The Annals of Statistics
1973, Vol. 1, No. 5, 872-887

LAWS OF THE ITERATED LOGARITHM FOR PERMUTED RANDOM VARIABLES AND REGRESSION APPLICATIONS[1]

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In this paper Laws of the Iterated Logarithm for maximums of absolute values of partial sums of permuted random variables are derived under conditions that are the same as or similar to conditions used by Kolmogorov, Hartman and Wintner, Petrov and Csáki in deriving Laws of the Iterated Logarithm for sums of random variables or semimartingales. These results are then applied to obtain logarithmic convergence rates for estimators of non-decreasing regression functions and integral regression functions.

1. Introduction and summary. Iterated logarithm convergence rates are established for maximums of absolute values of partial sums of permuted random variables under conditions that are the same or similar to conditions used by Kolmogorov [7], Hartman and Wintner [6], Petrov [11] and Csáki [3] in studying Laws of the Iterated Logarithm for sums of random variables or semimartingales. This type of maximum has been studied by Brunk [1, 2], who proved a Strong Law of Large Numbers for these maximums and showed that convergence rates for these maximums yielded convergence rates for certain estimators of non-decreasing regression functions and integral regression functions. New convergence rates will be obtained for these regression estimators.

The definition of maximums of absolute values of partial sums of permuted random variables depends upon a certain type of sequence of permutations of positive integers which is called order preserving. An order preserving sequence of permutations is a sequence of permutations that satisfies the following conditions. The nth permutation in the sequence is a permutation of the integers $1, 2, \cdots, n$. The first permutation is the identity permutation. And when n is an integer greater than one, the order of the integers $1, 2, \cdots, n-1$, as they appear in the permutation corresponding to n is the same as their order of appearance in the permutation corresponing to $n-1$.

We now define the sequence of maximums of absolute values of partial sums of permuted random variables that arises from any sequence $\{X_n\}$ of random variables and any order preserving sequence of permutations, where the permutation

Received December 1971; revised January 1973.

[1] The content of this paper is a revised version of part of the author's Ph. D. dissertation which was prepared under the supervision of Drs. Tim Robertson and Jonathan Cryer at the University of Iowa.

AMS 1970 *subject classifications.* Primary 60F99, 60G45; Secondary 60G50.

Key words and phrases. Iterated logarithm, order preserving permutation, maximum of partial sum, semimartingale, integral regression, non-decreasing regression, Galtonian regression.

of 1, 2, $\cdots$, n is given by $i_{1n}, i_{2n}, \cdots, i_{nn}$. Let $S_n = \sum_{v=1}^{n} X_v$ and $S_{jn} = \sum_{v=1}^{j} X_{i_{vn}}$. Then set $R_n = \max_{j \leqq n} S_{jn}$ and $R_n' = \max_{j \leqq n} |S_{jn}|$. The sequence $\{R_n'\}$ is called the sequence of maximums of absolute values of partial sums of permuted random variables that arises from $\{X_n\}$ and the sequence of order preserving permutations. Observe that if the order preserving permutations are the identity permutations, then $S_{jn} = S_j$, $R_n = \max_{j \leqq n} S_j$ and $R_n' = \max_{j \leqq n} |S_j|$.

Throughout the remainder of this paper, $\{X_n\}$ will be a sequence of independent random variables that are centered at expectations and have finite positive variances. Let $s_n{}^2$ and s_{jn}^2 denote the variances of S_n and S_{jn} respectively, and set $r_n = (2 \log\log s_n{}^2)^{\frac{1}{2}}$.

Laws of the Iterated Logarithm concerning $\{S_n\}$, that is, statements of the form

$$P[\limsup_n S_n/(s_n r_n) = 1] = 1 \tag{1.1}$$

have been obtained under various assumptions. It is well known that if equation (1.1) holds for $\{S_n\}$ and $\lim_n s_n{}^2 = +\infty$, then

$$P[\limsup_n \max_{j \leqq n} S_j/(s_n r_n) = 1] = 1$$

also holds for $\{X_n\}$. We will be concerned with establishing convergence rates of the form

$$P[1 \leqq \limsup_n \max_{j \leqq n} |S_{jn}|/(s_n r_n) \leqq \mathscr{K}] = 1$$

where $\mathscr{K}$ denotes a known positive constant.

2. Iterated logarithm convergence rates for $\max_{j \leqq n} |S_{jn}|$ **under Kolmogorov's and Hartman and Wintner's assumptions.** Consider the following condition concerning $\{X_n\}$.

CONDITION K. $\lim_n s_n{}^2 = +\infty$ and

$$s_n{}^{-1} \operatorname{ess\,sup} |X_n| = o(r_n{}^{-1}) . \tag{2.1}$$

Kolmogorov [7] proved that if Condition K is satisfied by $\{X_n\}$, then (1.1) obtains. It will now be shown that a Law of the Iterated Logarithm holds for R_n and R_n' under Condition K.

THEOREM 1. *Let* $\{X_n\}$ *be a sequence of random variables satisfying Condition* K. *Then*

$$P[1 \leqq \limsup_n R_n/(s_n r_n) \leqq 4(2)^{\frac{1}{2}}] = 1 \tag{2.2}$$

$$P[1 \leqq \limsup_n R_n'/(s_n r_n) \leqq 4(2)^{\frac{1}{2}}] = 1 . \tag{2.3}$$

OUTLINE OF THE PROOF. We first demonstrate that (2.2) holds. Since $R_n \geqq S_n$, we can conclude by (1.1) that

$$P[1 \leqq \lim_n R_n/(s_n r_n)] = 1 .$$

If it can be shown that

$$P[\limsup_n R_n/(s_n r_n) \leqq 4(2)^{\frac{1}{2}}] = 1 , \tag{2.4}$$

then (2.2) will follow. Since $\lim_n s_{n+1}^2/s_n{}^2 = 1$, for every number c that exceeds one there is a sequence $\{n_k\}$ of positive integers, eventually increasing, such that $s_{n_k} \sim c^k$. In what follows c is an arbitrary number that exceeds one. Later on c will be fixed and its value will depend upon positive numbers δ and δ'.

Following the method of Brunk [1], we now define terms which will be used in the remainder of this proof. We can assume that the sequence $\{n_k\}$ described above is increasing. Set $n_0 = 0$ and $n_1 = 1$. For $k = 1, 2, 3, \cdots$ arrange the terms X_i having indices i such that $n_{k-1} < i \leqq n_k$ in the order given by the permutation for n_k, and let $\Upsilon(k)$ denote the family of partial sums containing the first of these terms, the sum of the first two, and the sum of the first three, etc. Now consider partial sums S_{jn} when $j \leqq n$. For each n, choose $k = k(n)$ so that $n_{k-1} < n \leqq n_k$. Let $Z_1, Z_2, \cdots, Z_{n_k - n}$ denote the random variables $X_{n+1}, X_{n+2}, \cdots, X_{n_k - n}$ written in the order given by the permutation for n_k. Let $U(n)$ denote the family of partial sums $\{Z_1, Z_1 + Z_2, \cdots, Z_1 + Z_2 + \cdots + Z_{n_k - n}\}$.

For fixed j and n, and for $v = 1, 2, \cdots, k-1$, let $L_v = L_v(j, n)$ denote the sum of terms X_i which appear in the sum S_{jn} and which have indices i such that $n_{v-1} < i \leqq n_v$. Then L_v is an element of $\Upsilon(v)$ for $v = 1, 2, \cdots, k-1$. Let $L_k = L_k(j, n)$ denote the minimal member of $\Upsilon(k)$ containing all terms appearing in S_{jn} whose indices i satisfy $n_{k-1} < i \leqq n_k$ (minimal in the sense of containing the fewest possible terms.) Let $\mathscr{U} = \mathscr{U}(j, n)$ be the sum of terms appearing in $L_k(j, n)$ of index greater than n; then $\mathscr{U} \in U(n)$ and $S_{jn} = \sum_{v=1}^{k} L_v - \mathscr{U}$. Let $\mathscr{V}(k)$ denote the family of all sums of the form $\sum_{v=1}^{k} W_v$, where for $v = 1, 2, \cdots, k$, W_v is an element of $\Upsilon(v)$. Let $V = V(j, n) = \sum_{v=1}^{k} L_v(j, n)$. Then V is an element of $\mathscr{V}(k)$ and $S_{jn} = V - \mathscr{U}$. Finally, let max V_k and max W_i stand for $\max_{V \in \mathscr{V}(k)} V$ and $\max_{W_i \in \Upsilon(i)} W_i$, respectively.

Equation (2.4) holds if for every positive δ

$$P[R_l > (4(2)^{\frac{1}{2}} + \delta)s_l r_l \text{ i.o.}] = 0 .$$

Set $D_k = s_{n_k} r_{n_k}$. It will be established that $P[R_l > (4(2)^{\frac{1}{2}} + \delta)s_l r_l \text{ i.o.}]$ vanishes by bounding the above probability by $2P[\max V_k > (4(2)^{\frac{1}{2}} + \delta/2)D_{k-1} \text{ i.o.}]$ and then showing that this number equals zero.

We now verify that

$$P[R_l > (4(2)^{\frac{1}{2}} + \delta)s_l r_l \text{ i.o.}] \leqq 2P[\max V_k > (4(2)^{\frac{1}{2}} + \delta/2)D_{k-1} \text{ i.o.}] . \tag{2.5}$$

Set $A_l = [R_l > (4(2)^{\frac{1}{2}} + \delta)s_l r_l]$, $B_l = [-\min_{j \leq l} \mathscr{U}(j, l) \leqq (\delta/2)s_l r_l]$ and $C_l = [\max V_{k(l)} > (4(2)^{\frac{1}{2}} + \delta/2)s_l r_l]$.

Using Kolmogorov's Inequality, the fact that

$$-\min_{j \leq l} \mathscr{U}(j, l) = \max_{j \leq l} (-\mathscr{U}(j, l))$$

is equal to a nested partial sum of X_n's, and the relationships $s_{n_k} \sim c^k$ and $\lim_l n_{k(l)} = +\infty$, it can be shown that $\lim_l P(B_l) = 1$. By this equation, the set containment $A_l B_l \subset C_l$, and the Lemma for Events ([8], page 246), it follows that $P(\bigcup_{v=l}^{\infty} A_v) \leqq 2P(\bigcup_{v=l}^{\infty} A_v B_v) \leqq 2P(\bigcup_{v=l}^{\infty} C_l)$ for l sufficiently large. Equation (2.5) follows easily from this inequality.

Let $0 < \delta' < \delta/(4(2)^{\frac{1}{2}})$. Then we can select a number $c > 2^{\frac{1}{2}}$ such that $(4(2)^{\frac{1}{2}} + \delta/2)/c > 4 + 2\delta'$. By this inequality and the equation $\lim_k cD_{k-1}/D_k = 1$ it follows that

$$P[\max V_k > (4(2)^{\frac{1}{2}} + \delta/2)D_{k-1} \text{ i.o.}] = P[\max V_k > (4 + \delta')D_k \text{ i.o.}] .$$

If $P[\max V_k > (4 + \delta')D_k]$ is a general term of a convergent series then by the Borel–Cantelli Lemma we can conclude that $P[\max V_k > (4 + \delta')D_k \text{ i.o.}]$ vanishes and hence that (2.2) holds. Accordingly, asymptotic bounds will now be derived for $P[\max W_i > (4 + \delta')D_k/(k[k/2])]$, when $1 \leqq i \leqq [k/2]$, and $P[\max W_i > (4 + \delta')(D_i - D_{i-1})(k - 1)/k]$ when $[k/2] + 1 \leqq i \leqq k$, in order to deduce that both members of the relation

$$\begin{aligned} &P[\max V_k > (4 + \delta')D_k] \\ &\quad \leqq \textstyle\sum_{i=1}^{[k/2]} P[\max W_i > (4 + \delta')D_k/(k[k/2])] \\ &\qquad + \textstyle\sum_{i=[k/2]+1}^{k} P[\max W_i > (4 + \delta')(D_i - D_{i-1})(k - 1)/k] \end{aligned} \tag{2.6}$$

are general terms of convergent series.

We first consider $P[\max W_i > (4 + \delta')D_k/(k[k/2])]$ when $1 \leqq i \leqq [k/2]$. Since $\{X_n\}$ is a sequence of independent random variables that have finite variances and $\max W_i$ is a maximum of nested partial sums of X_n's, we can apply Kolmogorov's Inequality to $\max W_i$, concluding that

$$\max_{1 \leqq i \leqq [k/2]} P[\max W_i > (4 + \delta')D_k/(k[k/2])] \leqq 1/k^4 \tag{2.7}$$

for k sufficiently large.

Now an asymptotic bound will be derived for

$$P[\max W_i > (4 + \delta')(D_i - D_{i-1})(k - 1)/k]$$

when $[k/2] + 1 \leqq i \leqq k$. Let P_i be this probability. Then for any positive number η less than δ', P_i is bounded above by $P[\max W_i > (4 + \eta)(D_i - D_{i-1})]$ for k sufficiently large.

Let $p_i = (s_{n_i}^2 - s_{n_{i-1}}^2)^{\frac{1}{2}}$. By applying the Remark given by Loève ([8], page 248) to $\max W_i$ we conclude that $P[\max W_i > (4 + \eta)(D_i - D_{i-1})]$ is bounded above by $2P[S_{n_i} - S_{n_{i-1}} > (4 + \eta)(D_i - D_{i-1}) - 2^{\frac{1}{2}}p_i]$.

Let $0 < \eta' < \eta$. Then for k sufficiently large we have

$$P_i \leqq 2P[(S_{n_i} - S_{n_{i-1}})/p_i > (4 + \eta')(D_i - D_{i-1})/p_i] . \tag{2.8}$$

One result of Loève's Remark A, ([8], page 254), is that when $\{X_n\}$ is a sequence of independent random variables that are centered at expectations and g is a positive number such that $gc \leqq 1$ where $c = \max_{k \leqq n} \text{ess sup } |X_k|/s_n$, then

$$P[S_n/s > g] < \exp[(-g^2/2)(1 - gc/2)] .$$

We can apply this result to (2.8) by setting g_i equal to $(4 + \eta')(D_i - D_{i-1})/p_i$ and c_i equal to $\max_{n_{i-1} < i \leqq n_i} \text{ess sup } |X_i|/p_i$. We then conclude that $P[(S_{n_i} - S_{n_{i-1}})/p_i > g_i]$ is bounded above by $\exp[-(g_i^2/2)(1 - g_i c_i/2)]$.

Let $0 < \delta'' < \eta'$. Since $\lim_i g_i c_i = 0$, for k sufficiently large and $[k/2] \leqq i \leqq k$, we have $(4 + \eta')^2(1 - g_i c_i/2) \geqq 16 + \delta''$. By the last two inequalities we write:

$$(2.9) \qquad P[(S_{n_i} - S_{n_{i-1}})/p_i > g_i] < \exp\left[\frac{-(16 + \delta'')[s_{n_i}(\log\log s_{n_i}^2)^{\frac{1}{2}} - s_{n_{i-1}}(\log\log s_{n_{i-1}}^2)^{\frac{1}{2}}]^2}{s_{n_i}^2 - s_{n_{i-1}}^2}\right].$$

Using the Mean Value Theorem for Derivatives, we can bound the term $[s_{n_i}(\log\log s_{n_i}^2)^{\frac{1}{2}} - s_{n_{i-1}}(\log\log s_{n_{i-1}}^2)^{\frac{1}{2}}]^2$ below by

$$(\log\log v_i)^2(s_{n_i}^2 - s_{n_{i-1}}^2)/(4v_i \log\log v_i)$$

for i sufficiently large. And by using the relation $s_{n_i} \sim c^i$ and $c > 2^{\frac{1}{2}}$ this latter term can be bounded below by $\log[(2i - 2)\log c - 1]/8$ for i sufficiently large. Consequently, by the use of (2.9) and these lower bounds, we deduce that $P[(S_{n_i} - S_{n_{i-1}})/p_i < g_i]$ is bounded above by $1/\exp[(2+\delta''/8)\log[(2i-2)\log c - 1]]$ for i sufficiently large.

It follows from this last inequality that for k sufficiently large,

$$\max_{[k/2]+1 \leqq i \leqq k} P[(S_{n_i} - S_{n_{i-1}})/p_i > g_i] < 4/(2k\log c)^{2+\delta''/8}.$$

This inequality and inequality (2.8) yield the conclusion

$$\max_{[k/2]+1 \leqq i \leqq k} P_i \leqq 8/(2k\log c)^{2+\delta''/8}$$

for k sufficiently large. From this equation and (2.6) and (2.7) it follows that

$$P[\max V_k > (4 + \delta')D_k] \leqq \mathscr{C}/k^{1+\xi}$$

for some positive constants $\mathscr{C}$ and ξ. Hence, the above probability is a general term of a convergent series and (2.2) is established.

Equation (2.3) follows from (2.2) applied to $\max_{j \leqq n}(-S_{jn})$.

Hartman and Wintner [6] showed that (1.1) holds under certain assumptions which we shall call Condition HW.

Condition HW. $\{X_n\}$ is a sequence of independent random variables having cumulative distribution functions $Q_n(\cdot)$. These random variables are centered at expectations and have finite variances σ_n^2. s_n^2/n is bounded away from zero and there is a cumulative distribution function $\mathscr{T}$, satisfying $\int x^2\, d\mathscr{T}(x) < \infty$, such that

$$(2.10) \qquad \sup_n \int_{[|y| \geqq x]} dQ_n(y) = O\left(\int_{[|y| \geqq x]} d\mathscr{T}(y)\right).$$

We now use Condition HW to investigate R_n and R_n'.

Remark 2. Let $\{X_n\}$ satisfy Condition HW. Then

$$(2.11) \qquad P[1 \leqq \limsup_n R_n/(s_n r_n) \leqq 4(2)^{\frac{1}{2}}] = 1$$

$$(2.12) \qquad P[1 \leqq \limsup_n R_n'/(s_n r_n) \leqq 4(2)^{\frac{1}{2}}] = 1.$$

Proof. We establish (2.11) and (2.12) together. Hartman and Wintner [6] show there are sequences $\{Z_n\}$ and $\{Y_n\}$ of random variables and a sequence $\{\alpha_n\}$

of real numbers such that $\{Z_n\}$ satisfies Condition K,

(2.13) $$\sum_{j=1}^{n} |Y_j| = o[(n \log\log n)^{\frac{1}{2}}] ,$$

(2.14) $$\sum_{j=1}^{n} |\alpha_j| = o[(n \log\log n)^{\frac{1}{2}}] ,$$

and $X_n = Z_n + Y_n + \alpha_n$.

The last equation can be used to show that

$$\max_{j \leqq n} |\sum_{v=1}^{j} X_{i_{vn}}| \leqq \max_{j \leqq n} |\sum_{v=1}^{j} Z_{i_{vn}}| + \sum_{v=1}^{n} |Y_v| + \sum_{v=1}^{n} |\alpha_v| .$$

By this inequality, (2.13) and (2.14) and the fact that $s_n{}^2/n$ is bounded away from zero, we find that

$$R_n' \leqq \max_{j \leqq n} |\sum_{v=1}^{j} Z_{i_{vn}}| + o(s_n r_n) .$$

The conclusion follows from this inequality, the application of Theorem 1 to $\{Z_n\}$, the application of Hartman and Wintner's Law of the Iterated Logarithm to $\{S_n\}$, and the relation $S_n \leqq R_n \leqq R_n'$.

3. Iterated logarithm convergence rates for $\max_{j \leqq n} |S_{jn}|$ **under a normal convergence criterion.** Consider the following condition concerning $\{X_n\}$.

Condition P. $\{X_n\}$ is a sequence of independent random variables that are centered at expectations and have finite positive variances $\sigma_n{}^2$. Furthermore, $\lim_n s_n{}^2 = +\infty$ and $\lim_n s_{n+1}^2/s_n{}^2 = 1$. Finally, there is a positive number δ such that $\Delta_n = O[1/(\log s_n{}^2)^{1+\delta}]$ where $\Delta_n = \sup_x |F_n(x) - N(x)|$, $F_n(x) = P[S_n \leqq x s_n]$ and $N(x) = \int_{-\infty}^{x} (2\Pi)^{-\frac{1}{2}} \exp(-t^2/2)\, dt$.

Petrov [11] showed that Condition P is sufficient for (1.1). Following Petrov, we introduce the following Condition P′. We will show that under Condition P′ a Law of the Iterated Logarithm obtains for R_n'. An example has been given in [9] to show that Condition P′ is sufficient but not necessary for Condition P, as well as Condition P with $O[1/(\log s_n{}^2)^{1+\delta}]$ replaced by $O[1/(\log s_n{}^2)^{2+\delta}]$.

Condition P′. $\{X_n\}$ satisfies Condition P and in addition there is a positive number ε such that

(3.1) $$\limsup_n \sup_{k \in \mathscr{P}} [\log (s_{k+n}^2 - s_k{}^2)]^{2+\varepsilon} \Delta_{nk} < \infty$$

where $\Delta_{nk} = \sup_x |F_{nk}(x) - N(x)|$, $F_{nk}(x) = P[S_{k+n} - S_k \leqq (s_{k+n}^2 - s_k{}^2)^{\frac{1}{2}} x]$, and $\mathscr{P}$ denotes the positive integers.

Theorem 3. *If* $\{X_n\}$ *satisfies Condition* P′, *then*

(3.2) $$P[1 \leqq \limsup_n R_n/(s_n r_n) \leqq 2(6)^{\frac{1}{2}}] = 1$$

(3.3) $$P[1 \leqq \limsup_n R_n'/(s_n r_n) \leqq 2(6)^{\frac{1}{2}}] = 1 .$$

Outline of the Proof. We first establish (3.2). Since Condition P′ is sufficient for Condition P, (1.1) holds for S, and thus $P[1 \leqq \limsup_n R_n/(s_n r_n)] = 1$. If it can be shown that

(3.4) $$P[\limsup_n R_n/(s_n r_n) \leqq 2(6)^{\frac{1}{2}}] = 1 ,$$

then the conclusion (3.2) will follow.

A large part of this proof is identical to part of the proof of Theorem 1, and will not be repeated here. Any symbol occurring in this proof that was previously used in the proof of Theorem 1 has the same definition of the one given in the earlier argument.

Equation (3.4) will follow if we can show that

$$P[R_l > (2(6)^{\frac{1}{2}} + \delta)s_l r_l \text{ i.o.}] \leqq 2P[\max V_k > (2(6)^{\frac{1}{2}} + \delta/2)D_{k-1} \text{ i.o.}] .$$

We wish to show that the right-hand member of this inequality equals zero. Given a positive $\delta' < \delta/8$, we can select c, $2 < c < 3$, such that $(2(6)^{\frac{1}{2}} + \delta/2)/c > (6)^{\frac{1}{2}} + 2\delta'$. Then

$$P[\max V_k > (2(6)^{\frac{1}{2}} + \delta/2)D_{k-1} \text{ i.o.}] \leqq P[\max V_k > (6^{\frac{1}{2}} + \delta')D_k \text{ i.o.}] .$$

We now show that $P[\max V_k > (6^{\frac{1}{2}} + \delta')D_k]$ is a general term of a convergent series. The following inequality is needed:

$$\begin{aligned} &P[\max V_k > (6^{\frac{1}{2}} + \delta')D_k] \\ &\quad \leqq \textstyle\sum_{i=1}^{[k/2]} P[\max W_i > (6^{\frac{1}{2}} + \delta')D_k/(k[k/2])] \\ &\qquad + \textstyle\sum_{i=[k/2]+1}^{k} P[\max W_i > (6^{\frac{1}{2}} + \delta')(D_i - D_{i-1})(k-1)/k] . \end{aligned} \tag{3.5}$$

Asymptotic bounds must be derived for each type of summand of (3.5). The first type is bounded above by $1/k^4$ for k sufficiently large, as before.

Next consider

$$P[\max W_i > (6^{\frac{1}{2}} + \delta')(D_i - D_{i-1})(k-1)/k] .$$

Let P_i' be the above probability, and let $0 < \delta'' < \delta'/2$. As before, we conclude that

$$P_i' \leqq 2P[S_{n_i} - S_{n_{i-1}} \geqq (6^{\frac{1}{2}} + \delta'')(D_i - D_{i-1})]$$

whenever $[k/2] + 1 \leqq i \leqq k$ for k sufficiently large. By this inequality and (3.1) of Condition P′ we can write

$$\begin{aligned} P_i' &\leqq 2P[S_{n_i} - S_{n_{i-1}} \geqq [(6^{\frac{1}{2}} + \delta'')(D_i - D_{i-1})/p_i]p_i] \\ &= 2\int_{(6^{\frac{1}{2}}+\delta'')(D_i-D_{i-1})/p_i}^{\infty} \exp(-t^2/2)/(2\pi)^{\frac{1}{2}}\, dt + O[(1/\log p_i^2)^{2+\varepsilon}] , \end{aligned}$$

Following Petrov [11] we conclude that

$$P_i' \leqq \frac{2\exp[-(6^{\frac{1}{2}} + \delta'')^2(D_i - D_{i-1})^2/(2p_i^2)]}{(6^{\frac{1}{2}} + \delta'')(D_i - D_{i-1})/p_i} + \mathscr{I}/(\log p_i^2)^{2+\varepsilon}$$

for some positive constant $\mathscr{I}$ for i sufficiently large. There are positive numbers η, η', and η'' such that $(6^{\frac{1}{2}} + \delta'')^2 > 6^{\frac{1}{2}} + \eta$ and $(1 - \eta')^2(2 + \eta/3) > 2 + \eta''$. These inequalities and the relation $s_{n_i} \sim c^i$ enable us to deduce that $P_i' \leqq 1/(2i)^{2+\eta''} + \mathscr{I}_1/(2i)^{2+\varepsilon}$ for some positive constant $\mathscr{I}_1$, for k sufficiently large. The conclusions follow.

We now give two results that specify conditions under which Condition P′ holds.

REMARK 4. Let $\{X_n\}$ be a sequence of independent random variables that are centered at expectations, such that $0 < \inf_n EX_n^2$ and $\sup_n E|X_n|^3 < \infty$; then these random variables satisfy Condition P′.

This result can be obtained by use of Hölder's inequality and the Normal Approximation Theorem, given by Loève ([8], page 288). The next result, Remark 5, generalizes what we have just established. The assumptions used here do not, however, have the simplicity of the assumptions used in Remark 4.

REMARK 5. Let $\{X_n\}$ be a sequence of independent random variables centered at expectations. Let g be a positive function satisfying: g is even and non-decreasing on $(0, \infty)$ with $\lim_x g(x) = +\infty$; the function $x/g(x)$ is defined for all real numbers x and is non-decreasing on $(0, \infty)$, and $\lim_{x\to\infty} [\log x]^{2+\delta}/g(x) = 0$. Assume further that

$$0 < \inf_n EX_n^2 \leqq \sup_n EX_n^2 < \infty \quad \text{and} \quad 0 < \sup_n E[X_n^2 g(X_n)] < \infty .$$

Then Condition P′ holds for $\{X_n\}$.

This Remark is easily justified using a result given by Petrov [10].

4. Semimartingales. Set $R_n'' = (R_n')^2$. It will first be shown that $\{R_n\}$, $\{R_n'\}$, and $\{R_n''\}$ are semimartingales. We can then apply semimartingale results of Csáki [3] and Darling and Robbins [4] to study convergence rates for $\{R_n\}$, $\{R_n'\}$, and $\{R_n''\}$.

THEOREM 6. *Let $\{X_n\}$ be a sequence of independent random variables.*

(i) *If these random variables have nonnegative expectations, then $\{R_n\}$ is a semimartingale.*

(ii) *If these random variables are centered at expectations, then $\{R_n'\}$ and $\{R_n''\}$ are semimartingales.* (The proof of this theorem does not require the variance of X_n to be finite.)

PROOF. We first establish that (i) holds. For any random variables $Z_1, Z_2, \cdots, Z_n$, let $\sigma\{Z_1, \cdots, Z_n\}$ denote the sigma field generated by $Z_1, Z_2, \cdots, Z_n$. We wish to show that

$$E(R_n \mid R_1, \cdots, R_{n-1}) \geqq R_{n-1} \quad \text{a.s.}, \tag{4.1}$$

or equivalently that

$$\int_A E(R_n \mid R_1, \cdots, R_{n-1})\, dP \geqq \int_A R_{n-1}\, dP \tag{4.2}$$

for all $A \in \sigma\{R_1, \cdots, R_{n-1}\}$. The verification of (4.2) can be accomplished in three parts, corresponding to the cases where the integer n is given the first, second through $(n-1)$st, or nth place by the permutation $i_{1n}, i_{2n}, \cdots, i_{nn}$ of $1, 2, \cdots, n$.

Assume first that $n = i_{l+1n}$ for some positive $l < n - 1$. Then

$$R_n = \max\{S_{n-11}, S_{n-12}, \cdots, S_{n-1l}, S_{n-1l} + X_n, \cdots, S_{n-1\,n-1} + X_n\} .$$

Set $H = [R_{n-1} > S_{ln-1}, \cdots, R_{n-1} > S_{n-1\,n-1}]$ and $H' = \Omega - H$. Then for $A \in \sigma\{R_1, \cdots, R_{n-1}\}$, we write

$$\begin{aligned} \int_A E(R_n \mid R_1, \cdots, R_{n-1})\, dP &= \int_A R_n\, dP = \int_{AH} R_n\, dP + \int_{AH'} R_n\, dP \\ &\geqq \int_{AH} R_{n-1}\, dP + \int_{AH'} (R_{n-1} + X_n)\, dP \\ &\geqq \int_A R_{n-1}\, dP\,. \end{aligned}$$

Thus (4.2) and (4.1) follow in this case. The remaining cases are easily verified.

The proof of Proposition (ii) will now be outlined. First we consider $\{R_n'\}$. Set $\mathscr{F} = \{S_{11}, S_{12}, S_{22}, \cdots, S_{n-1\,n-1}\}$ and $R_n^0 = \max_{j \leqq n} (-S_{jn})$. The arguments that $E(R_n' \mid \mathscr{F}) \geqq E(R_n \mid \mathscr{F}) \geqq R_{n-1}$ a.s. and $E(R_n' \mid \mathscr{F}) \geqq R_{n-1}^0$ a.s. are similar to the proof of (4.1). From these ineqalities and

$$E(R_n' \mid R_1', \cdots, R_{n-1}') = E[E(R_n' \mid \mathscr{F}) \mid R_1', \cdots, R_{n-1}'] \geqq R_{n-1}' \quad \text{a.s.}$$

one deduces that $\{R_n'\}$ is a semimartingale.

Since $R_n'' = (R_n')^2$, $\{R_n''\}$ is a semimartingale and the proof of Theorem 1 is concluded.

We will now study convergence rates for R_n' by application of a Law of the Iterated Logarithm that holds for a type of semimartingale. Csáki [3] proved the following theorem.

THEOREM 7. *Let $\{Z_n\}$ be a semimartingale such that the moment generating functions of $\{Z_n\}$ exist in an interval of the form $[0, a]$, $a > 0$. Furthermore assume that*

$$E[\exp(tZ_n)] \leqq \mathscr{I}[\Psi(t)]^n\,, \tag{4.3}$$

for t in $[0, \alpha]$, where $\mathscr{I} > 0$ and $\Psi(t)$ is a function such that

$$\Psi(t) = 1 + (A^2/2)t^2 + O(t^3)\,; \qquad t \to 0+ \tag{4.4}$$

for some constant $A > 0$. Then

$$P[\limsup_n (2n \log\log n)^{-\frac{1}{2}} Z_n \leqq A] = 1\,. \tag{4.5}$$

By use of Csáki's result we can obtain a Law of the Iterated Logarithm for R_n' under the following condition.

CONDITION C′. $\{X_n\}$ is a sequence of independent random variables that are centered at expectations and have moment generating functions existing in an interval of the form $[-a, a]$ where $a > 0$. Furthermore,

$$\max\{\sup_n E[\exp(tX_n)], \sup_n E[\exp(-tX_n)]\} \leqq \Psi(t)$$

for t in $[0, a]$, where Ψ satisfies (4.4).

REMARK 8. Let $\{X_n\}$ be a sequence of random variables that satisfies Condition C′ for some $A > 0$. Then

$$P[\limsup_n (2n \log\log n)^{-\frac{1}{2}} R_n' \leqq A] = 1\,. \tag{4.6}$$

The proof can be accomplished by applying Csáki's theorem to R_n and R_n^0 separately. The argument given in [9] utilizes two results given by Doob [5], page 317 and page 295, part (iii) .

We will now use the fact that $\{R_n''\}$ is a semimartingale and the following theorem, due to Darling and Robbins [4], to derive bounds for certain probabilities involving R_n'. Darling and Robbins [4] and Robbins [12] give statistical applications of these kinds of bounds (as well as a bibliography of the literature in this area), which can be used in connection with Brunk's [1], [2] work on the estimation of integral regression and non-decreasing regression functions.

THEOREM 9. *If $\{Z_n\}$ is a nonnegative semimartingale with $EZ_n \leqq b_n$, and if $\{a_n\}$ is any non-decreasing sequence of positive constants, then*

$$P(Z_n \geqq a_n \text{ for some } m \leqq n \leqq k) \leqq b_m/a_m + \sum_{n=m+1}^{k} (b_n - b_{n-1})/a_n .$$

REMARK 10. Let $\{X_n\}$ be a sequence of independent ràndom variables, centered at expectations, such that $\sup_n EX_n^2 \leqq \sigma^2$ for some number σ^2. Let $\{a_n\}$ be a non-decreasing sequence of positive numbers. Then

$$\begin{aligned} P[R_n' \geqq \sqrt{a_n} \text{ for some } n \geqq m] &= P[R_n'' \geqq a_n \text{ for some } n \geqq m] \\ &\leqq 4m\sigma^2/a_m + \textstyle\sum_{n=m+1}^{\infty} 4\sigma^2/a_n . \end{aligned}$$

A straightforward argument utilizing Theorem 9 is given in [9].

We shall now evaluate the bound given in Remark 10 for two particular sequences $\{a_n\}$. In both of these examples $\{X_n\}$ is a sequence of independent random variables that are centered at expectations.

EXAMPLE 1. Brunk [1] proved that if $\{X_n\}$ satisfied an rth order Kolmogorov condition for some number $r \geqq 1$, then $P[\lim_n (1/n)R_n' = 0] = 1$. This equation is equivalent to statement that for every $\delta > 0$,

$$\lim_m P[(1/n)R_n' \geqq \delta \text{ for some } n \geqq m] = 0 .$$

If we further assume that $\sup_n EX_n^2 \leqq \sigma^2$, for some number σ^2, then by Remark 10, we shall derive a rate of convergence to zero of $P[(1/n)R_n' \geqq \delta$ for some $n \geqq m]$. Set $a_n = \sigma^2 n^2$. Then by Remark 9,

$$\begin{aligned} &P[(1/n)R_n' \geqq \delta \text{ for some } n \geqq m] \\ &\qquad \leqq 4\sigma^2/(m\delta^2) + \textstyle\sum_{n=m+1}^{\infty} 4\sigma^2/(n^2\sigma^2) = o(1/m^\varepsilon) \end{aligned}$$

whenever $0 < \varepsilon < 1$.

EXAMPLE 2. For any positive numbers $\mathscr{I}$ and δ, set $a_n = \mathscr{I} n(\log n)^{1+\delta}$. Then by Remark 10,

$$\begin{aligned} &P[[n(\log n)^{1+\delta}]^{-\frac{1}{2}} R_n' \geqq \mathscr{I}^{\frac{1}{2}} \text{ for some } n \geqq m] \\ &\qquad \leqq 4\sigma^2/[\mathscr{I}(\log m)^{1+\delta}] + \textstyle\sum_{n=m}^{\infty} 4\sigma^2/[\mathscr{I} n(\log n)^{1+\delta}] \\ &\qquad = o[\mathscr{I}/(\log m)^\varepsilon] \end{aligned}$$

whenever $0 < \varepsilon < \delta$.

The preceding Laws of the Iterated Logarithm will be used in the next two sections to study integral regression functions and non-decreasing regression functions.

5. Integral regression functions. The following discussion and notation is very similar to that given by Brunk [1].

Suppose that associated with each point t of the unit interval there is a univariate distribution $D(t)$ with mean $\mu(t)$; $\mu(\cdot)$ is called the regression function. Let $\{t_n\}$ be a sequence of numbers in $[0, 1]$, not necessarily distinct, to be called observation points. For each n, let $Y_n(t_n)$ denote a random variable having the distribution associated with t_n, so that $EY_n(t_n) = \mu(t_n)$; and let the random variables $\{Y_n(t_n)\}$ be independent.

Set

$$h_j(t) = \mathscr{I}_{[t_j,\infty)}(t) \qquad \text{and} \qquad S_n(t) = \sum_{j=1}^n Y_j(t_j)h_j(t)$$

for each $t \in [0, 1]$. Let $F(\cdot)$ denote the "empirical distribution function" of the set $\{t_1, \cdots, t_n\}$:

$$F_n(t) = \sum_{j=1}^n h_j(t)/n .$$

For a given probability distribution function F with support in $[0, 1]$, set $M(t) = \int_{[0,t]} \mu(v)\, dF(v)$ for each t in $[0, 1]$. M is called the integral regression function. Also let

$$M_n(t) = ES_n(t)/n = \int_{[0,t]} \mu(v)\, dF_n(v) .$$

For any function $f(t)$, $\sup_t f(t)$ will be written in place of $\sup_{0\leqq t\leqq 1} f(t)$. We will take $S_n(t)/n$ as our estimator of $M(t)$. Brunk [1] has established that when μ is continuous on the unit interval and $\{Y_n(t_n)\}$ is a sequence of random variables having bounded variances, then

$$P[\lim_n \sup_t |S_n(t)/n - M_n(t)| = 0] = 1 .$$

Brunk [1] has also shown that if in addition to the above conditions F_n converges uniformly to F, then

$$P[\lim_n \sup_t |S_n(t)/n - M(t)| = 0] = 1 .$$

Now under stronger assumptions than Brunk's, we will show rates of convergence to zero of $\sup_t |S_n(t)/n - M_n(t)|$ and $\sup_t |S_n(t) - M(t)|$.

REMARK 11. Let $\{t_n\}$ be a sequence of observation points in [0. 1] and let $\mu(\cdot)$ be a continuous regression function. Assume that the sequence of random variables $\{Y_n(t_n) - \mu(t_n)\}$ satisfies

$$\sup_n \operatorname{Var} Y_n(t_n) \leqq \sigma^2 , \tag{5.1}$$

for some positive constant σ^2, as well as Condition K, HW, P′ or C′.

Then, for some constant $\mathscr{I}$ (whose value depends upon σ^2 and the parameter A of Condition C′),

$$P[\limsup_n (n/\log\log n)^{\frac{1}{2}} \cdot \sup_t |S_n(t)/n - M_n(t)| \leqq \mathscr{I}] = 1 .$$

The proof, given in [9], is similar to Brunk's proof of Proposition 2.2 ([2], page 179).

We now consider the problem of obtaining a rate of convergence to zero of

$\sup_t |S_n(t)/n - M(t)|$. Through the next three results, we obtain rates of convergence of this statistic under assumptions that become progressively easier to work with.

COROLLARY 12. *If in addition to the assumptions of Remark* 11, *the empirical distribution function* $F_n(\cdot)$ *of* $\{t_1, t_2, \cdots, t_n\}$ *satisfies*

$$(5.2) \qquad \limsup_n (n/\log\log n)^{\frac{1}{2}} \cdot \sup_t |M(t) - M_n(t)| \leqq \mathscr{F}$$

for some positive constant $\mathscr{F}$ *and probability distribution function* F, *then*

$$(5.3) \qquad P[\limsup_n (n/\log\log n)^{\frac{1}{2}} \sup_t |S_n(t) - M(t)| \leqq \mathscr{F}_1] = 1$$

for some positive constant $\mathscr{F}_1$.

Corollary 12 follows from the previous result and the Triangle Inequality. We next consider a condition that is sufficient for (5.2).

LEMMA 13. *Let* $\{t_n\}$ *be a sequence of observation points and assume that the regression function* $\mu(\cdot)$ *is a continuous function on* [0, 1].

Let $\rho(n)$ *be the number of values of a step function* $\mu_n(\cdot)$ *on* [0, 1] *that uniformly approximates* $\mu(\cdot)$ *to within* $(\log\log n/n)^{\frac{1}{2}}$. *If the empirical distribution function* $F_n(\cdot)$ *and a probability distribution function* $F(\cdot)$ *satisfy*

$$\limsup_n \rho(n)(n/\log\log n)^{\frac{1}{2}} \sup_t |F_n(t) - F(t)| \leqq \mathscr{F}$$

for some positive constant $\mathscr{F}$, *then* (5.2) *holds.*

The easy argument, given in [9], is omitted here.

The $\rho(n)$ factor used in Lemma 13 appears hard to work with in applications. In the next result, we see that by means of an extra condition on μ the factor $\rho(n)$ can be dispensed with altogether.

The function $\mu(\cdot)$ is said to satisfy a Lipschitz condition if there is a constant $\mathscr{F}$ such that

$$(5.4) \qquad |\mu(u) - \mu(v)| \leqq \mathscr{F}|u - v|$$

for all elements u and v of [0, 1]. This condition will hold in particular if μ has a bounded derivative.

REMARK 14. Assume that the regression function μ satisfies the Lipschitz condition (5.4) and that

$$(5.5) \qquad \limsup_n (n/\log\log n)^{\frac{1}{2}} \sup_t |F_n(t) - F(t)| \leqq \mathscr{F}_1$$

for some positive constant $\mathscr{F}_1$. Then (5.2) holds.

The argument, given in [9], follows from Lemma 13 by actually constructing the aforementioned step function such that $\rho(n) \leqq 2\mathscr{F}(n/\log\log n)^{\frac{1}{2}}$. The next Remark is a result of Remarks 11 and 14, together with the Triangle Inequality.

REMARK 15. Let $\{t_n\}$ be a sequence of observation points in [0, 1], and let $\mu(\cdot)$ be a regression function that satisfies the Lipschitz condition (5.4). Assume that F_n and F satisfy (5.5), and that $\{Y_n(t_n) - \mu(t_n)\}$ satisfies (5.1), as well as Condition K, HW, P′ or C′. Then (5.3) holds.

These results can be generalized to the case where the observation points T_n are random variables. The new situation, called an independent observations regression model, is discussed by Brunk [2]. The generalization is outlined in [9]. The final part of this generalization is that the fundamental convergence rate of Corollary 12, equation (5.3), can hold in an independent observations regression model. This result, Corollary 17, is given here. Its justification utilizes Theorem 16, proven by Csáki [3]. All symbols occurring in these results are defined in [2] and in [9].

THEOREM 16. *If $T_1, T_2, \cdots$ are independent identically distributed random variables having a continuous distribution function F and empirical distribution function F_n, then*

$$P[\limsup_n (n/\log\log n)^{\frac{1}{2}} \sup_t |F_n(t) - F(t)| = 2^{-\frac{1}{2}}] = 1 .$$

COROLLARY 17. *Let (T, Y) be an independent observations regression model and let $T_1, T_2, \cdots$ be independent identically distributed random variables having a continuous cumulative distribution function F and empirical distribution function F_n.*

Furthermore, let A_1 be a subset of Ω_1 of P_1 probability one such that for each of its elements $\omega_1 = \{t_n\}$ the random variables $\{Y_n(t_n) - \mu(t_n)\}$ satisfy (5.1) *and Condition* K, HW, P′ *or* C′. *Finally assume that μ satisfies* (5.4). *Then*

$$P[\limsup_n (n/\log\log n)^{\frac{1}{2}} \sup_t |S_n(t)/n - M(t)| \leqq \mathscr{I}] = 1$$

for some positive constant $\mathscr{I}$.

6. Increasing regression functions. We will now obtain convergence rates for estimators of increasing regression functions. To conform to the current usage of Brunk [1], we will let $\Theta(x)$ be an increasing regression function. The following introduction to this situation is the same, except for minor changes, as that given by Brunk [1].

Let a probability distribution $D(x)$ with mean $\Theta(x)$ be associated with each $x \in [0, 1]$. Let $\{x_r\}$ be a sequence of "observation points" in [0, 1], not necessarily distinct. An estimator of $\Theta(\cdot)$ appropriate to a situation in which $\Theta(\cdot)$ is known to be non-decreasing in [0, 1] will now be described, and the rate of convergence at observation points will be studied.

Let $\{Z_r(x_r)\}$ be a sequence of independent random variables such that $Z_r(x_r)$ has the distribution $D(x_r)$; in particular, $EZ_r(x_r) = \Theta(x_r)$. Let $a(\cdot)$ be a given bounded positive function on [0, 1] bounded away from 0; $a(x_r)$ is to be interpreted as a weighting factor to be applied to $Z_r(x_r)$. For a fixed positive integer r, let $0 < x_{r,1} < x_{r,2} < \cdots < x_{r,k(r)}$ be the $k = k(r)$ distinct numbers among $x_1, \cdots, x_r$, arranged in increasing order. Let $m_i = m_{r,i}$ denote the number of numbers among $\{x_1, \cdots, x_r\}$ which are equal to $x_{r,i}$, and let $w_i = w_{r,i}$ denote the sum of weights at $x_{r,i}$. Set

$$Z_r'(x_{ri}) = \textstyle\sum_{j:x_j=x_{ri}} Z_j(x_{ri})/m_i .$$

Brunk [1] gives a formula for an estimator Θ_r' for Θ:

$$\Theta_r'(x) = \max_{x_{rq} \leqq x} \min_{x_{rs} \geqq x} [\textstyle\sum_{i=q}^{s} w_i Z_r'(x_{ri})]/\sum_{i=q}^{s} w_i . \tag{6.1}$$

For use in the following theorems, for each r let $N_r(J)$ denote the number of numbers among $x_1, x_2, \cdots, x_r$ which are elements of the set J.

Brunk [1] proved the following:

THEOREM 18. *Let $a(\cdot)$ be a bounded positive function on* $[0, 1]$, *bounded away from* 0. *Let $\Theta(\cdot)$ be continuous and non-decreasing on* $[0, 1]$. *Let $\{x_r\}$ be a sequence of observation points, not necessarily distinct, such that for each interval $J \subset (0, 1)$, $r/N_r(J)$ is bounded. Let the variances of the independent observed random variables $Z_r(x_r)$ be bounded. Let $0 < a < b < 1$, then*

$$P[\lim_r \sup_{a \leqq x \leqq b} |\Theta_r'(x) - \Theta(x)| = 0] = 1 .$$

By use of iterated logarithm convergence rates given earlier we now obtain a rate of convergence for $|\Theta_r'(x) - \Theta(x)|$ at each observation point, even when Θ is not necessarily continuous.

REMARK 19. Let $a(\cdot)$ be a bounded positive function on $[0, 1]$, bounded away from 0, and let $\Theta(\cdot)$ be non-decreasing on $[0, 1]$ (but not necessarily continuous). Let $\{x_r\}$ be a sequence of observation points, such that for each point x of $\{x_r\}$, $\liminf_r N_r(\{x\})/r$ is positive. Let $\{a(x_j)[Z_j(x_j) - \Theta(x_j)]\}$ be a sequence of independent random variables having variances bounded above by a number σ^2 such that every subsequence satisfies Condition K, HW, P′ or C′. Then for each number x in $\{x_r\}$, there is a positive number $\mathscr{I}$ such that

$$P[\limsup_r (r/\log\log r)^{\frac{1}{2}} |\Theta_r'(x) - \Theta(x)| \leqq \mathscr{I}] = 1 . \tag{6.2}$$

PROOF. Let x be an element of $\{x_r\}$. Then there exists an integer h such that $x = x_h$. For fixed r at least this large, define q by $x_{rq} = x$. We will show first that for some negative constant $\mathscr{I}_1$,

$$P[\liminf_r (r/\log\log r)^{\frac{1}{2}} (\Theta_r'(x_{ri}) - \Theta(x)) \geqq \mathscr{I}_1] = 1 . \tag{6.3}$$

Using (6.1), we write

$$\Theta_r'(x) - \Theta(x) \geqq \min_{s \geqq q} \textstyle\sum_{i=q}^{s} w_i(Z_r'(x_{ri}) - \Theta(x_{ri}))/\sum_{i=q}^{s} w_i .$$

Since $a(\cdot)$ is a bounded positive function the inequality

$$\textstyle\sum_{i=q}^{s} w_i / \sum_{i=q}^{s} m_i \geqq \delta > 0$$

holds; here δ is a lower bound on $a(\cdot)$. Using $\alpha \wedge \beta$ to denote the smaller of two numbers α and β, by the last two inequalities we then have

$$\begin{aligned} \Theta_r'(x) - \Theta(x) &= \Theta_r'(x_{rq}) - \Theta(x_{rq}) \\ &\geqq \min_{s \geqq q} (1/\delta)(0 \wedge \textstyle\sum_{i=q}^{s} w_i[Z_r'(x_{ri}) - \Theta(x_{ri})]/\sum_{i=q}^{s} m_i) . \end{aligned} \tag{6.4}$$

In order to use previously obtained Laws of the Iterated Logarithm, for $n = 1, 2, \cdots$ let $r(n)$ be the index of the nth number among $x_1, x_2, \cdots$ which is at least as large as x. Set $t_n = x_{r(n)}$, $Y_n(t_n) = a(t_n)[Z_{r(n)} - \Theta(t_n)]$ and $S_n(t) = \sum_{j=1}^{n} Y_j(t_j)h_j(t)$.

Let $i_{1n}, i_{2n}, \cdots, i_{nn}$ be the permutation of $1, 2, \cdots, n$ such that $t_{i_{1n}} \leqq t_{i_{2n}} \leqq \cdots \leqq t_{i_{nn}}$ (where in case exactly two of the t's are the same, say t_i and t_j, with $i < j$, we let $i = i_{ln}$ and $j = i_{l+1n}$ for the appropriate integer l. We adopt a similar convention in the case of three or more identical t's). Since the value of t_n does not affect the values of $t_1, t_2, \cdots, t_{n-1}$, the sequence of permutations whose nth permutation is represented by $i_{1n}, i_{2n}, \cdots, i_{nn}$ is an order preserving sequence of permutations. Thus,

$$\sup_t |S_n(t)| = \max_{j \leqq n} |S_{jn}| = R_n'$$

where $S_{jn} = \sum_{v=1}^{j} Y_{i_{vn}}(t_{i_{vn}})$.

We now show using iterated logarithm convergence rates for R_n' and a lower bound of $\Theta_r'(x) - \Theta(x)$ in terms of R_n' that (6.3) obtains.

For $x_{r(n)s} \leqq t < x_{r(n)s \leqq 1}$, we write

$$S_n(t) = \sum_{i=q}^{s} w_i[Z'_{r(n)}(x_{r(n)i}) - \Theta(x_{r(n)i})] .$$

Using this equation and the hypothesis that $r/N_r(\{x\})$ is bounded it can be shown that there exists a constant $\mathscr{I}_2$ such that

$$(6.5) \qquad 0 \wedge \sum_{i=q}^{s} w_i[Z'_{r(n)}(x_{r(n)i}) - \Theta(x_{r(n)i})]/\sum_{i=q}^{s} m_i \geqq \mathscr{I}_2(0 \wedge [S_n(t)/r(n)])$$

if $x_{r(n)s} \leqq t < x_{r(n)s+1}$.

By inequalities (6.4) and (6.5), we conclude that

$$(r(n)/\log\log r(n))^{\frac{1}{2}}[\Theta'_{r(n)}(x) - \Theta(x)] \geqq (\mathscr{I}_2/\delta)(0 \wedge S_n(t))/(n \log\log n)^{\frac{1}{2}} .$$

Observe that the right-hand member of this inequality is constant for r between $r(n)$ and $r(n+1)$. Thus,

$$(6.6) \qquad (r/\log\log r)^{\frac{1}{2}}[\Theta_r'(x) - \Theta(x)] \geqq (\mathscr{I}_2/\delta)(0 \wedge S_n(t))/(n \log\log n)^{\frac{1}{2}}$$

for any integer r in $[r(n), r(n+1))$.

Since $\{Y_n(t_n)\}$ satisfies Condition K, HW, P′ or C′, by Theorems 1 and 3 and Remarks 2 and 8, it follows that

$$P[\limsup_n (n \log\log n)^{-\frac{1}{2}} \cdot \sup_t |S_n(t)| \leqq \mathscr{I}_3] = 1$$

for some constant $\mathscr{I}_3 > 0$.

Equation (6.3) can then be derived using (6.6) and the last equation above.

By a similar argument we can conclude that

$$P[\limsup_r (r/\log\log r)^{\frac{1}{2}} \cdot [\Theta_r'(x) - \Theta(x)] \leqq \mathscr{I}_4] = 1$$

for some positive constant $\mathscr{I}_4$. This equation and (6.3) are sufficient for the conclusion (6.2).

It should be noted that the requirement (stated in the hypothesis) that every subsequence of $\{a(x_r)[Z_r(x_r) - \Theta(x_r)]\}$ satisfies Condition K, HW, P′ or C′ follows in many cases from the statement that the sequence $\{a(x_r)[Z_r(x_r) - \Theta(x_r)]\}$ itself satisfies these conditions in the presence of reasonable added assumptions.

For instance, if the above sequence of random variables satisfies Condition HW, and the variances of these random variables are bounded away from zero, then Condition HW holds for every subsequence.

Furthermore, if these random variables have second and absolute third moments bounded above and bounded away from zero, then by Remark 4, Condition P′ obtains for every subsequence.

Finally, if Condition C′ holds for the above sequence of random variables, it holds for every subsequence.

7. Acknowledgment. I wish to thank Drs. Tim Robertson and Jonathan Cryer for their patience and encouragement during the preparation of the author's dissertation that led to this paper.

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