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ON A NEW TYPE OF m-CLASS CYCLIC ASSOCIATION SCHEME AND DESIGNS BASED ON THE SCHEME

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A new type of m-class cyclic association scheme is defined for v objects, where v is either a prime or power of a prime. The two cases, when v is even and when v is odd are dealt with separately. A simple method of construction of PBIB designs based on this scheme is proposed. The PBIB designs presented in this paper have cyclic solutions and are obtainable through more than one initial block. The proposed scheme has been called an NC_m association scheme and the number of independent p_{jk}'s for this scheme is obtained.

0. Summary. A new type of m-class cyclic association scheme is defined for $v = p^n$ objects, where p is a prime and n, any positive integer. A simple method of construction of PBIB designs based on this scheme is proposed. The PBIB designs presented in this paper have cyclic solutions and are obtainable through more than one initial block.

1. Introduction. Since the introduction of PBIB designs by Bose and Nair [2], there has been considerable theoretical work on the subject. The concept of association schemes of PBIB designs was first given by Bose and Shimamoto [3]. A study of cyclic designs was made by John [6], who also presented a table of these designs in the useful range of $5 \leqq v \leqq 15$. The two class cyclic association scheme of Bose and Shimamoto [3] has been generalized to higher associate classes by Nandi and Adhikari [8] and Adhikari [1].

This paper introduces a new type of m-class cyclic association scheme and proposes a simple method of constructing PBIB designs, with more than one initial block, based on this scheme. The proposed method of construction is based on a method of construction of BIB designs obtainable through more than one initial block. (See Das and Kulshreshtha [4].) For brevity we use the symbol NC_m for the new m-class cyclic association scheme.

2. Association schemes and PBIB designs. Given v treatments, $1, 2, \cdots, v$, a relation satisfying the following conditions is said to be an association scheme with m classes:

(a) Any two treatments are either 1st, 2nd, $\cdots$, or mth associates, the relation of association being symmetric, i.e. if treatment α is the ith associate of treatment β, then β is the ith associate of treatment α.

(b) Each treatment has n_i, ith associates, the number n_i being independent of the treatment taken.

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(c) If any two treatments α and β are ith associates, then the number of treatments which are jth associates of α and kth associates of β is p^i_{jk} and is independent of the pair of ith associates α and β.

The numbers v, n_i, p^i_{jk}, $1 \leqq i, j, k \leqq m$, are the parameters of the association scheme.

If we have an association scheme with m-classes, then we get a PBIB design with r replications, b blocks, and block size k based on the association scheme, provided we can arrange the v treatments into b blocks such that

(i) each block contains k distinct treatments;
(ii) each treatment is contained in r blocks;
(iii) if two treatments α and β are ith associates, then they occur together in λ_i blocks (not all λ_i's equal), the number λ_i being independent of the particular pair of ith associates α and β $(1 \leqq i \leqq m)$.

3. The NC_m-association scheme. Let v be a prime or power of a prime so that a Galois field GF (v) always exists. Let x be a primitive element of GF (v). Then, it is known that $x^{v-1} = 1$ and that all the elements of GF (v) can be expressed as

$$0, x^0, x^1, \cdots, x^{v-2} .$$

Let C be a multiplicative group of the Galois field GF (v). Let $m \geqq 2$ divide $v - 1$, say $m = (v-1)/s$. Then x^m has order s. Let $C_1 = \{x^{qm} \mid 0 \leqq q \leqq s-1\}$ be the cyclic subgroup generated by x^m, and $C_1, C_2, \cdots, C_m$ be the cosets of the factor subgroup C/C_1. Then $C_j = x^{j-1}C_1 = \{x^{j-1+qm} \mid 0 \leqq q \leqq s-1\}$, $1 \leqq j \leqq m$.

Designating the v elements of GF (v) as v treatments, we have the following:

DEFINITION 3.1. For any two distinct treatments α and β; $\alpha, \beta \in$ GF (v), β will be said to be an ith associate of α if $(\alpha - \beta) \in C_i$, $1 \leqq i \leqq m$. Such an association relation will be called an NC_m-association relation.

We now prove the following in relation to the NC_m association relation.

LEMMA 1. *If $p^i_{jk}(\alpha, \beta)$ denotes the number of common treatments between the jth associates of α and the kth associates of β, where β is ith associate of α, then*

$$p^i_{jk}(\alpha, \beta) = p^i_{jk}(\alpha_1, \beta_1) , \quad \alpha \neq \beta,\ \alpha_1 \neq \beta_1;\ \alpha, \beta, \alpha_1, \beta_1 \in \text{GF}(v) . \tag{3.1}$$

PROOF. Denote by $\alpha + C_j$ the set $\{\alpha + y \mid y \in C_j\}$ and by αC_j the set $\{\alpha y \mid y \in C_j\}$. It is then known that for all i, j and nonzero elements α, β in GF (v), we have

$$\alpha C_j = \beta C_j \quad \text{if and only if} \quad \alpha, \beta \in C_i , \qquad 1 \leqq i, j \leqq m . \tag{3.2}$$

Now, $p^i_{jk}(\alpha, \beta) = |(\alpha + C_j) \cap (\beta + C_k)|$, where $|\cdot|$ denotes the number of elements in the set. Evidently

$$\begin{aligned} |(\alpha + C_j) \cap (\beta + C_k)| &= |(\overline{\alpha - \beta} + C_j) \cap C_k| \\ &= |(1 + u^{-1}C_j) \cap u^{-1}C_k| , \end{aligned} \tag{3.3}$$

where $u = \alpha - \beta$. Similarly, setting $w = \alpha_1 - \beta_1$ we have

$$p^i_{jk}(\alpha_1, \beta_1) = |(1 + w^{-1}C_j) \cap w^{-1}C_k| .$$

Then (3.1) follows from (3.2) and the fact that u^{-1} and w^{-1} belong to the same coset, for u and w do so. Hence the lemma.

LEMMA 2. *The NC_m relation is symmetric if v and m satisfy the following condition*:

$$(3.4) \qquad \begin{aligned} (v-1)/2 &\equiv 0 \pmod m \qquad \text{if } v \text{ is odd};\\ v - 1 &\equiv 0 \pmod m \qquad \text{if } v \text{ is even}. \end{aligned}$$

PROOF. Let $\alpha - \beta = x^Q$. Then,

$$\begin{aligned} \beta - \alpha &= x^{(v-1)/2+Q}, \qquad \text{if } v \text{ is odd},\\ &= x^{(v-1)+Q}, \qquad \text{if } v \text{ is even}. \end{aligned}$$

Thus, by virtue of (3.4), if β is ith associate of α, then α is also ith associate of β. Hence the lemma.

The above two lemmas lead us to the following

THEOREM 1. *For any v, a prime or power of a prime, and a positive integer $m \geqq 2$ which divides $(v - 1)$, satisfying* (3.4), *an NC_m association relation is an association scheme with the parameters v, $n_i = s$, p^i_{jk}; $1 \leqq i, j, k \leqq m$.*

This association scheme will be called NC_m association scheme. It may be noted that under (3.4) C_i, $1 \leqq i \leqq m$, is simply the ith associate class of treatment 0, which shows that the NC_m association scheme is of cyclical nature and hence its nomenclature.

4. Evaluation of p^i_{jk}'s of the NC_m association scheme. Without loss of generality, setting $u = x^{i-1}$, it immediately follows from (3.3) that

$$(4.1) \qquad p^i_{jk} = |(1 + C_{j-i+1}) \cap C_{k-i+1}| ,$$

where $(j - i)$ and $(k - i)$ are to be reduced mod m. One can now calculate p^i_{jk}'s easily since this amounts to calculating the cosets. Since there are m cosets, only one matrix of dimension $m \times m$ will give all the p^i_{jk} values of the association scheme. Such matrices, hereinafter called $T_{v,m}$ matrices, can easily be constructed for given v and m. Evidently, the (u, u')th entry $t_{uu'}$ of $T_{v,m}$ is $|(1 + C_u) \cap C_{u'}|$, $u, u' = 1, 2, \cdots, m$. Thus, for given i, j, k, $1 \leqq i, j, k \leqq m$, $p^i_{jk} = t_{uu'}$, where $u - 1 \equiv j - i \pmod m$ and $u' - 1 \equiv k - i \pmod m$. It may be noted that since $p^i_{jk} = p^i_{kj}$, $t_{uu'} = t_{u'u}$, and hence $T_{v,m}$ matrices are symmetrical.

The matrices $T_{v,2}$ are known in another context. When v is an even prime or prime power, we cannot have $m = 2$, for $v - 1 \neq 0 \bmod 2$. Further when v is an odd prime or power of a prime and is of the form $4t + 3$, we again cannot have $m = 2$, for $(v - 1)/2 \neq 0 \bmod 2$. Finally, when v is an odd prime or power of a prime of the form $4t + 1$, the general expressions for $t_{uu'}$ in terms of v only are well known. (See Mesner [7].)

To illustrate the method we construct the matrix $T_{13,3}$.

EXAMPLE 1. For $v = 13$, $x = 2$ is a primitive element of GF (13), and $m = 3$ divides $v - 1$. The cosets C_1, C_2 and C_3 are given by

$$C_1 = (x^0, x^3, x^6, x^9) = (1, 8, 12, 5)\,,$$
$$C_2 = (x^1, x^4, x^7, x^{10}) = (2, 3, 11, 10)\,,$$
$$C_3 = (x^2, x^5, x^8, x^{11}) = (4, 6, 9, 7)\,.$$

Also,

$$1 + C_1 = (2, 9, 0, 6)\,,$$
$$1 + C_2 = (3, 4, 12, 11)\,,$$
$$1 + C_3 = (5, 7, 10, 8)\,.$$

Therefore,

$$T_{13,3} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Suppose we wish to evaluate p_{23}^1. Here $i = 1, j = 2, k = 3$. So, $u = 2$ and $u' = 3$. Therefore, $p_{23}^1 = t_{23} = 1$. All the p_{jk}^i parameters of the NC_3 association scheme for 13 treatments are as follows:

$$P_1 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad P_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

In a similar manner, it can be shown that

$$T_{16,3} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}; \qquad T_{49,4} = \begin{bmatrix} 5 & 2 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 2 & 4 & 2 & 4 \\ 2 & 4 & 4 & 2 \end{bmatrix}$$

$$T_{71,5} = \begin{bmatrix} 0 & 5 & 4 & 2 & 2 \\ 5 & 2 & 2 & 3 & 2 \\ 4 & 2 & 2 & 3 & 3 \\ 2 & 3 & 3 & 4 & 2 \\ 2 & 2 & 3 & 2 & 5 \end{bmatrix}; \qquad T_{25,6} = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

REMARK 1. It is easy to see that the matrix $T_{v,m}$ is the same as the matrix $P_1 = (p_{jk}^1)$. The number of independent cell entries in $T_{v,m}$ is, therefore,

$$(m - 1) + (m - 2) + \cdots + 1 = m(m - 1)/2\,,$$

because of the relations $\sum_{k=1}^{m} p_{jk}^1 = n_j - \delta_{1j}$ where δ_{1j} is the Kronecker delta. Thus the number of independent p_{jk}^i parameters of the NC_m association scheme is equal to $m(m - 1)/2$ which is always less than $m(m^2 - 1)/6$, unless $m = 2$, in which case they are equal, where $m(m^2 - 1)/6$ is the number of independent p_{jk}^i's in any association scheme in general.

REMARK 2. Like all the existing association schemes, the NC_m association scheme will also find its use in obtaining plans for partial diallel crosses following the method given by Das and Sivaram [5]. The point that deserves special mention here is that we may obtain, in some cases, plans with a smaller number of crosses by suitably choosing a smaller value for s, as the total number of crosses in these plans is equal to $vn_i/2 = vs/2$ for v parental lines.

5. Construction of NC_m-type PBIB designs. A PBIB design based on an NC_m association scheme will be called an NC_m-type PBIB design. We describe in this section a simple method of construction of NC_m-type PBIB designs through more than one initial block. The method is based on the procedure of Das and Kulshreshtha [4].

When we say a design D is generated by a *basic* initial block $I = (a_1, a_2, \cdots, a_k)$ and a set $M = (e_1, e_2, \cdots, e_t)$ of t multipliers, denoted by $D = [I, M]$, we mean the formation of tv blocks each of size k, obtainable from t initial blocks I_1, $I_2, \cdots, I_t$, where $I_j = e_j \cdot I = (e_j a_1, e_j a_2, \cdots, e_j a_k) \bmod v$. Here the a_i's are $k(k < v)$ distinct elements of GF (v) and the e_j's are t distinct nonzero elements of GF (v).

Let f_i denote the number of differences in the basic initial block I, belonging to the coset C_i, so that $\sum_{i=1}^{m} f_i = k(k-1)$. We shall denote by I^* in what follows a basic initial block consisting of k distinct elements of GF (v) such that not all f_i's are equal. One can then easily prove the following

THEOREM 2. *When v is an even prime power such that an NC_m association scheme with the parameters v, $n_i = s$, p^i_{jk} exists, the design $D_1 = [I^*, M_1]$, $M_1 = (x^0, x^m, x^{2m}, \cdots, x^{(s-1)m})$ is an NC_m-type* PBIB *design with the parameters of the second kind as*

$$b = sv\,, \qquad r = sk, k\,, \qquad \lambda_i = f_i\,, \qquad 1 \leqq i, j, k \leqq m\,.$$

Since for any v which is an odd prime power,

$$C_1 = (x^0, x^m, \cdots, x^{m(s-2)/2}, -x^0, -x^m, \cdots, -x^{m(s-2)/2})\,,$$

the following result can be established.

THEOREM 3. *When v is an odd prime power such that an NC_m association scheme with the parameters v, $n_i = s$, p^i_{jk} exists, the design $D_2 = [I^*, M_2]$, $M_2 = (x^0, x^m, x^{2m}, \cdots, x^{m(s-2)/2})$ is an NC_m-type* PBIB *design with parameters of the second kind as*

$$b = sv/2\,, \qquad r = sk/2, k\,, \qquad \lambda_i = f_i/2\,, \qquad 1 \leqq i, j, k \leqq m\,.$$

EXAMPLE 2. Let $v = 2^4 = 16$; $m = 3$ divides $v - 1$ giving $s = (v-1)/m = 5$ and makes $(v - 1) \equiv 0 \bmod m$. Thus an NC_3 association scheme exists for 16 treatments. Let x stand for the primitive element of GF (2^4) which has a minimum function $x^4 + x^3 + 1 \equiv 0 \bmod 2$. We now choose $I^* = (x^0, x^1, x^2)$ which has $f_1 = 4, f_2 = 2, f_3 = 0$. From Theorem 2, we have $M_1 = (x^0, x^3, x^6, x^9, x^{12})$. Then $D_1 = [I^*, M_1]$ gives us an NC_3-type PBIB design with the following parameters:

$$v = 16\,, \qquad n_1 = n_2 = n_3 = 5\,, \qquad p^i_{jk}\text{'s as given in } T_{16,3} \text{ in Section } 4\,,$$
$$b = 80\,, \qquad r = 15\,, \qquad k = 3\,, \qquad \lambda_1 = 4\,, \qquad \lambda_2 = 2\,, \qquad \lambda_3 = 0\,.$$

EXAMPLE 3. Let $v = 13$; $x = 2$ is a primitive element of GF (13) and $m = 2$ satisfies (i) $(v - 1)/2 \equiv 0 \bmod m$ and (ii) $s = (v - 1)/m = 6$, an integer. Thus an NC_2 association scheme is existent for 13 treatments. We take $I^* = (1, 2, 4)$ which has $f_1 = 4$ and $f_2 = 2$. By Theorem 3, $M_2 = (1, 4, 3)$. Then $D_2 = [I^*, M_2]$ gives an NC_2-type PBIB design with the following parameters:

$$v = 13\,, \quad n_1 = n_2 = 6\,, \quad P_1 = \begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3 & 3 \\ 3 & 2 \end{bmatrix};$$
$$b = 39\,, \quad r = 9\,, \quad k = 3\,, \quad \lambda_1 = 2\,, \quad \lambda_2 = 1\,.$$

REMARK 3. While a large number of designs of two associate classes can be constructed through the methods presented above, it does not seem feasible to construct a new design with two associate classes having $r \leqq 10$ and $3 \leqq k \leqq 10$ through the proposed methods.

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