The Annals of Statistics
1973, Vol. 1, No. 4, 740–744

AN EXTENSION OF THE HÁJEK–RÉNYI INEQUALITY FOR ONE MAXIMUM OF PARTIAL SUMS

BY DOMINIK SZYNAL

M.C.S. University of Lublin

This note gives an extension of the Hájek–Rényi inequality without the assumptions of independence and of the moment conditions. It is a generalization of results of Kounias and Weng [3].

1. Introduction. The well-known Hájek–Rényi inequality [1], which generalizes the famous Kolmogorov's inequality ([4], page 235), has been recently extended without moment conditions [2], [5], and without the assumption of independence [3], [6]. This note gives an extension of the Hájek–Rényi inequality and of the Kolmogorov inequality free from both of these two assumptions. This extension strengthens both mentioned inequalities in the case of independent random variables, and the one given in [3] in the case of dependent random variables. Our extension allows the establishment of the almost sure convergence of certain series or sequences of random variables when other inequalities do not.

2. Inequalities. Let $\{X_k, k \geqq 1\}$ be a sequence of random variables, and let $S_n = \sum_{k=1}^n X_k$.

LEMMA 1. *If $\{X_k, k \geqq 1\}$ is any sequence of random variables, then for arbitrary $\varepsilon > 0$,*

$$\text{(I)} \qquad P[\max_{1\leqq k\leqq n} |S_k| \geqq 2\varepsilon] \leqq 2(\textstyle\sum_{k=1}^n E^{1/s}[|X_k|^r/(\varepsilon^r + |X_k|^r)])^s ,$$

where $s = 1$ if $0 < r \leqq 1$ and $s = r$ if $1 \leqq r$.

PROOF. Let

$$X_k^\bullet = X_k \mathscr{T}_{[|X_k|<\varepsilon]} , \qquad X_k^{\bullet\bullet} = X_k \mathscr{T}_{[|X_k|\geqq\varepsilon]} ,$$

where $\mathscr{T}_{[A_k]}$ is the indicator of the event A_k.

It is obvious that

$$\text{(1)} \qquad P[\max_{1\leqq k\leqq n} |S_k| \geqq 2\varepsilon] \leqq P[\max_{1\leqq k\leqq n} |S_k^\bullet| \geqq \varepsilon] + P[\max_{1\leqq k\leqq n} |S_k^{\bullet\bullet}| \geqq \varepsilon] ,$$

where $S_k^\bullet = \sum_{j=1}^k X_j^\bullet$ and $S_k^{\bullet\bullet} = \sum_{j=1}^k X_j^{\bullet\bullet}$. The inequality of Kounias and Weng [3] gives

$$P[\max_{1\leqq k\leqq n} |S_k^\bullet| \geqq \varepsilon] \leqq (\textstyle\sum_{k=1}^n E^{1/s}[|X_k^\bullet|^r/\varepsilon^r])^s ,$$

where $s = 1$ if $0 < r \leqq 1$ and $s = r$ if $r \geqq 1$.

Since

$$E|X_k^\bullet|^r/\varepsilon^r \leqq 2E[|X_k^\bullet|^r/(\varepsilon^r + |X_k^\bullet|^r)] ,$$

we have

$$\text{(2)} \qquad P[\max_{1\leqq k\leqq n} |S_k^\bullet| \geqq \varepsilon] \leqq 2(\textstyle\sum_{k=1}^n E^{1/s}[|X_k|^r/(\varepsilon^r + |X_k|^r)]\mathscr{T}_{[|X_k|<\varepsilon]})^s ,$$

where s and r are as in above.

Received April 1971; revised August 1972.

We now give bounds for $P[\max_{1\leqq k\leqq n}|S_k^{\bullet\bullet}|\geqq\varepsilon]$. Note that for $r>0$

$$(3)\qquad \begin{aligned} P[\max_{1\leqq k\leqq n}|S_k^{\bullet\bullet}|\geqq\varepsilon] &\leqq \sum_{k=1}^n P[|X_k|\geqq\varepsilon] \\ &\leqq 2\sum_{k=1}^n E[|X_k|^r/(\varepsilon^r+|X_k|^r)]\mathscr{T}_{[|X_k|\geqq\varepsilon]} \\ &\leqq 2(\sum_{k=1}^n E^{1/s}[|X_k|^r/(\varepsilon^r+|X_k|^r)]\mathscr{T}_{[|X_k|\geqq\varepsilon]})^s . \end{aligned}$$

Taking into account the inequalities (1)—(3) and the inequality

$$(4)\qquad (\sum_{k=1}^n a_k^{1/s})^s+(\sum_{k=1}^n b_k^{1/s})^s\leqq(\sum_{k=1}^n(a_k+b_k)^{1/s})^s ,$$

where a_k, b_k are positive constants and $s\geqq 1$, we have (I).

Lemma 1 allows us to obtain a more general result than one given in [3].

LEMMA 2. *If $\{X_k, k\geqq 1\}$ is any sequence of random variables and $\{c_k, k\geqq 1\}$ is a non-increasing sequence of positive numbers, then for any positive integers m, n with $m<n$, and arbitrary $\varepsilon>0$, we have*

$$(\mathrm{II})\qquad \begin{aligned} P[\max_{m\leqq k\leqq n}c_k|S_k|\geqq 3\varepsilon] \leqq 2(&\sum_{i=1}^m E^{1/s}[|X_i|^r/((b_m\varepsilon)^r+|X_i|^r)] \\ &+\sum_{i=m+1}^n E^{1/s}[|X_i|^r/((b_i\varepsilon)^r+|X_i|^r)])^s , \end{aligned}$$

where $s=1$ if $0<r\leqq 1$, and $s=r$ if $r\geqq 1$, and $b_i=1/c_i$.

PROOF. Put $X_i^*=X_i\mathscr{T}_{[|X_i|<b_i\varepsilon]}$, $X_i^{**}=X_i\mathscr{T}_{[|X_i|\geqq b_i\varepsilon]}$, $S_k^*=\sum_{i=1}^k X_i^*$, and $S_k^{**}=\sum_{i=1}^k X_i^{**}$.

To abbreviate the proof, set $Y_i^r=|X_i|^r/[(b_i\varepsilon)^r+|X_i|^r]$.

By means of the inequality of Kounias and Weng [3] and the inequality

$$\varepsilon^{-r}c_i^rE|X_i^*|^r\leqq 2EY_i^r\mathscr{T}_{[|X_i|<b_i\varepsilon]} ,$$

we obtain

$$\begin{aligned} P[\max_{m\leqq k\leqq n}c_k|S_k^*|\geqq\varepsilon]\leqq 2(&c_m^{r/s}\sum_{i=1}^m b_i^{r/s}E^{1/s}Y_i^r\mathscr{T}_{[|X_i|<b_i\varepsilon]} \\ &+\sum_{i=m+1}^n E^{1/s}Y_i^r\mathscr{T}_{[|X_i|<b_i\varepsilon]})^s , \end{aligned}$$

where $s=1$ if $0<r\leqq 1$ and $s=r$ if $r\geqq 1$.

But

$$\begin{aligned} c_m^{r/s}&\sum_{i=1}^m b_i^{r/s}E^{1/s}Y_i^r\mathscr{T}_{[|X_i|<b_i\varepsilon]} \\ &=\sum_{i=1}^m b_m^{-r/s}b_i^{r/s}E^{1/s}[|X_i|^r/((b_i\varepsilon)^r+|X_i|^r)]\mathscr{T}_{[|X_i|<b_i\varepsilon]} \\ &\leqq\sum_{i=1}^m E^{1/s}[|X_i|^r/((b_m\varepsilon)^r+|X_i|^r)]\mathscr{T}_{[|X_i|<b_i\varepsilon]} \end{aligned}$$

since $b_m^rb_i^{-r}\geqq 1$ for $i=1,2,\cdots,m$. Thus

$$(5)\qquad \begin{aligned} &P[\max_{m\leqq k\leqq n}c_k|S_k^*|\geqq\varepsilon] \\ &\quad\leqq 2(\sum_{i=1}^m E^{1/s}[|X_i|^r/((b_m\varepsilon)^r+|X_i|^r)]\mathscr{T}_{[|X_i|<b_i\varepsilon]} \\ &\qquad+\sum_{i=m+1}^n E^{1/s}[|X_i|^r/((b_i\varepsilon)^r+|X_i|^r)]\mathscr{T}_{[|X_i|<b_i\varepsilon]})^s , \end{aligned}$$

where r and s are as previously.

We now give bounds for $P[\max_{m\leqq k\leqq n}c_k|S_k^{**}|\geqq 2\varepsilon]$.

Note that

$$\begin{aligned} (6) \qquad & P[\max_{m\leqq k\leqq n} c_k|S_k^{**}| \geqq 2\varepsilon] \\ &= P[c_m|S_m^{**}| \geqq 2\varepsilon] \\ &\quad + \textstyle\sum_{k=m+1}^n P\{\bigcap_{i=m}^{k-1} ([c_i|S_i^{**}| < 2\varepsilon]) \cdot [c_k|S_k^{**}| \geqq 2\varepsilon]\} \\ &\leqq P[c_m|S_m^{**}| \geqq 2\varepsilon] + \textstyle\sum_{i=m+1}^n P[|X_i| \geqq b_i\varepsilon] . \end{aligned}$$

Using inequality (I) and putting $b_m\varepsilon$ instead of ε, we obtain

$$\begin{aligned} P[|S_m^{**}| \geqq 2b_m\varepsilon] &\leqq P[\max_{1\leqq k\leqq m} |S_k^{**}| \geqq 2b_m\varepsilon] \\ &\leqq 2(\textstyle\sum_{i=1}^m E^{1/s}[|X_i|^r/((b_m\varepsilon)^r + |X_i|^r)]\mathscr{T}_{[|X_i|\geqq b_i\varepsilon]})^s . \end{aligned}$$

Thus by (6) we have

$$\begin{aligned} (7) \qquad P[\max_{m\leqq k\leqq n} c_k|S_k^{**}| \geqq 2\varepsilon] &\leqq 2\{(\textstyle\sum_{i=1}^m E^{1/s}[|X_i|^r/((b_m\varepsilon)^r + |X_i|^r)]\mathscr{T}_{[|X_i|\geqq b_i\varepsilon]})^s \\ &\quad + (\textstyle\sum_{i=m+1}^n E^{1/s} Y_i^r \mathscr{T}_{[|X_i|\geqq b_i\varepsilon]})^s\} . \end{aligned}$$

By means of (4), (5), (7) and the inequality

$$\begin{aligned} P[\max_{m\leqq k\leqq n} c_k|S_k| \geqq 3\varepsilon] &\leqq P[\max_{m\leqq k\leqq n} c_k|S_k^*| \geqq \varepsilon] \\ &\quad + P[\max_{m\leqq k\leqq n} c_k|S_k^{**}| \geqq 2\varepsilon] , \end{aligned}$$

we obtain (II).

COROLLARY 1. *Under the conditions of Lemma* 2

$$\text{(III)} \qquad P[\max_{m\leqq k\leqq n} c_k|S_k| \geqq 3\varepsilon] \leqq 2(\textstyle\sum_{i=1}^n E^{1/s}[|X_i|^r/((b_m\varepsilon)^r + |X_i|^r)])^s .$$

From the inequality (I) we obtain the following generalization of Tchebyschev's inequality.

COROLLARY 2. *If* $\{X_k, k \geqq 1\}$ *is any sequence of random variables*, $\{b_n\}$ *is a sequence of positive numbers, and r and s are as above, then for arbitrary* $\varepsilon > 0$,

$$\text{(IV)} \qquad P[|S_n| \geqq 2b_n\varepsilon] \leqq 2(\textstyle\sum_{i=1}^n E^{1/s}[|X_i|^r/((b_n\varepsilon)^r + |X_i|^r)])^s .$$

If we assume $\{X_k, k \geqq 1\}$ to be a sequence of independent random variables symmetric with respect to the origin or generally of symmetrized random variables ([4], page 245), we obtain a stronger result than (III).

LEMMA 3. *If* $\{X_k, k \geqq 1\}$ *is a sequence of independent symmetrized random variables, and* $\{c_k, k \geqq 1\}$ *is a non-increasing sequence of positive numbers, then for any positive integers m, n with* $m < n$, *and arbitrary* $\varepsilon > 0$, *we have*

$$\text{(V)} \qquad P[\max_{m\leqq k\leqq n} c_k|S_k| \geqq 3\varepsilon] \leqq 2 \textstyle\sum_{i=1}^n E[X_i^2/((b_m\varepsilon)^2 + X_i^2)] .$$

After using the results of [3], considerations analogous to the previous proofs lead to (V).

3. Applications. The inequalities (I)—(V), and their particular cases, play a fundamental role in establishing almost sure convergence and convergence in probability.

THEOREM 1. *If* $\sum_{n=1}^{\infty} E^{1/s}[|X_n|^r/(1+|X_n|^r)] < \infty$, *where* $s = 1$ *when* $0 < r \leqq 1$ *and* $s = r$ *when* $r \geqq 1$, *then* $\{S_n\}$ *converges almost surely.*

PROOF. From (I) we have

$$P[\max_{k\geqq 1} |S_{m+k} - S_m| \geqq 2\varepsilon] \leqq 2(\sum_{i=m+1}^{m+k} E^{1/s}[|X_i|^r/(\varepsilon^r + |X_i|^r)])^s .$$

Hence,

$$\lim_{m\to\infty} P[\max_{k\geqq 1} |S_{m+k} - S_m| \geqq 2\varepsilon] = 0 ,$$

which implies the almost sure convergence of S_n ([4], page 113).

THEOREM 2. *If* $b_n \uparrow \infty$ *and either*

$$\sum_{n=1}^{\infty} E[|X_n|^r/(b_n{}^r + |X_n|^r)] < \infty \qquad \text{for} \quad 0 < r \leqq 1$$

or

$$\sum_{n=1}^{\infty} E^{1/r}[|X_n|^r/(b_n{}^r + |X_n|^r)] < \infty \qquad \text{for} \quad 1 \leqq r ,$$

then $S_n/b_n \to_{\text{a.s.}} 0$ *as* $n \to \infty$.

PROOF. Since

$$\lim_{m\to\infty} \sum_{k=m+1}^{\infty} E[|X_k|^r/(b_k{}^r + |X_k|^r)] = 0$$

as the tail of convergent sequence, so by Lemma 1, putting X_k/b_k instead of X_k, we conclude that $\sum_{k=1}^{\infty} X_k/b_k$ converges almost surely. Kronecker's lemma ([4], page 238) implies $S_n/b_n \to_{\text{a.s.}} 0$ since $b_n \to \infty$.

The proof for $1 \leqq r$ is similar.

Corollary 1 allows us to prove

THEOREM 3. *If* $b_n \uparrow \infty$ *and either*

$$\lim_{n\geqq m\to\infty} \sum_{i=1}^{n} E[|X_i|^r/(b_m{}^r + |X_i|^r)] = 0 \qquad \text{for} \quad 0 < r \leqq 1$$

or

$$\lim_{n\geqq m\to\infty} \sum_{i=1}^{n} E^{1/r}[|X_i|^r/(b_m{}^r + |X_i|^r)] = 0 \qquad \text{for} \quad 1 \leqq r ,$$

then $S_n/b_n \to_{\text{a.s.}} 0$ *as* $n \to \infty$.

From the inequality (IV) we obtain

THEOREM 4. *If* $b_n \uparrow \infty$ *and either*

$$\lim_{n\to\infty} \sum_{i=1}^{n} E[|X_i|^r/(b_n{}^r + |X_i|^r)] = 0 \qquad \text{for} \quad 0 < r \leqq 1$$

or

$$\lim_{n\to\infty} \sum_{i=1}^{n} E^{1/r}[|X_i|^r/(b_n{}^r + |X_i|^r)] = 0 \qquad \text{for} \quad 1 \leqq r ,$$

then $S_n/b_n \to_P 0$ *as* $n \to \infty$.

From Lemma 3 we obtain the following theorem.

THEOREM 5. *If* $\{X_k, k \geqq 1\}$ *is a sequence of independent symmetrized random variables*, $b_n \uparrow \infty$ *and*

$$\lim_{n\geqq m\to\infty} \sum_{i=1}^{n} E[X_i^2/(b_m{}^2 + X_i^2)] = 0 ,$$

then $S_n/b_n \to_{\text{a.s.}} 0$ *as* $n \to \infty$.

4. Remarks. The following examples elucidate the relations between the results given in [3] and our results.

EXAMPLE 1. Let $P[X_i = i] = 1/i \log^{1+\delta}(1 + i)$ and

$$P[X_i = 0] = 1 - 1/i \log^{1+\delta}(1 + i),$$

where $\delta > 0$. Then, for $r > 0$, $E|X_i|^r = 1/i^{1-r} \log^{1+\delta}(1 + i)$ and we cannot verify, using the results given in [3], whether $\{S_n\}$ converges almost surely. However, since $E[|X_i|^r/(1 + |X_i|^r)] = 1/(1 + i^r)i^{1-r} \log^{1+\delta}(1 + i)$, it follows, by the first part of Theorem 1 ($0 < r \leqq 1$), that $\{S_n\}$ converges almost surely.

As a consequence, one can obtain similar examples concerning the strong law of large numbers.

It is easy to see that the obtained bounds, being sharper than the Hájek-Rényi inequality, or its cited versions, furnishes more accurate information about the rate of the almost sure convergence of series or sequences of random variables than other inequalities.

EXAMPLE 2. Let $P[X_i = i] = 1/i^{1+\delta}$ and $P[X_i = 0] = 1 - 1/i^{1+\delta}$, where $\delta > 0$. Then, for $r > 0$, $E|X_i|^r = 1/i^{1+\delta-r}$, and $E[|X_i|^r/(1 + |X_i|^r)] = 1/(1 + i^r)i^{1+\delta-r}$. Taking into account that $\sum_{i=1}^{\infty} E|X_i|^r < \infty$ for $r < \delta$, and $\sum_{i=1}^{\infty} E[|X_i|^r/(1 + |X_i|^r)] < \infty$ for every $0 < r \leqq 1$, we see that the almost sure convergence of $\{S_n\}$ is established by both the Kounias and Weng result [2] and ours. However, this note provides more accurate information about the order of the rate of the mentioned convergence.

5. Acknowledgment. The author is grateful to the referee for helpful suggestions and for comments which led to an improved presentation.

REFERENCES

[1] HÁJEK, J. and RÉNYI, A. (1955). Generalization of an inequality of Kolmogorov. *Acta Math. Sci. Acad. Hungar.* **6** 281-283.

[2] HEYDE, C. C. (1968). An extension of the Hájek-Rényi inequality for the case without moment conditions. *J. Appl. Probability* **5** 481-483. Erratum **8** 430-431.

[3] KOUNIAS, E. G. and WENG, TENG-SHAN (1969). An inequality and almost sure convergence. *Ann. Math. Statist.* **40** 1091-1093.

[4] LOÈVE, M. (1955). *Probability Theory.* Van Nostrand, Princeton.

[5] TOMKINS, R. J. (1971). On an inequality of Heyde. *J. Appl. Probability* **8** 428-429.

[6] TOMKINS, R. J. (1972). Some probability inequalities related to the law of large numbers. *Ann. Math. Statist.* **43** 230-235.

UNIVERSITY OF LUBLIN
LUBLIN, UL. NOWOTKI 10
POLAND