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# MULTIPLE PRODUCTS OF POLYKAYS USING ORDERED PARTITIONS[1]

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Methods for multiplication of polykays have been given by various authors. Carney used ordered partitions in considering double products. In this paper, relations between polykays, symmetric means, augmented monomial functions and unrestricted power sums are studied. The method of ordered partitions is extended to obtain multiple products of polykays. Kronecker products of ordered partitions are introduced and certain rules are suggested by whose application the size of the problem is reduced considerably. The method is illustrated on triple products of weight 7.

**1. Introduction and summary.** Multiplication of polykays has been treated by Fisher [5], Tukey [12], [13], Wishart [14], Dwyer and Tracy [4], Tracy [10], [11] and Carney [2]. Carney [2], using ordered partitions and following Hooke [7], developed relations connecting polykays with symmetric means and unrestricted sums. He also developed a method to obtain products of two polykays using ordered partitions. In this paper we obtain relations of polykays with other symmetric functions as in [1], and extend Carney's method [2] for double products of polykays to multiple products. Carney, in his method [2], encounters a matrix of which he can eliminate certain rows and add certain columns together. This is applicable for multiple products as well, where the matrices involved are quite large. We formally describe these rules and devise some further rules to systematize the computation and reduce the sizes of the matrices and to find least upper bounds of ordered partitions. An interesting outcome is the possibility of writing the inverse of a matrix directly without having first to obtain this matrix itself. Formulae for triple polykay products of weight 7 are obtained by way of illustration.

**2. Preliminaries.** In this section, following mainly Carney [1], [2] and Hooke [7], we briefly review the notation needed in the paper. The reader is referred to Fig. 1 of [1] for an example.

2.1 An ordered partition $\alpha$ of weight $m$ is a list of $m$ symbols $\alpha_1\alpha_2 \cdots \alpha_m$, any two of which are either identical or distinct. Thus for each pair $(i, j)$, we have either $\alpha_i = \alpha_j$ or $\alpha_i \neq \alpha_j$. Ordered partition $\alpha$ characterizes the partition $\alpha^*$ of weight $m$ with sets of identical symbols as parts of $\alpha^*$. For example, 12213345, 12213435, $\cdots$ characterize the 5-part partition 22211 of weight 8. The correspondence between ordered partitions and partitions is many-to-one.

The number of distinct symbols in $\alpha$ is denoted by $\phi(\alpha)$. Clearly $\phi(\alpha)$ is the number of parts of $\alpha^*$. In the example above, $\phi(\alpha) = 5$.

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We use $(\alpha)$ for the polykay corresponding to ordered partition $\alpha$, $\langle\alpha\rangle$ for the symmetric mean, $[\alpha]$ for the augmented monomial symmetric function and $\{\alpha\}$ for the unrestricted sum (power sum). Actually, order is not pertinent in these symmetric functions. Thus (12213345), (12213435) represent the same polykay $k_{22211}$. Therefore corresponding to partition $\alpha^*$, we can define $(\alpha^*)$, $\langle\alpha^*\rangle$, $[\alpha^*]$ and $\{\alpha^*\}$ with the understanding that, just as between ordered partitions and partitions, the correspondence between $(\alpha)$'s and $(\alpha^*)$'s is many-to-one, and similarly for other symmetric functions. Though this leads to certain redundancy, such distinction between $(\alpha)$'s needs to be retained till a certain stage in order to carry out the method given below for obtaining their products. The redundancy is eliminated in the final steps of the method.

We use $\boldsymbol{\alpha}$ to denote the vector of all ordered partitions of weight $m$ and $\boldsymbol{\alpha}^*$ for the corresponding vector of partitions. If $M$ and $\mu$ denote the number of ordered partitions and the number of partitions of weight $m$ respectively, $\boldsymbol{\alpha}$ is an $M$-vector and $\boldsymbol{\alpha}^*$ a $\mu$-vector. Similarly we can assemble $M$-vectors $(\boldsymbol{\alpha})$, $\langle\boldsymbol{\alpha}\rangle$, $[\boldsymbol{\alpha}]$, $\{\boldsymbol{\alpha}\}$ and $\mu$-vectors $(\boldsymbol{\alpha}^*)$, $\langle\boldsymbol{\alpha}^*\rangle$, $[\boldsymbol{\alpha}^*]$, $\{\boldsymbol{\alpha}^*\}$. We list values of $M$ and $\mu$ for some weights $m$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $M$ | 1 | 2 | 5 | 15 | 52 | 203 | 877 | 4050 |
| $\mu$ | 1 | 2 | 3 | 5 | 7 | 11 | 15 | 22 |

Values $M$ can be obtained by adding the entries in the last row of Table 1.1 of David, Kendall and Barton [3], whereas $\mu$ is the number of rows in the same table.

2.2 Let $\alpha = \alpha_1\alpha_2 \cdots \alpha_m$ and $\beta = \beta_1\beta_2 \cdots \beta_m$ be ordered partitions of weight $m$. Then $\alpha$ is said to be an ordered subpartition of $\beta$ ($\alpha \leqq \beta$) if and only if $\alpha_i = \alpha_j$ implies $\beta_i = \beta_j$ for all pairs $(i, j)$; $i, j = 1, 2, \cdots, m$.

EXAMPLES.

(1) Every ordered partition is an ordered subpartition of itself.

(2) 11233455 is an ordered subpartition of 11122333.

Two ordered partitions $\alpha$ and $\beta$ of weight $m$ are the same if $\alpha \leqq \beta$ and $\beta \leqq \alpha$.

2.3 In an ordered partition $\alpha = \alpha_1\alpha_2 \cdots \alpha_m$, we say that $\alpha_{i+1}\alpha_{i+2} \cdots \alpha_{i+r}$, $(i = 0, 1, 2, \cdots, m-1;\ r = 1, 2, \cdots, m-i)$, is a run of length $r$ if $\alpha_i \neq \alpha_{i+i} = \cdots = \alpha_{i+r} \neq \alpha_{i+r+1}$.

2.4 Let $\boldsymbol{\alpha}_i$ denote the column vector formed by the set of ordered partitions of weight $m_i$. Then the symbolic product $\boldsymbol{\alpha}_1 \otimes \boldsymbol{\alpha}_2 \otimes \cdots \otimes \boldsymbol{\alpha}_n$ is the column vector whose components are the ordered partition

$$\underbrace{11\cdots 1}_{m_1}\ \underbrace{22\cdots 2}_{m_2}\cdots\underbrace{nn\cdots n}_{m_n}$$

of weight $\sum^n m_i$ (with runs of length $m_1, m_2, \cdots, m_n$) and all ordered subpartitions.

EXAMPLE.

$$\begin{bmatrix}11\\12\end{bmatrix}\otimes\begin{bmatrix}11\\12\end{bmatrix}=\begin{bmatrix}1122\\1123\\1233\\1234\end{bmatrix}$$

2.5 Let $\alpha$ be an ordered partition of weight $m$. Then the $m$th degree polykay $\langle\alpha\rangle$ is defined implicitly by the equation

$$\langle\alpha\rangle = \sum(\alpha_i)$$

where the summation extends over all ordered subpartitions $\alpha_i$ of $\alpha$, and the symmetric mean $\langle\alpha\rangle = [\alpha]/n^{(\phi(\alpha))}$.

For each symmetric mean there is only one such equation, since when all $m$th degree symmetric means are assembled in $\langle\alpha\rangle$, and similarly all $m$th degree polykays in $(\alpha)$, we have

$$\langle\alpha\rangle = \Lambda(\alpha)$$

where the incidence matrix $\Lambda$ is nonsingular, its determinant being unity, cf. [7, page 60].

2.6 Let $\alpha = \alpha_1 \cdots \alpha_m$ and $\beta = \beta_1 \cdots \beta_m$ be ordered partitions of weight $m$. Their l.u.b. $\delta = \alpha \vee \beta = \delta_1\delta_2 \cdots \delta_m$ is defined as an ordered partition of weight $m$ such that

(i) $\alpha \leqq \delta$, $\beta \leqq \delta$,

(ii) if $\lambda$ is any other ordered partition of weight $m$ such that $\alpha \leqq \lambda$, $\beta \leqq \lambda$, then $\delta \leqq \lambda$.

Ordered partition $\delta$ is formed by setting $\delta_i = \delta_j$ if $\alpha_i = \alpha_j$ or $\beta_i = \beta_j$ or $\delta_i$ and $\delta_j$ are linked by partitions which are equal in $\alpha$ or $\beta$ [2, page 1749]. Construction of l.u.b. has been demonstrated by Carney [1, page 645].

EXAMPLE. Let $\alpha = 11232434$, $\beta = 12223123$. We want to determine $\delta = \alpha \vee \beta = \delta_1\delta_2 \cdots \delta_8$. Here $\alpha_1 = \alpha_2$ and $\beta_2 = \beta_3 = \beta_4 = \beta_7$, thus $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_7$. Set $\delta_1 = 1$, so $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_7 = 1$. Again $\alpha_3 = \alpha_5$, $\beta_5 = \beta_8$ and $\alpha_8 = \alpha_6$, hence $\delta_3 = \delta_5 = \delta_8 = \delta_6$. Since $\delta_3 = 1$, we have $\delta_i = 1$ for $i = 1, 2, \cdots, 8$, i.e. $\delta = 11111111$ and $\phi(\delta) = 1$.

2.7 Let the elements of the set $S_m$ of ordered partitions of weight $m$ be $\alpha^1, \alpha^2, \cdots, \alpha^M$, arranged in such a way that $\alpha^i \leqq \alpha^j \Rightarrow i > j$. $S_m$, with this partial ordering, is a lattice [1].

Now define matrix $\Lambda$ as

$$\begin{aligned}\lambda_{ij} &= 1 \qquad \text{if} \quad \alpha^j \leqq \alpha^i\,,\\ &= 0 \qquad \text{otherwise}\,.\end{aligned}$$

Also define a diagonal matrix $N$ with elements $n_{ii} = n^{(\phi(\alpha^{ij}))}$. Then

THEOREM (Carney [2, page 1750]). *Let $A = \Lambda' N \Lambda$. Then the elements of $A$ are $a_{ij} = n^{\phi(\alpha^i \vee \alpha^j)}$.*

*Let* $\boldsymbol{\alpha}$ *be the vector of ordered partitions of weight m. Then we have* [1, page 649]

$$(\boldsymbol{\alpha}) = (\Lambda' N \Lambda)^{-1}\{\boldsymbol{\alpha}\} = A^{-1}\{\boldsymbol{\alpha}\} \tag{1}$$

*where* $A^{-1}$ *is a symmetric matrix of dimension M.*

Each row of $A^{-1}$ corresponds to a polykay $(\alpha)$, each column to an unrestricted sum $\{\alpha\}$, and each element is the coefficient of an unrestricted sum when a polykay is expressed as a linear combination of unrestricted sums by means of equation (1). When various $\{\alpha\}$'s characterizing the same $\{\alpha^*\}$ are pooled together, columns in $A^{-1}$ corresponding to such $\{\alpha\}$'s are added together. We notice now that rows representing all $(\alpha)$'s characterizing the same $(\alpha^*)$ are identical. To eliminate such redundancy, from various ordered partitions $\alpha$ characterizing the same partition $\alpha^*$, we choose just one and compute the corresponding row. When the required columns of $A^{-1}$ are added together, and repeated rows eliminated, it is denoted by $(A^{-1})^*$. Since $A^{-1}$ is symmetric, one need not compute all the elements of $(A^{-1})^*$. A method to compute the required elements of $(A^{-1})^*$ without first having to compute all the elements of $A = (a_{ij})$ is suggested below.

Let $(A_{ij})$ denote the adjugate of $A$. Let $\phi(\alpha^i \vee \alpha^j)$ be denoted by $c_{ij}$, thus $a_{ij} = n^{c_{ij}}$. Then

*Case* I. When $i \neq j$,

$$A_{ij} = P + Q$$

where

$$P = (-1)^{i+j} n^{c_{ij} + \sum_{r \neq i,j} c_{rr}}$$

and $Q$ is the sum of all the terms obtained by permuting second suffixes in the expression $(c_{ij} + \sum_{r \neq i,j} c_{rr})$ in all possible ways. The sign of a term in $Q$ is the same or opposite of $P$ according as it involves an even or odd permutation of the second suffixes in $P$ (i.e., $k$, the number of transpositions, is even or odd).

*Case* II. When $i = j$,

$$A_{ii} = P_1 + Q_1$$

where

$$P_1 = n^{\sum_{r \neq i} c_{rr}}$$

and $Q_1$ is obtained from $P_1$ in the same manner as $Q$ from $P$.

Using the rules of Section 4 below, the values of $c_{ij}$ are procured easily.

**3. Generalization of some known results.** In this section we give some results which extend the results of Carney [1], [2].

3.1 *Relation of polykays with other symmetric functions.* We express polykays of degree $d$ in terms of symmetric means and unrestricted sums and vice versa. Let $s + 1 \leqq d \leqq s + n(= m)$, where $s$ in an arbitrary integer such that $0 \leqq s \leqq d - 1$. Let the ordered partitions of weight $s + 1$ be denoted by $\alpha^{11}, \alpha^{21}, \cdots, \alpha^{M_1 1}$, those of weight $s + 2$ by $\alpha^{M_1+1,2}, \cdots, \alpha^{M_1+M_2,2}$, and so on.

If $(\langle\alpha^{ij}\rangle)$, $((\alpha^{ij}))$ denote the matrices of symmetric means and polykays respectively, then

$$(\langle\alpha^{ij}\rangle) = H((\alpha^{ij})) \tag{2}$$

where $H$ is a nonsingular $(M_1 + \cdots + M_n) \times (M_1 + \cdots + M_n)$ upper triangular matrix with diagonal elements unity. One can write

$$H = (h_{ij}) = (\alpha^{ik} \cap \alpha^{jt})$$

where

$$\begin{aligned} \alpha^{ik} \cap \alpha^{jt} &= 0 \quad \text{if} \quad k \neq t\,, \quad \text{or} \quad k = t \quad \text{but} \quad \alpha^{ik} \nleqq \alpha^{jt}\,, \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

Since $H$ is nonsingular, we have

$$((\alpha^{ij})) = H^{-1}(\langle\alpha^{ij}\rangle)\,. \tag{3}$$

For infinite populations, the symmetric means become moment products and the polykays become cumulant products, so we get relations between such products.

Further, $\langle\alpha\rangle = [\alpha]/n^{(\phi(\alpha))}$ and unrestricted sum $\{\alpha\} = \sum_i [\alpha_i]$, where summation extends over all ordered partitions of which $\alpha$ is an ordered subpartition. Thus

$$(\{\alpha^{ij}\}) = H'([\alpha^{ij}]) \tag{4}$$

and

$$([\alpha^{ij}]) = T(\langle\alpha^{ij}\rangle) \tag{5}$$

where $T$ is a diagonal matrix with elements $t_{ii} = n^{(\phi(\alpha^{ji}))}$ for a fixed $j, j \in \{1, 2, \cdots, m\}$, $i \in \{M_1 + \cdots + M_{j-1} + 1, M_1 + \cdots + M_{j-1} + 2, \cdots, M_1 + \cdots + M_{j-1} + M_j\}$. Putting (3), (4), and (5) together,

$$((\alpha^{ij})) = (H'TH)^{-1}(\{\alpha^{ij}\})\,. \tag{6}$$

Clearly Carney's ([1], pages 647, 649) results become particular cases of the above results when $d$ is a fixed integer.

3.2 *Some results involving Kronecker products.*

Result 1. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be column vectors formed by the set of ordered partitions of weights $m_1$, $m_2$ respectively, so that $\boldsymbol{\alpha} \otimes \boldsymbol{\beta} = \boldsymbol{\rho}'$, the column vector composed of the ordered partition

$$\underbrace{11\cdots1}_{m_1}\ \underbrace{22\cdots2}_{m_2}$$

and all its ordered subpartitions. Then $\{\boldsymbol{\alpha}\} \otimes \{\boldsymbol{\beta}\} = \{\boldsymbol{\rho}'\}$.

Proof.

$$\begin{aligned} \{\boldsymbol{\alpha}\} \otimes \{\boldsymbol{\beta}\} &= \Lambda_1'[\boldsymbol{\alpha}] \otimes \Lambda_2'[\boldsymbol{\beta}]\,, \quad \text{where} \quad \Lambda_1, \Lambda_2 \quad \text{are upper triangular matrices with diagonal elements unity,} \\ &= (\Lambda_1' \otimes \Lambda_2')([\boldsymbol{\alpha}] \otimes [\boldsymbol{\beta}]) \\ &= \Lambda'[\boldsymbol{\alpha} \otimes \boldsymbol{\beta}] \\ &= \Lambda'[\boldsymbol{\rho}'] \\ &= \{\boldsymbol{\rho}'\}\,. \end{aligned}$$

The above result can easily be extended to

$$\{\boldsymbol{\alpha}_1\} \otimes \{\boldsymbol{\alpha}_2\} \otimes \cdots \otimes \{\boldsymbol{\alpha}_n\} = \{\boldsymbol{\rho}'\}$$

where $\boldsymbol{\alpha}_i$ is a column vector composed of ordered partitions of weight $m_i$ and $\boldsymbol{\rho}'$ is the column vector composed of the ordered partition

$$\underbrace{11 \cdots 1}_{m_1} \ \underbrace{22 \cdots 2}_{m_2} \cdots \underbrace{nn \cdots n}_{m_n}$$

and all its ordered subpartitions. In the following section this subset of ordered partitions in $\boldsymbol{\rho}$ is referred to as a sublattice since it forms a sublattice of the lattice of ordered partitions of weight $m_1 + m_2 + \cdots + m_n$ (proved in [6]).

RESULT 2. Let $\boldsymbol{\alpha}_1 \otimes \boldsymbol{\alpha}_2 \otimes \cdots \otimes \boldsymbol{\alpha}_n = \boldsymbol{\rho}'$. Then

$$(\boldsymbol{\alpha}_1) \otimes (\boldsymbol{\alpha}_2) \otimes \cdots \otimes (\boldsymbol{\alpha}_n) = (A_1 \otimes A_2 \otimes \cdots \otimes A_n)^{-1} B(\boldsymbol{\rho}) \tag{7}$$

where $A_i = \Lambda_i' N_i \Lambda_i$, $B$ is the matrix such that $\{\boldsymbol{\rho}'\} = B\{\boldsymbol{\rho}\}$, $\boldsymbol{\rho}'$ is the column vector composed of the ordered partition

$$\underbrace{11 \cdots 1}_{m_1} \ \underbrace{22 \cdots 2}_{m_2} \cdots \underbrace{nn \cdots n}_{m_n}$$

and all its ordered subpartitions, and $\boldsymbol{\rho}$ is the column vector of the ordered partitions of weight $m_1 + m_2 + \cdots + m_n$.

PROOF. Let $(\boldsymbol{\alpha})$ be the column vector formed by the products of polykays $(\boldsymbol{\alpha}_1) \otimes (\boldsymbol{\alpha}_2) \otimes \cdots \otimes (\boldsymbol{\alpha}_n)$. Then

$$\begin{aligned}
(\boldsymbol{\alpha}) &= (\boldsymbol{\alpha}_1) \otimes (\boldsymbol{\alpha}_2) \otimes \cdots \otimes (\boldsymbol{\alpha}_n) \\
&= A_1^{-1}\{\boldsymbol{\alpha}_1\} \otimes A_2^{-1}\{\boldsymbol{\alpha}_2\} \otimes \cdots \otimes A_n^{-1}\{\boldsymbol{\alpha}_n\} \\
&= (A_1^{-1} \otimes A_2^{-1} \otimes \cdots \otimes A_n^{-1})(\{\boldsymbol{\alpha}_1\} \otimes \{\boldsymbol{\alpha}_2\} \otimes \cdots \otimes \{\boldsymbol{\alpha}_n\}) \\
&= (A_1 \otimes A_2 \otimes \cdots \otimes A_n)^{-1}\{\boldsymbol{\rho}'\}, \qquad \text{by Result 1 above,} \\
&= (A_1 \otimes A_2 \otimes \cdots \otimes A_n)^{-1} B(\boldsymbol{\rho}) .
\end{aligned}$$

Using Rules 1 and 2 below, $A_1^{-1} \otimes A_2^{-1} \otimes \cdots \otimes A_n^{-1}$ and $B$ are reduced to $(A_1^{-1} \otimes A_2^{-1} \otimes \cdots \otimes A_n^{-1})^*$ and $B^*$, matrices of size $\nu \times \nu$, $\nu \times \mu$ respectively, where $\nu$ represents the number of distinct polykay products in $(\boldsymbol{\alpha})$ and $\mu$ the number of partitions of weight $m_1 + m_2 + \cdots + m_n$.

**4. Rules helpful in computation.** In this section we provide certain rules which are helpful in expressing products of polykays as linear combinations of the same using relation (7) above. Rules 1 and 2 help to reduce the sizes of the matrices $B$ and $A_1^{-1} \otimes A_2^{-1} \otimes \cdots \otimes A_n^{-1}$ respectively by eliminating rows and combining columns corresponding to ordered partitions $\alpha$ characterizing the same partition $\alpha^*$. Rules 3 to 7 are helpful in obtaining the l.u.b. of ordered partitions, i.e., the elements of the matrices $B$, $A_1^{-1}$, $A_2^{-1}$, $\cdots$, $A_n^{-1}$.

RULE 1. In matrix $B$ compute rows corresponding to the partitions $\alpha^*$'s rather than the ordered partitions $\alpha$'s. Add together the columns corresponding to all ordered partitions $\alpha$'s characterizing the same partition $\alpha^*$.

EXAMPLE. Let us consider the triple polykay product $(\boldsymbol{\alpha}_1) \otimes (\boldsymbol{\alpha}_2) \otimes (\boldsymbol{\alpha}_3)$ of weight 7 (see Section 5), where $(\boldsymbol{\alpha}_1) = [(111), (112), (121), (211), (123)]'$ and $(\boldsymbol{\alpha}_2) = (\boldsymbol{\alpha}_3) = [(11), (12)]'$. The columns of $B$ correspond to the 877 ordered partitions of weight 7 and the rows to the 20 ordered partitions of the vector $\boldsymbol{\rho}' = \boldsymbol{\alpha}_1 \otimes \boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_3$, i.e., 1112233 and all its ordered subpartitions. To eliminate the redundancy in $B$, compute rows of $B$ corresponding only to the ordered partitions 1112233, 1112234, 1112345, 1123344, 1123345, 1123456, 1234455, 1234456, 1234567 (only one ordered partition characterizing each partition). Each of these represents a distinct partition $\alpha^*$.

Add together the columns in $B$ which correspond to ordered partitions characterizing the fifteen partitions 7, 61, 52, 43, 511, 421, 331, 322, 4111, 3211, 2221, 31111, 22111, 211111 and 1111111, i.e., the columns to be added are 1, next 7, next 21, next 35, $\cdots$ (being entries in [3, Table 1.1.7]). Thus $B$ is reduced from a $20 \times 877$ matrix to a $9 \times 15$ matrix $B^*$.

This matrix $B^*$, by applying the rules concerning l.u.b. below, can be written without obtaining many of the 877 columns individually.

RULE 2. In the matrix $A_1^{-1} \otimes A_2^{-1} \otimes \cdots \otimes A_n^{-1}$ retain only one row corresponding to each polykay product. Add together the columns representing the same polykay product.

In order to determine which columns are to be added one has to consider the column vector of polykay products

$$(\boldsymbol{\alpha}) = (\boldsymbol{\alpha}_1) \otimes (\boldsymbol{\alpha}_2) \otimes \cdots \otimes (\boldsymbol{\alpha}_n) .$$

Let us suppose that $r$th and $s$th ($r < s$) components of $(\boldsymbol{\alpha})$ represent the same polykay product (there may be more than one $s$). Then add $s$th column of $A_1^{-1} \otimes A_2^{-1} \otimes \cdots \otimes A_n^{-1}$ to its $r$th column.

EXAMPLE. Let us consider again the above example. Thus $(\boldsymbol{\alpha}) = (\boldsymbol{\alpha}_1) \otimes (\boldsymbol{\alpha}_2) \otimes (\boldsymbol{\alpha}_3)$, where $(\boldsymbol{\alpha})$ has 20 components as below:

(8)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | (111) | (11) | (11) | 11. | (121) | (12) | (11) |
| 2. | (111) | (11) | (12) | 12. | (121) | (12) | (12) |
| 3. | (111) | (12) | (11) | 13. | (211) | (11) | (11) |
| 4. | (111) | (12) | (12) | 14. | (211) | (11) | (12) |
| 5. | (112) | (11) | (11) | 15. | (211) | (12) | (11) |
| 6. | (112) | (11) | (12) | 16. | (211) | (12) | (12) |
| 7. | (112) | (12) | (11) | 17. | (123) | (11) | (11) |
| 8. | (112) | (12) | (12) | 18. | (123) | (11) | (12) |
| 9. | (121) | (11) | (11) | 19. | (123) | (12) | (11) |
| 10. | (121) | (11) | (12) | 20. | (123) | (12) | (12) . |

Clearly certain components of $(\boldsymbol{\alpha})$ represent the same polykay product. For example component 1 is the only one representing $k_3k_2^2$, components 2, 3 both represent $k_3k_2k_{11}$, and so on. The components (indicated by their serial number in (8) above) belonging to the same set below represent the same polykay product.

$$\{1\}, \quad \{2, 3\}, \quad \{4\}, \quad \{5, 9, 13\}, \quad \{6, 7, 10, 11, 14, 15\}, \quad \{8, 12, 16\}, \quad \{17\}, \quad \{18, 19\}, \quad \{20\}. \tag{9}$$

Thus, by eliminating the redundant polykay products in $(\boldsymbol{\alpha})$, we get $(\boldsymbol{\alpha})^*$, a vector with 9 components. Further to remove the redundancy in $A_1^{-1} \otimes A_2^{-1} \otimes A_3^{-1}$, retain only 9 rows corresponding to the first element of each set in (9). Also add together the columns corresponding to each set in (9). Thus we obtain a reduced $9 \times 9$ matrix $(A_1^{-1} \otimes A_2^{-1} \otimes A_3^{-1})^*$.

Using Rules 1 and 2, (7) reduces to $(\boldsymbol{\alpha})^* = (A_1^{-1} \otimes A_2^{-1} \otimes A_3^{-1})^* B^*(\boldsymbol{\rho}^*)$, where $(\boldsymbol{\rho}^*)$ is the column vector of distinct polykays of degree $\sum_i m_i$.

REMARK. The matrix $(A_1^{-1} \otimes A_2^{-1} \otimes \cdots \otimes A_n^{-1})^*$ may also be obtained as follows:

*Case* I. *When all matrices* $A_1, A_2, \cdots, A_n$ *are of different dimension.* As described in Section 2.7, reduce the matrices $A_1^{-1}, A_2^{-1}, \cdots, A_n^{-1}$ to $(A_1^{-1})^*, (A_2^{-1})^*, \cdots, (A_n^{-1})^*$ and then obtain the Kronecker product of the reduced matrices, i.e.,

$$(A_1^{-1} \otimes A_2^{-1} \otimes \cdots \otimes A_n^{-1})^* = (A_1^{-1})^* \otimes (A_2^{-1})^* \otimes \cdots \otimes (A_n^{-1})^* .$$

*Case* II. *When some matrices are of same dimension.* Let $A_i, A_j; A_r, A_s, A_t; \cdots$ be of the same dimension. Reduce $(A_i^{-1} \otimes A_j^{-1}), (A_r^{-1} \otimes A_s^{-1} \otimes A_t^{-1}), \cdots$ to $(A_i^{-1} \otimes A_j^{-1})^*, (A_r^{-1} \otimes A_s^{-1} \otimes A_t^{-1})^*, \cdots$. Then the Kronecker product of the reduced matrices gives $(A_1^{-1} \otimes A_2^{-1} \otimes \cdots \otimes A_n^{-1})^*$.

It is important that the order of $\boldsymbol{\alpha}_i$'s in $((\boldsymbol{\alpha}_1) \otimes (\boldsymbol{\alpha}_2) \otimes \cdots \otimes (\boldsymbol{\alpha}_n))$ be the same as that of $A_i^{-1}$'s in $(A_1^{-1} \otimes A_2^{-1} \otimes \cdots \otimes A_n^{-1})$.

RULE 3. If ordered partition $\alpha$ of the lattice characterizes a one- or two-part partition, break it into parts of the same lengths as those of the runs of the ordered partition $\beta$ of the sublattice. If there is at least one part containing both the symbols, $\phi(\delta) = \phi(\alpha \vee \beta) = 1$, otherwise $\phi(\delta) = 2$.

EXAMPLE. Let $\alpha = 1111222$, $\beta = 1122233$. Write $\alpha = 11|112|22$, thus $\phi(\alpha \vee \beta) = 1$.

RULE 4. If ordered partition $\beta$ of the sublattice characterizes a two-part partition, break the ordered partition $\alpha$ of the lattice into two parts of the same lengths as those of the runs of $\beta$. Thus $\phi(\delta) = 1$ or 2 according as the two parts of $\alpha$ have a common symbol or not.

EXAMPLE. Let $\alpha = 1112223$, $\beta = 1112222$. Write $\alpha = 111|2223$. Clearly $\phi(\delta) = 2$ since there is no common symbol in the two parts of $\alpha$.

RULE 5. If each ordered partition of the lattice is broken up into parts of the same length as those of the runs of the ordered partition of the sublattice, then the ordered partitions obtained by permuting

(i) runs of the same length,
(ii) non-runs of the same length,
(iii) symbols within non-runs,

yield identical elements in the rows.

EXAMPLE. Let $\alpha = 1122345$, $\beta = 1112233$. Write $\alpha = 112\,|\,23\,|\,45$. Then the ordered partitions 1122345, 1124523, 1123245 yield identical elements in the row corresponding to the ordered partition $\beta$.

RULE 6. Let the first $r(>1)$ symbols of the ordered partition $\beta$ of the sublattice be identical and the rest $m - r$ distinct. Let $\alpha$'s of the lattice be broken up into two parts, the first of length $r$ and the second of length $m - r$. Let the number of distinct symbols in the second part, which have not already occurred in the first part, be equal ($t$, say). Then the $\alpha$'s yield identical elements and $\phi(\delta) = t + 1$.

EXAMPLE. Let $\alpha_1 = 1231145$, $\alpha_2 = 1123344$ and $\beta = 1112345$. Thus $\alpha_1 = 123\,|\,1145$, $\alpha_2 = 112\,|\,3344$; the second part in each case contains two distinct symbols which do not occur in the first part. So $\phi(\alpha_1 \vee \beta) = \phi(\alpha_2 \vee \beta) = 2 + 1 = 3$.

RULE 7. For the last row, i.e. the row corresponding to the ordered partition $\beta = 123 \cdots m$ of the sublattice, $\phi(\delta) = \phi(\alpha)$ for all $\alpha \in$ the lattice $S_m$, since $\alpha \vee \beta = \alpha$.

**5. Illustration.** We illustrate the method by using it to obtain triple polykay products of weight 7. The formulae so obtained extend the results in [3], [8] and [14].

Let $(\boldsymbol{\alpha}_1) = [(111)\ (112)\ (121)\ (211)\ (123)]'$, $(\boldsymbol{\alpha}_2) = (\boldsymbol{\alpha}_3) = [(11)\ (12)]'$. Then, we have

$$A_1 = \begin{bmatrix} n & n & n & n & n \\ n & n^2 & n & n & n^2 \\ n & n & n^2 & n & n^2 \\ n & n & n & n^2 & n^2 \\ n & n^2 & n^2 & n^2 & n^3 \end{bmatrix}, \qquad A_2 = A_3 = \begin{bmatrix} n & n \\ n & n^2 \end{bmatrix}.$$

Now $(\boldsymbol{\alpha}) = (\boldsymbol{\alpha}_1) \otimes (\boldsymbol{\alpha}_2) \otimes (\boldsymbol{\alpha}_3)$ is a column vector with 20 components, only 9 of which are distinct as in (9) above. Retaining only distinct polykay prodncts in $(\boldsymbol{\alpha})$, we have

$$\begin{aligned}(\boldsymbol{\alpha})^* = [&(111)(11)(11) \quad (111)(11)(12) \quad (111)(12)(12) \\ &(112)(11)(11) \quad (112)(11)(12) \quad (112)(12)(12) \\ &(123)(11)(11) \quad (123)(11)(12) \quad (123)(12)(12)]'.\end{aligned}$$

Thus, using Rule 2, $(A_1^{-1} \otimes A_2^{-1} \otimes A_3^{-1})^*$ equals $1/n^{(3)}n^{(2)}n^{(2)}$ times

$$\begin{bmatrix} n^4 & -2n^3 & n^2 & -3n^3 & 6n^2 & 2n^2 & -3n & -4n & 2 \\ -n^3 & n^2(n+1) & -n^2 & 3n^2 & -3n(n+1) & -2n & 3n & 2(n+1) & -2 \\ n^2 & -2n^2 & n^2 & -3n & 6n & 2 & -3n & -4 & 2 \\ -n^3 & 2n^2 & -n & n^2(n+1) & -2n(n+1) & -n^2 & n+1 & 2n & -1 \\ n^2 & -n(n+1) & n & -n(n+1) & (n+1)^2 & n & -(n+1) & -(n+1) & 1 \\ -n & 2n & -n & n+1 & -2(n+1) & -1 & n+1 & 2 & -1 \\ 2n^2 & -4n & 2 & -3n^2 & 6n & n^2 & -3 & -2n & 1 \\ -2n & 2(n+1) & -2 & 3n & -3(n+1) & -n & 3 & n+1 & -1 \\ 2 & -4 & 2 & -3 & 6 & 1 & -3 & -2 & 1 \end{bmatrix} .$$

Further, $B$ is a $20 \times 877$ matrix, which can be directly written in $20 \times 15$ form by applying Rules 1 and 3—7. This matrix can be further reduced by Rule 1 to $B^*$, which is $9 \times 15$. Denoting the column vector of distinct polykays of weight 7 by $(\boldsymbol{\rho}^*)$, we have

$$(\boldsymbol{\alpha})^* = (A_1^{-1} \otimes A_2^{-1} \otimes A_3^{-1})^* B^* (\boldsymbol{\rho}^*) .$$

Thus with $p = 1/n$, $q = 1/n(n-1)$, $r = 1/(n-1)$, $t = 1/(n-1)(n-2)$ and replacing (111)(11)(11), (111)(11)(12), $\cdots$, by (3)(2)(2), (3)(2)(11), $\cdots$ etc., we obtain Table 1 below, where the first column represents $(\boldsymbol{\alpha})^*$ and entries in $(\boldsymbol{\rho}^*)$ appear in the first row.

TABLE 1

| | (7) | (61) | (52) | (43) | (511) | (421) | (331) |
|---|---|---|---|---|---|---|---|
| (3)(2)(2) | $p^2$ | 0 | $2(n+7)q$ | $(n^2+22n-35)qr$ | 0 | 0 | 0 |
| (3)(2)(11) | 0 | $2p^2$ | $-10pq$ | $2(n^2-5n+10)q^2$ | $p$ | $2(n+5)q$ | $2(n+5)q$ |
| (3)(11)(11) | 0 | 0 | $4pq$ | $-12q^2$ | $4p^2$ | $8(n-4)pq$ | $-24pq$ |
| (21)(2)(2) | 0 | $p^2$ | $(n-5)pq$ | $2(n^2-4n+5)q^2$ | 0 | $3(n+3)q$ | $4(n-2)qr$ |
| (21)(2)(11) | 0 | 0 | $2pq$ | $2(n-3)q^2$ | $2p^2$ | $4(n-4)pq$ | $4(n^2-3n+3)q^2$ |
| (21)(11)(11) | 0 | 0 | 0 | $4q^2$ | 0 | $12pq$ | $4(2n-3)q^2$ |
| $(1^3)$(2)(2) | 0 | 0 | 0 | 0 | $3p^2$ | $-24pq$ | $6(n^2-2n+3)q^2$ |
| $(1^3)$(2)(11) | 0 | 0 | 0 | 0 | 0 | $12pq$ | $-12q^2$ |
| $(1^3)$(11)(11) | 0 | 0 | 0 | 0 | 0 | 0 | $12q^2$ |

| | (322) | $(41^3)$ | (3211) | $(2^31)$ | $(31^4)$ | $(221^3)$ | $(21^5)$ | $(1^7)$ |
|---|---|---|---|---|---|---|---|---|
| (3)(2)(2) | $(n^2+12n+35)r^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (3)(2)(11) | $-8(n+5)qr$ | 0 | $(n+5)r$ | 0 | 0 | 0 | 0 | 0 |
| (3)(11)(11) | $2(n^2-n+24)q^2$ | $4p$ | $4(n-4)q$ | 0 | 1 | 0 | 0 | 0 |
| (21)(2)(2) | $2(n^2-13)qr$ | 0 | 0 | $(n^2+4n+3)r^2$ | 0 | 0 | 0 | 0 |
| (21)(2)(11) | $4(n^2-n+6)q^2$ | $p$ | $(5n+1)q$ | $2(n^2-2n-3)qr$ | 0 | $(n+1)r$ | 0 | 0 |
| (21)(11)(11) | $12(n-3)q^2$ | $4p^2$ | $12(2n-3)pq$ | $2(5n^2-13n+12)q^2$ | $4p$ | $4(2n-3)q$ | 1 | 0 |
| $(1^3)$(2)(2) | $12(n-3)q^2$ | $p$ | $6(n+1)q$ | $-12(n+1)qr$ | 0 | $(n+1)r$ | 0 | 0 |
| $(1^3)$(2)(11) | $6(n-5)q^2$ | $6p^2$ | $18(n-3)pq$ | $6(n^2-3n+6)q^2$ | $5p$ | $2(3n-7)q$ | 1 | 0 |
| $(1^3)$(11)(11) | $24q^2$ | 0 | $60pq$ | $24(2n-3)q^2$ | $12p^2$ | $2(31n-24)pq$ | $16p$ | 1 |

Table 1 gives the nine triple polykay products as linear combinations of the polykays of weight 7. For example, the first formula reads

$$(3)(2)(2) = (7)/n^2 + 2(n+7)(52)/n(n-1) + (n^2+22n-35)(43)/n(n-1)^2 \\ + (n+5)(n+7)(322)/(n-1)^2$$

which checks with [3], [14]. The formulae for the other eight triple products, which are all new, can similarly be read.

For completeness double products of weight 7 are worked out in the same manner by the authors and presented above. It may be remarked that only four of these have appeared earlier in print (besides [9]), viz., (5)(2), (4)(3) in [14] and (32)(2), (22)(3) in [8].

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