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SOME LIMIT THEOREMS FOR MAXIMA OF NONSTATIONARY GAUSSIAN PROCESSES[1]

BY CHANDRAKANT M. DEO

University of California, Davis

Let $Z_n = \max_{1\leq j\leq n} X_j$ where $\{X_n : 1 \leqq n < \infty\}$ is a Gaussian process. Some known limit theorems for Z_n when $\{X_n\}$ is stationary are extended to the nonstationary case.

1. Introduction. Let $\{X_n : 1 \leqq n < \infty\}$ be a discrete-parameter Gaussian process with $E(X_i) = 0$, $E(X_i^2) = 1$ and $E(X_i X_j) = r(i, j)$ for all i, j. Let $Z_n = \max_{1\leq j\leq n} X_j$. The asymptotic behavior of Z_n, for stationary Gaussian process $\{X_n\}$, has been investigated in [1], [3] and [4]. In this note it is shown that some theorems in [1] and [3] continue to hold for nonstationary sequences under suitable conditions. The techniques used are essentially those of Berman (1964). After this note was submitted for publication I found that Theorem 3 here has been obtained earlier by P. I. Yuditskaya (1970). Since a proof of this theorem is given in [5] no proof is given here.

2. Results. Let $\delta_n = \sup_{|i-j|\geqq n} |r(i, j)|$ and $\delta_n' = \sup_{|i-j|\geqq n} r(i, j)$.

THEOREM 1. *Suppose* $\delta_n \to 0$ *and for some* $\gamma > 0$,

$$\textstyle\sum_{1\leqq i<j\leqq n} |r(i, j)| = O(n^{2-\gamma}) ,$$

then $Z_n - (2 \log n)^{\frac{1}{2}} \to 0$ a.s.

It is easy to see that the conditions of Theorem 1 are satisfied if

$$n^\alpha \delta_n \to 0 \qquad \text{for some} \quad \alpha > 0 . \tag{1}$$

A simple application of Schwarz's inequality shows that the conditions of Theorem 1 are also implied by

$$\textstyle\sum \delta_n^2 < \infty . \tag{2}$$

However, under either (1) or (2), it is possible to prove a stronger theorem about the almost sure behavior of Z_n. It is shown in [2] that in this case,

$$Z_n^* = (\log \log n)^{-1}(2 \log n)^{\frac{1}{2}}(Z_n - (2 \log n)^{\frac{1}{2}})$$

has, with probability one, lim sup equal to $\frac{1}{2}$ and lim inf equal to $-\frac{1}{2}$. For stationary sequences such an iterated logarithm type result was first established in Pickands (1969).

The proof of Theorem 1 is based on the following lemmas.

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LEMMA 1 (Berman [1]). *Let the random variables $X_1, X_2, \cdots, X_n$ have joint Gaussian distribution with zero means, unit variances and correlations $\{r(i,j): 1 \leqq i < j \leqq n\}$ under probability measure P; and let them be independent, standard normal under probability P^*. Then for any $c > 0$,*

$$(3) \qquad |P\{\max_{1\leqq j\leqq n} X_j \leqq c\} - P^*\{\max_{1\leqq j\leqq n} X_j \leqq c\}| \leqq D_n(c) = \sum_{1\leqq i<j\leqq n} [1 - r^2(i,j)]^{-\frac{1}{2}} |r(i,j)| \exp\{-c^2/(1 + |r(i,j)|)\} .$$

In the remainder of this paper A denotes a generic constant which is finite and strictly positive.

LEMMA 2. *If each X_n has (marginally) standard normal distribution then for any $\varepsilon > 0$, we have,*

$$P\{\limsup_{n\to\infty} (Z_n - (2\log n)^{\frac{1}{2}}) \leqq \varepsilon\} = 1 .$$

PROOF. This is proved in [3]. Indeed, with probability one,

$$X_n > (2\log n)^{\frac{1}{2}} + \varepsilon$$

only finitely often since,

$$\sum_n P\{X_n > (2\log n)^{\frac{1}{2}} + \varepsilon\} \leqq \sum_n \exp\{-\tfrac{1}{2}((2\log n)^{\frac{1}{2}} + \varepsilon)^2\} \leqq A \sum \frac{1}{n(\log n)^2} < \infty .$$

From this, the conclusion of the lemma follows easily.

LEMMA 3. *If $\{X_n\}$ are* i.i.d. *standard normal variables then,*

$$(\log n)^2 P\{Z_n \leqq (2\log n)^{\frac{1}{2}} - \varepsilon\} \to 0$$

for any $\varepsilon > 0$.

PROOF. This is given in [3], page 199.

PROOF OF THEOREM 1. In view of Lemma 2 we need only show that

$$(4) \qquad P\{\liminf (Z_n - (2\log n)^{\frac{1}{2}}) \geqq 0\} = 1 .$$

Let $\varepsilon > 0$. Making use of the elementary fact that $(2\log 2^{n+1})^{\frac{1}{2}} - 2\varepsilon$ is eventually less than $(2\log 2^n)^{\frac{1}{2}} - \varepsilon$, it is easy to see that the statements (5) and (6) below are equivalent.

$$(5) \qquad \forall \varepsilon > 0: Z_n \leqq (2\log n)^{\frac{1}{2}} - \varepsilon \quad \text{for only finitely many } n\text{'s.}$$

$$(6) \qquad \forall \varepsilon > 0: Z_{2^n} \leqq (2\log 2^n)^{\frac{1}{2}} - \varepsilon \quad \text{for only finitely many } n\text{'s.}$$

Hence we need merely show that

$$(7) \qquad \sum P\{Z_{2^n} \leqq (2\log 2^n)^{\frac{1}{2}} - \varepsilon\} < \infty , \qquad \text{for} \quad \varepsilon > 0 .$$

Now (7) is clearly implied by

$$(8) \qquad (\log n)^2 P\{Z_n \leqq (2\log n)^{\frac{1}{2}} - \varepsilon\} \to 0 .$$

In view of Lemmas 1 and 3, to establish (8) it suffices to show

$$(9) \qquad (\log n)^2 D_n(c_n) \to 0\,, \qquad \text{where} \quad c_n = (2\log n)^{\frac{1}{2}} - \varepsilon\,.$$

Now assume for a moment that $\delta_1 < 1$. Then,

$$(10) \qquad (\log n)^2 D_n(c_n) \leqq (\log n)^2 (1-\delta_1^2)^{-\frac{1}{2}} \times \sum_{1\leqq i<j\leqq n} |r(i,j)| \exp\{-c_n^2/(1+|r(i,j)|)\}\,.$$

Let λ be any number such that $0 < \lambda < (1-\delta_1)(1+\delta_1)^{-1}$. Write the sum appearing in (10) as $S_1 + S_2$ where S_1 is over all i, j for which $|i-j| \leqq n^\lambda$ and S_2 over all remaining i, j. Then,

$$\begin{aligned}(\log n)^2 S_1 &\leqq A(\log n)^2 n^{1+\lambda} \exp\{-c_n^2/(1+\delta_1)\}\\ &\leqq A(\log n)^2 n^{\lambda-(1-\delta_1)(1+\delta_1)^{-1}} \exp(2\varepsilon(2\log n)^{\frac{1}{2}})\\ &\to 0 \quad \text{as} \quad n\to\infty\,.\end{aligned}$$

Furthermore,

$$(11) \qquad \begin{aligned}(\log n)^2 S_2 &\leqq A(\log n)^2 \exp\{(-2\log n + 2\varepsilon(2\log n)^{\frac{1}{2}})(1+\delta(n^\lambda))^{-1}\}\\ &\qquad \times \sum_{1\leqq i<j\leqq n} |r(i,j)|\\ &\leqq An^{-2/(1+\delta(n^\lambda))}\{\exp(2\varepsilon(2\log n)^{\frac{1}{2}}\}(\log n)^2 n^{2-\gamma}\,;\end{aligned}$$

where we have written $\delta(k)$ for δ_k for typographical convenience. Now since $\delta(n^\lambda) \to 0$ it is clear that $2-\gamma-2(1+\delta(n^\lambda))^{-1}$ becomes eventually less than, say, $-\gamma/2$. Thus the expression on the right side of (11) goes to zero as $n\to\infty$.

This proves the theorem under the assumption $\delta_1 < 1$. To remove this assumption, note that since $\delta_n \to 0$ we can find a constant k_0 such that $\delta_{k_0} < 1$. Then for each $j < k_0$ the theorem applies to the sequence $\{X_{nk_0+j} : 0 \leqq n < \infty\}$. Now $(2\log n)^{\frac{1}{2}} - (2\log[n/k_0])^{\frac{1}{2}} \to 0$ as $n\to\infty$. Writing Z_n as the maximum of submaxima in terms of the sequences $\{X_{nk_0+j} : 0 \leqq n < \infty\}$ it follows easily that $Z_n - (2\log n)^{\frac{1}{2}} \to 0$ a.s. and the proof is complete.

THEOREM 2. *Let the conditions of Theorem* 1 *be satisfied and, in addition, suppose that* $\delta_1 < 1$. *Then the limit law of* $a_n^{-1}(Z_n - b_n)$ *is the extreme value distribution* $\exp(-e^{-x})$, *where* $a_n = (2\log n)^{-\frac{1}{2}}$ *and* $b_n = (2\log n)^{\frac{1}{2}} - \frac{1}{2}(2\log n)^{-\frac{1}{2}}(\log\log n + \log 4\pi)$.

PROOF. Let x be a real number and write $c_n' = a_n x + b_n$. The theorem is true if $\{X_n\}$ are i.i.d. standard normal. Hence in view of Lemma 1 it suffices to prove that $D_n(c_n') \to 0$, under the hypotheses of this theorem. Note that

$$c_n'^2 = 2\log n - \log\log n + O_n(1)\,.$$

Also $\delta_1 < 1$ by assumption. Hence

$$(12) \qquad \begin{aligned}D_n(c_n') &\leqq A \sum_{1\leqq i<j\leqq n} |r(i,j)| \exp\{-c_n'^2/(1+|r(i,j)|)\}\\ &\leqq A(\log n) \sum_{1\leqq i<j\leqq n} |r(i,j)| \exp\{-2\log n(1+|r(i,j)|)^{-1}\}\,.\end{aligned}$$

Now the last expression on the right side of (12) is dominated by the expression on the right side of (10) and the latter was shown to go to zero, under the hypotheses of this theorem, in the course of the proof of Theorem 1. Hence $D_n(c_n') \to 0$ and the proof is complete.

Finally we mention

THEOREM 3 (P. I. Yuditskaya (1970)). *If* $\delta_n' \to 0$ *then* $(2 \log n)^{-\frac{1}{2}} Z_n \to 1$ a.s.

3. Concluding remarks. (A) The condition $\delta_1 < 1$ cannot be completely removed from Theorem 2. This can be seen by considering the nonstationary Gaussian sequence $\{X_n\}$, where $X_{2n-1} = X_{2n} = Y_n$, $n = 1, 2, \cdots$; $\{Y_n\}$ being a sequence of i.i.d. standard normal variables.

(B) Let $\{X_n\}$ be stationary and write $r_n = E(X_1 X_{n+1})$. If $r_n \to 0$ then $(2 \log n)^{-\frac{1}{2}} Z_n \to 1$ a.s. Thus $r_n \to 0$ is a sufficient condition for $(2 \log n)^{-\frac{1}{2}} Z_n \to 1$ a.s. Using the arguments in [1] it can be seen that a necessary condition for $(2 \log n)^{-\frac{1}{2}} Z_n \to 1$ a.s. is the ergodicity of $\{X_n\}$. Thus it would be interesting to find a necessary and sufficient condition for Theorem 3 to hold at least in the stationary case.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
DAVIS, CALIFORNIA 95616