \tag{15}$$

THEOREM 2. *Let* (3) *be satisfied for* ϕ *given by* (13). *Then, under the hypothesis of randomness,*

$$\lim_{N\to\infty} \frac{1}{N} \log P\left(\frac{1}{N} S_N > \lambda K(\bar{f}, \bar{g})\right) = -\bar{J}(f, g, \lambda), \tag{16}$$

where $\bar{J}(f, g, \lambda)$ *is given by* (15), *and* $K(\bar{f}, \bar{g})$ *by* (14).

PROOF. For ϕ and ρ given by (13) and (14), the equations (11) and (12) become

$$\int_1^0 [1 + R(\bar{g}/\bar{f})^H]^{-1}\, du = \lambda, \tag{17}$$

$$\int_0^1 \log \frac{\bar{f}}{\bar{g}} [1 + R(\bar{g}/\bar{f})^H]^{-1}\, du = \lambda(K(\bar{f}, \bar{g})), \tag{18}$$

which is solved by $H = 1$ and $R = (1 - \lambda)/\lambda$, as may be easily seen. Substituting for ρ, ϕ, H and R into (1), we obtain

$$\begin{aligned} b(\lambda, \rho) &= \lambda K(f, g) + (1 - \lambda) \log [(1 - \lambda)/\lambda] \\ &\quad - \int_0^1 \log [\bar{f}/\bar{g} + (1 - \lambda)/\lambda]\, du - \lambda \log \lambda - (1 - \lambda) \log (1 - \lambda) \\ &= \lambda K(\bar{f}, \bar{g}) + \int_0^1 \log \bar{g}\, du = \bar{J}(f, g, \lambda). \end{aligned}$$

This completes the proof of Theorem 2.

Putting

$$L_N(t) = P\left(\frac{1}{N} S_N \geqq t\right)$$

we are now able to compute the exact slope of the S_N-test for the (f, g, λ) alternative. See Bahadur (1967) for details.

COROLLARY 1. *Under conditions of Theorem* 2,

$$\lim_{N\to\infty} \frac{1}{N} \log L_N\left(\frac{1}{N} S_N\right) = -J(f, g, \lambda) \tag{19}$$

with probability 1 *under the* (f, g, λ)*-alternative described in Theorem* 1.

PROOF. We know that $(1/N)\log L_n(t) \to -b(\lambda, t)$ which is continuous in t. Further $(1/N)S_N \to K(\bar{f}, \bar{g})$ with probability 1 under the alternative and $b(\lambda, \lambda K(\bar{f}, \bar{g})) = \bar{J}(f, g, \lambda)$. This completes the proof of Corollary 1.

The most striking feature of Corollary 1 is that the exact slope for S_N is the best possible slope at all.

COROLLARY 2. *Under conditions of Theorem 2, $c = 2J(f, g, \lambda)$ is the best exact slope for testing the hypothesis of randomness against the alternative*

$$q(x_1, \cdots, x_n, y_1, \cdots, y_m) = \prod_{i=1}^{n} f(x_i) \prod_{j=1}^{m} g(y_j) .$$

PROOF. The hypothesis of randomness H_0 is composite and is satisfied whenever $X_1, X_2, \cdots, X_n, Y_1, \cdots, Y_n$ are independent and have a common distribution, which may be arbitrary but continuous. The best exact slope cannot be larger for H_0 against q than for p against q, where p is a particular member of H_0. Since the best exact slope for testing

$$p(x_1, \cdots, x_n, y_1, \cdots, y_m) = \prod_{i=1}^{n} f_0(x_i) \prod_{j=1}^{m} f_0(y_j)$$

against q is known to satisfy

$$c \leqq 2\left[\lambda \int_{-\infty}^{\infty} f \log \frac{f}{g_0}\, dx + (1-\lambda) \int_{-\infty}^{\infty} g \log \frac{g}{f_0}\, dx\right], \tag{20}$$

(Raghavachari (1970) Theorem 1) the least favorable p will correspond to f_0 that minimizes the right side of (20). It is easy to see that the minimum occurs for

$$f_0 = \lambda f + (1-\lambda) g$$

and that for this choice of f_0 the right side of (20) equals $2\bar{J}(f, g, \lambda)$. Thus $2\bar{J}(f, g, \lambda)$ actually is the best possible exact slope.

REMARK 2. The Raghavachari result [5] corresponds to a "one-sample" situation. However, it extends easily to a two-sample situation as well.

3. The rank-likelihood ratio statistic. We intend to prove that the rank-likelihood statistic also provides the best possible exact slope. If $(R_1, \cdots, R_N)$ is the vector of ranks and $r = (r_1, \cdots, r_N)$ is a particular permutation, we have under H_0

$$P(R = r) = 1/N! .$$

If the density q is true, let us denote the probability of the same event as $Q(R = r)$ and put

$$T_N(r) = \log [N!\, Q(R = r)] . \tag{21}$$

Let $U_N^{(i)}$ be the ith order statistic from a uniform sample over (0, 1) of size N. Consider again densities f, g and their "normed" counterparts $\bar{f}$ and $\bar{g}$. Put

$$a_N(i) = E\{\log [\bar{f}(U_N^{(i)})/\bar{g}(U_N^{(i)})]\} . \tag{22}$$

THEOREM 3. *If the scores are given by* (22) *and* T_N *is defined by* (21), *then for every* $R = (R_1, \cdots, R_N)$

$$\sum_{i=1}^{n} a_N(R_i) + N \int_0^1 \log \bar{g}(u)\, du \leqq T_N(R) . \tag{23}$$

PROOF. Introducing

$$\Lambda_N = \sum_{i=1}^{n} \log \bar{f}(X_i) + \sum_{j=1}^{m} \log \bar{g}(Y_i)$$

we have

$$T_N = \log E_1(e^{\Lambda_N} \mid R)$$

where $E_1(\cdot \mid R)$ refers to the conditioning given R via the uniform density $p(x_1, \cdots, x_n, y_1, \cdots, y_m) = 1,\ 0 \leqq x_1, \cdots, y_m \leqq 1$. Now

$$\begin{aligned} \log E_1(e^{\Lambda_N} R) &\geqq E_1(\Lambda_N \mid R) \\ &= \sum_{i=1}^{n} E_1\{\log \bar{f}(X_i) \mid R\} + \sum_{j=1}^{m} E_1\{\log \bar{g}(Y_j) \mid R\} \\ &= \sum_{i=1}^{n} a_N(R_i) + N \int_0^1 \log \bar{g}(u)\, du . \end{aligned}$$

This completes the proof of Theorem 3.

Theorem 3 entails

THEOREM 4. *The exact slope corresponding to the rank-likelihood ratio statistic* T_N *of* (21) *equals* $2\bar{J}(f, g, \lambda)$, *if testing the hypothesis of randomness against the alternative described in Theorem* 1, *and if* $\log \bar{f}(u)/\bar{g}(u)$ *is integrable and of bounded variation on every closed subinterval of* (0, 1).

PROOF. Let $\hat{L}_N$ be the level attained by T_N. Since T_N is a likelihood ratio statistic, we have

$$-\frac{1}{N} \log \hat{L}_N \geqq \frac{1}{N} T_N . \tag{24}$$

Now by Theorem 3,

$$\frac{1}{N} T_N \geqq \frac{1}{N} S_N + \int_0^1 \log \bar{g}\, du , \tag{25}$$

where S_N is given by (2) with scores of (22). From Theorem 1 we conclude that

$$\frac{1}{N} S_N \to \lambda \int_0^1 \phi(u) \bar{f}(u)\, du = \lambda \int_0^1 \bar{f}(u) \log \frac{\bar{f}(u)}{\bar{g}(u)}\, du \tag{26}$$

since (3) is satisfied for $\phi(u) = \log [\bar{f}(u)/\bar{g}(u)]$ in view of (22). (In order to prove it we may use the martingale argument of Lemma 6.1 in [4].) Putting (24) through (26) together, we obtain

$$\begin{aligned} \liminf_{N\to\infty} \left[-\frac{1}{N} \log \hat{L}_N \right] &\geqq \lambda \int_0^1 \bar{f}(u) \log \frac{\bar{f}(u)}{\bar{g}(u)}\, du \\ &\quad + \int_0^1 \log \bar{g}(u)\, du = \bar{J}(f, g, \lambda) . \end{aligned}$$

Again the Raghavachari theorem (Theorem 1 in [5]) implies that

$$\limsup_{N\to\infty} \left[-\frac{1}{N} \log \hat{L}_N \right] \leqq \bar{J}(f, g, \lambda) .$$

This completes the proof of Theorem 4.

Thus it was easier to establish the exact slope for the rank-likelihood statistic than for the linear rank statistic S_N: the only tools used were the strong law of large numbers for S_N, the Raghavachari theorem and inequality (23).

4. Composite alternatives. We shall now show that a rank statistic can serve efficiently any finite number of alternatives as tested against the hypothesis of randomness.

THEOREM 5. *Let* $\log \bar{f}(u)/\bar{g}(u)$ *be integrable and have bounded variation on every closed subinterval of* $(0, 1)$. *Let* $X_1, X_2, \cdots$ *have density* f_1 *and* $Y_1, Y_2, \cdots$ *density* g_1, *and* $n/N \to \lambda \varepsilon (0, 1)$.
Then

$$\liminf_{N\to\infty} \frac{1}{N} T_N \geqq \lambda \int_0^1 \bar{f}_1 \log \bar{f}\, du + (1-\lambda) \int_0^1 \bar{g}_1 \log \bar{g}\, du \tag{27}$$

with probability 1, *where again* $\bar{f}_1 = [F_1(H_1^{-1})]'$, $\bar{g}_1 = [G_1(H_1^{-1})]'$.

PROOF. Theorem 1 entails

$$\frac{1}{N} \sum_{i=1}^{n} a_N(R_i) + \int_0^1 \log \bar{g}\, du \to \lambda \int_0^1 \bar{f}_1 \log [\bar{f}/\bar{g}]\, du + \int_0^1 \log \bar{g}\, du$$
$$= \lambda \int_0^1 \bar{f}_1 \log \bar{f}\, du + \int_0^1 [1-\lambda \bar{f}_1] \log \bar{g}\, du\,.$$

Theorem 5 now follows by noting that $1-\lambda\bar{f}_1 = (1-\lambda)\bar{g}_1$ and by using inequality (23).

Consider a composite alternative given by k pairs

$$A = \{(f_j, g_j),\ 1 \leqq j \leqq k\}\,. \tag{28}$$

For each member of the alternative, let us establish the rank likelihood ratio statistic

$$T_{Nj}(r) = \log [N!\, Q_j(R=r)]\,, \qquad 1 \leqq j \leqq k$$

where Q_j corresponds to $X_1, \cdots, X_n$ having density f_j, and $Y_1, \cdots, Y_m$ density g_j. Put

$$U_N(r) = \max_{1\leq j\leq k} T_{Nj}(r)\,.$$

THEOREM 6. *Let* $\log \bar{f}_j(u)/\bar{g}_j(u)$, $1 \leqq j \leqq k$, *be integrable and have bounded variation on every closed subinterval of* $(0, 1)$, *and* $n/N \to \lambda\varepsilon(0, 1)$. *The exact slope of* U_N *equals*

$$c = 2\lambda \int_0^1 \bar{f}_j \log \bar{f}_j\, du + 2(1-\lambda)\int_0^1 \bar{g}_j \log \bar{g}_j\, du$$
$$= 2\bar{J}(f_j, g_j, \lambda)$$

for the jth *member of the alternative* A *of* (28), *and it is the best attainable exact slope.*

PROOF. We have, under the hypothesis of randomness,

$$P\left(\frac{1}{N} U_N \geqq t\right) \leqq k \max_{1\leq j\leq k} P\left(\frac{1}{N} T_{Nj} \geqq t\right) \leqq k e^{-Nt}$$

since T_{Nj} are log-rank likelihood ratios. Thus the level obtained by U_N, say $\hat{M}_N$, satisfies

$$\frac{1}{N}\log \hat{M}_N \leqq \frac{1}{N}\log k - \frac{1}{N} U_N .$$

On the other hand, if (f_j, g_j) obtains

$$\limsup_{N\to\infty}\left(-\frac{1}{N} U_N\right) \leqq \limsup_{N\to\infty}\left(-\frac{1}{N} T_{Nj}\right) \leqq -\bar{J}(\bar{f}_j, \bar{g}_j, \lambda)$$

according to (23) and Theorem 1. Consequently,

$$\limsup_{N\to\infty}\left\{\frac{1}{N}\log \hat{M}_N\right\} \leqq -\bar{J}(f_j, g_j, \lambda) .$$

However, the sharp inequality can hold only with probability 0, since $-\bar{J}(f_j, g_j, \lambda)$ is a lower bound according to the above mentioned result by Raghavachari (1970). This completes the proof of Theorem 6.

As a side product we obtain the following inequalities for the log-rank-likelihood ratio statistic.

THEOREM 7. *If under conditions of Theorem* 6, T_{Nj} *refers to* (f_j, g_j) *and* (f_i, g_i) *obtains, then with probability* 1

$$\begin{aligned} &\lambda \textstyle\int_0^1 \bar{f}_i \log \bar{f}_j \, du + (1-\lambda) \int_0^1 \bar{g}_i \log \bar{g}_j \, du \\ &\qquad \leqq \liminf_{N\to\infty}\left[\frac{1}{N} T_{Nj}\right] \leqq \limsup_{N\to\infty}\left[\frac{1}{N} T_{Nj}\right] \\ &\qquad \leqq \lambda \textstyle\int_0^1 \bar{f}_i \log \bar{f}_i \, du + (1-\lambda) \int_0^1 \bar{g}_i \log \bar{g}_i \, du . \end{aligned} \tag{29}$$

PROOF. The first inequality is obtained from Theorem 5, and the violation of the last inequality would conflict with the Raghavachari result, since for (f_i, g_i) the exact slope of U_N would exceed $2\bar{J}(f_i, g_i, \lambda)$.

REMARK 3. If $f_i = f_j$, $g_i = g_j$ the inequalities become equalities and we obtain a special case of the Berk–Savage Theorem [3] under somewhat relaxed conditions.

REMARK 4. The above result for the finite alternative A could be extended to infinite but compactifiable alternatives in a standard way. This is not done here.

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