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ASYMPTOTIC NON-NULL DISTRIBUTIONS OF THE LIKELIHOOD RATIO CRITERIA FOR COVARIANCE MATRIX UNDER LOCAL ALTERNATIVES[1]

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Asymptotic expansions of the non-null distributions of the likelihood ratio criteria for testing the equality of a covariance matrix, equality of a mean vector and a covariance matrix, independence between two sets of variates, equality of two covariance matrices, in multivariate normal distributions are derived under the sequence of alternative hypotheses converging to the null hypothesis when the sample size tends to infinity.

Numerical accuracies of the asymptotic formulas are also examined.

1. Introduction. Asymptotic expansions of the distributions of the likelihood ratio (=LR) criteria based on a random sample from a multivariate normal population under fixed alternative hypothesis have been derived by Sugiura [17], (1) for the equality of covariance matrix to a given matrix, (2) for the equality of mean vector and covariance matrix to a given vector and a given matrix, and also by Sugiura and Fujikoshi [18], (3) for testing the hypothesis of independence between two sets of variates. The limiting non-null distribution of the LR criterion (4) for the equality of several covariance matrices has been obtained by Sugiura [17], asymptotic expansion of which was obtained recently by Nagao [12]. These limiting non-null distributions always degenerate at the null hypothesis so that the asymptotic formulas do not give good approximations when the alternative hypothesis is near to the null hypothesis, as we have experienced in calculating the approximate powers of Bartlett's test for homogeneity of variances in Sugiura and Nagao [20].

In this paper, we shall derive limiting non-null distributions of the LR criteria for the problems (1) and (2) under sequences of alternatives converging to the null hypothesis with the rate of convergence $N^{-\gamma}$, where N means sample size, for arbitrary positive number γ and then asymptotic expansions of the non-null distributions in the case of $\gamma = \frac{1}{2}$ and $\gamma = 1$ in the next two sections. With the help of the hypergeometric function of matrix argument due to Constantine [3], we shall derive an asymptotic expansion of the distribution for the problem (3) in the case $\gamma = \frac{1}{2}$ in Section 4, and an asymptotic expansion of the distribution of the modified LR criterion, given in Sugiura and Nagao [19], for the equality of two covariance matrices under the sequence of alternatives with $\gamma = 1$ in Section 5. The formulas in this paper can be applied to compute the approximate power, when the alternative hypothesis is near to the null hypothesis. Some

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numerical examples are given to indicate the accuracy of our formulas, some of which are compared with the known results.

2. Asymptotic expansions of the modified LR statistic for $\Sigma = \Sigma_0$.

2.1. *Asymptotic expansions.* Let λ_1^* be the modified LR statistic, based on a random sample of size $n + 1 = N$, from a p-variate normal distribution, for testing the equality of a covariance matrix ($= \Sigma$) to a given matrix ($= \Sigma_0$). The unbiasedness and the monotonicity of the λ_1^*-test were established by Sugiura and Nagao [19], Nagao [11], and Das Gupta [4], respectively.

Let the sequence of alternative hypotheses be

$$K_\gamma : \Sigma_0^{-\frac{1}{2}} \Sigma \Sigma_0^{-\frac{1}{2}} = I + n^{-\gamma}\Theta \tag{2.1}$$

for $\gamma > 0$ and symmetric matrix Θ. When $\gamma \geqq \frac{1}{2}$, the characteristic function of $-2 \log \lambda_1^*$ under K_γ can be written from the moments of λ_1^* given by (2.3) in Sugiura [17] as

$$\begin{aligned}(2.2)\qquad &\left(\frac{n}{2e}\right)^{npit}[\Gamma_p(\tfrac{1}{2}n(1-2it))/\Gamma_p(\tfrac{1}{2}n)](1-2it)^{-np(1-2it)/2} \\ &\qquad \times |I + n^{-\gamma}\Theta|^{-nit}|I - 2itn^{-\gamma}\Theta/(1-2it)|^{-n(1-2it)/2} .\end{aligned}$$

The first factor is simply the characteristic function under the null hypothesis, which was expanded asymptotically for large n by Sugiura [17]. Especially it tends to $(1 - 2it)^{-f/2}$ for $f = p(p + 1)/2$ as $n \to \infty$. Applying the asymptotic formula for symmetric matrix Z:

$$-\log |I - n^{-1}Z| = \textstyle\sum_{j=1}^{l} n^{-j} \operatorname{tr} Z^j/j + O(n^{-l-1}) , \tag{2.3}$$

to the second factor of (2.2) we have

$$\begin{aligned}(2.4)\qquad &\exp [n^{1-2\gamma}it(1-2it)^{-1} \operatorname{tr} \Theta^2/2 + n^{1-3\gamma}(\operatorname{tr} \Theta^3/6) \\ &\quad \times \{2 - 3(1-2it)^{-1} + (1-2it)^{-2}\} + n^{1-4\gamma}(\operatorname{tr} \Theta^4/8) \\ &\qquad \times \{-3 + 6(1-2it)^{-1} - 4(1-2it)^{-2} + (1-2it)^{-3}\} \\ &\qquad\qquad + O(n^{1-5\gamma})] .\end{aligned}$$

It follows that the characteristic function of $-2 \log \lambda_1^*$ can be expressed by

$$\begin{aligned}(2.5)\qquad &(1-2it)^{-f/2} + o(1) , &&\text{when } \gamma > \tfrac{1}{2} \\ &(1-2it)^{-f/2} \operatorname{etr} \{\tfrac{1}{2}it\, \Theta^2(1-2it)^{-1}\} + o(1) , &&\text{when } \gamma = \tfrac{1}{2} ,\end{aligned}$$

which implies the first part of Theorem 2.1. We can also get the asymptotic expansion of the characteristic function (2.2) for any given γ. For $\gamma = 1$,

$$\begin{aligned}(2.6)\qquad &(1-2it)^{-f/2}[1 + n^{-1}(\tfrac{1}{4}s_2 + B_2)\{(1-2it)^{-1} - 1\} \\ &\quad + \tfrac{1}{6}n^{-2}\{3B_2^2 + 4B_3 - 6B_2^2(1-2it)^{-1} + (3B_2^2 - 4B_3)(1-2it)^{-2}\} \\ &\qquad + n^{-2} \textstyle\sum_{\alpha=0}^{2} g_{2\alpha}(1-2it)^{-\alpha}] + O(n^{-3}) ,\end{aligned}$$

where $s_j = \operatorname{tr} \Theta^j$ for abbreviation and

$$\begin{aligned}(2.7)\qquad &B_2 = p(2p^2 + 3p - 1)/24 , &&B_3 = -p(p^2 - 1)(p + 2)/32 , \\ &g_0 = \tfrac{1}{4}B_2 s_2 + \tfrac{1}{3}s_3 + \tfrac{1}{32}s_2^2 , &&g_2 = -2g_0 + \tfrac{1}{6}s_3 , \\ &g_4 = g_0 - \tfrac{1}{6}s_3 .\end{aligned}$$

For $\gamma = \frac{1}{2}$,

$$(2.8)\quad \begin{aligned} &(1-2it)^{-f/2}\,\mathrm{etr}\,\{\tfrac{1}{2}it\,\Theta^2(1-2it)^{-1}\}\\ &\quad\times[1+\tfrac{1}{6}n^{-\frac{1}{2}}\,\mathrm{tr}\,\Theta^3\{2-3(1-2it)^{-1}+(1-2it)^{-2}\}\\ &\qquad + n^{-1}\textstyle\sum_{\alpha=0}^{4} h_{2\alpha}(1-2it)^{-\alpha}] + O(n^{-\frac{3}{2}})\,, \end{aligned}$$

where

$$(2.9)\quad \begin{aligned} h_0 &= -B_2 - \tfrac{3}{8}s_4 + \tfrac{1}{18}s_3^2\,, & h_2 &= B_2 + \tfrac{3}{4}s_4 - \tfrac{1}{6}s_3^2\,,\\ h_4 &= -\tfrac{1}{2}s_4 + \tfrac{13}{72}s_3^2\,, & h_6 &= \tfrac{1}{8}s_4 - \tfrac{1}{12}s_3^2\,,\\ h_8 &= \tfrac{1}{72}s_3^2\,. \end{aligned}$$

Let $P(f, \delta^2)$ mean the lower tail of the probability distribution of the noncentral χ^2-distribution for f degrees of freedom and the noncentrality parameter δ^2. Put $P(f) = P(f, 0)$ and

$$(2.10)\quad \begin{aligned} P_H(-2\log\lambda_1^* < \chi^2) = P(f) &+ n^{-1}B_2\{P(f+2) - P(f)\}\\ &+ \tfrac{1}{6}n^{-2}\{(3B_2^2 - 4B_3)P(f+4)\\ &- 6B_2^2P(f+2) + (3B_2^2 + 4B_3)P(f)\}\\ &+ O(n^{-3})\,, \end{aligned}$$

which is the asymptotic expansion under the null hypothesis given in Sugiura [17]. Then inversion of the characteristic functions (2.6) and (2.8) yields:

THEOREM 2.1. *Under the sequence of alternatives K_γ defined by* (2.1), *the limiting distributions of the modified* LR *statistic* $-2\log\lambda_1^*$ *is* χ^2 *with* $f = p(p+1)/2$ *degrees of freedom, when* $\gamma > \frac{1}{2}$, *and noncentral* χ^2 *with f degrees of freedom and noncentrality parameter* $\delta^2 = \frac{1}{4}\,\mathrm{tr}\,\Theta^2$, *when* $\gamma = \frac{1}{2}$. *When* $\gamma = 1$,

$$(2.11)\quad \begin{aligned} P_\gamma(-2\log\lambda_1^* < \chi^2) = P_H(-2\log\lambda_1^* < \chi^2) &+ \tfrac{1}{4}n^{-1}s_2\{P(f+2) - P(f)\}\\ &+ n^{-2}\textstyle\sum_{\alpha=0}^{2} g_{2\alpha}P(f+2\alpha) + O(n^{-3})\,, \end{aligned}$$

where $g_{2\alpha}$ are given by (2.7). *When* $\gamma = \frac{1}{2}$, *we have*

$$(2.12)\quad \begin{aligned} P_\gamma(-2\log\lambda_1^* < \chi^2) = P(f, \delta^2) &+ \tfrac{1}{6}n^{-\frac{1}{2}}s_3\{P(f+4, \delta^2) - 3P(f+2, \delta^2)\\ &+ 2P(f, \delta^2)\} + n^{-1}\textstyle\sum_{\alpha=0}^{4} P(f+2\alpha, \delta^2)h_{2\alpha}\\ &+ O(n^{-\frac{3}{2}})\,, \end{aligned}$$

where the $h_{2\alpha}$ are given by (2.9).

Considering the characteristic function of $(-2\log\lambda_1^*)n^{\gamma-\frac{1}{2}}$, based on the moments of λ_1^*, when $\gamma < \frac{1}{2}$, we can conclude:

THEOREM 2.2. *Under K_γ for* $0 < \gamma < \frac{1}{2}$, *the statistic*

$$(2.13)\quad n^{\gamma-\frac{1}{2}}[-2\log\lambda_1^* - n\{\mathrm{tr}\,(\Sigma\Sigma_0^{-1} - I) - \log|\Sigma\Sigma_0^{-1}|\}]$$

has asymptotically normal distribution with mean zero and variance $2\,\mathrm{tr}\,\Theta^2$.

Noting that the asymptotic variance $2\,\mathrm{tr}\,\Theta^2$ is equal to $2\,\mathrm{tr}\,(\Sigma\Sigma_0^{-1} - I)^2 \cdot n^{2\gamma}$, we can see that the limiting distribution in Theorem 2.2 is of the same form as under fixed alternatives given in Sugiura [17].

2.2. *Numerical examples.* It may be useful to note that by applying the general inverse expansion formula of Hill and Davis [7] to the asymptotic null distribution of $-2\log\lambda_1^*$ given in Theorem 2.1 in Sugiura [17], we can get an asymptotic formula of the 100α % point of $-2\log\lambda_1^*$ in terms of the 100α % point of the χ^2 distribution with f degrees of freedom, giving

$$u + \frac{2B_2}{nf}u + \frac{1}{3n^2}\left\{\frac{u^2}{f^2(f+2)}(-6B_2^2 - 4fB_3) + \frac{u}{f^2}(6B_2^2 - 4fB_3)\right\} + n^{-3}(u^3 g_3 + u^2 g_2 + u g_1) + O(n^{-4}) , \tag{2.14}$$

where u is so chosen that $P(\chi_f^2 > u) = \alpha$ and

$$\begin{aligned} g_1 &= \frac{4}{3f^3}(f^2B_4 - 2fB_2B_3 + B_2^3) , \\ g_2 &= \frac{4}{3f^2(f+2)}(f^2B_4 - 2fB_2B_3 - 5B_2^3) , \\ g_3 &= \frac{4}{3f^3(f+2)(f+4)}(f^2B_4 + 4fB_2B_3 + 4B_2^3) , \end{aligned} \tag{2.15}$$

with B_2, B_3 in (2.7) and $B_4 = p(6p^4 + 15p^3 - 10p^2 - 30p + 3)/480$. Some 5 % and 1 % points of λ_1^* have been computed recently by Korin [10], using the slightly different asymptotic expression of the distribution.

When $p = 1$ and $n = 10$, the exact 5 % point of $-2\log\lambda_1^*$ can be obtained from Table I by Pachares [12] as 3.9682. The asymptotic formula (2.14) gives 3.9683, which shows good approximation for the percentage point. Also for $p = 2$, $n = 100$ and $\alpha = 0.05$, the formula (2.14) gives 7.87173. Specifying the alternatives K as $\Sigma_0^{-\frac{1}{2}}\Sigma\Sigma_0^{-\frac{1}{2}} = (1 + \Delta)I$, we have $\delta^2 = n\Delta^2/2$ and the following case 1 ($\Delta = 0.5$) is computed by the formula (2.16) in Sugiura [17] with the normal distribution function and its derivatives, as well as cases 2 ($\Delta = 0.1$) and 3 ($\Delta = 0.02$) by the formulas (2.12) and (2.11) respectively.

approximate powers, when $p = 2$ and $n = 100$

	$\Delta = 0.5$	$\Delta = 0.1$	$\Delta = 0.02$
δ^2	12.5	0.5	0.02
first term	0.8651	0.1134	0.05
second term	0.0714	-0.0033	0.00229
third term	0.0052	0.0018	0.00001
approximate power	0.942	0.112	0.0523

3. Asymptotic expansions of the LR statistic for $\Sigma = \Sigma_0$ and $\mu = \mu_0$. Let λ_2 be the LR statistic for testing the equality of the covariance matrix Σ and the mean vector μ of a p-variate normal population to a given Σ_0 and a given μ_0, based on a random sample of size N. The unbiasedness of the LR criterion without modification was proved by Sugiura and Nagao [19] and Das Gupta [4]. The asymptotic expansions of $-2\log\lambda_2$ both under the null hypothesis and under a fixed alternative hypothesis have been derived by Sugiura [17].

Specifying the sequence of alternatives

$$K_\gamma : \mu - \mu_0 = N^{-\gamma}\Sigma_0^{\frac{1}{2}}\nu , \qquad \Sigma_0^{-\frac{1}{2}}\Sigma\Sigma_0^{-\frac{1}{2}} = I + N^{-\gamma}\Theta , \tag{3.1}$$

and exactly the same computation as in the previous section, based on the moments of λ_2 given in Sugiura [17], yields the following theorems:

THEOREM 3.1. *Under K_γ for $0 < \gamma < \frac{1}{2}$, the limiting distribution of the statistic*

$$N^{\gamma-\frac{1}{2}}[-2 \log \lambda_2 - N\{\mathrm{tr}\,(\Sigma\Sigma_0^{-1} - I) - \log |\Sigma\Sigma_0^{-1}| + (\mu - \mu_0)'\Sigma_0^{-1}(\mu - \mu_0)\}] \tag{3.2}$$

is normal with mean zero and variance $2\,\mathrm{tr}\,\Theta^2 + 4\nu'\nu$ *as N tends to infinity.*

THEOREM 3.2. *Under K_γ for $\gamma > \frac{1}{2}$, the* LR *statistic* $-2 \log \lambda_2$ *has asymptotically χ^2-distribution with $f = p + p(p + 1)/2$ degrees of freedom.*

When $\gamma = 1$, putting $s_j = \mathrm{tr}\,\Theta^j$ and $t_j = \nu'\Theta^j\nu$,

$$\begin{aligned} &P_\gamma(-2 \log \lambda_2 < \chi^2) \\ &\quad = P_H(-2 \log \lambda < \chi^2) + N^{-1}(\tfrac{1}{4}s_2 + \tfrac{1}{2}t_0)\{P(f + 2) - P(f)\} \\ &\quad\quad + N^{-2} \textstyle\sum_{\alpha=0}^{2} g'_{2\alpha}\, P(f + 2\alpha) + O(N^{-3}) , \end{aligned} \tag{3.3}$$

where the first term $P_H(-2 \log \lambda < \chi^2)$ is the asymptotic expansion under the null hypothesis given by

$$\begin{aligned} &P(f) + N^{-1}B_2'\{P(f + 2) - P(f)\} + \tfrac{1}{6}N^{-2}\{(3B_2'^2 - 4B_3')P(f + 4) \\ &\quad - 6B_2'^2P(f + 2) + (3B_2'^2 + 4B_3')P(f)\} + O(N^{-3}) , \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &B_2' = p(2p^2 + 9p + 11)/24 , \qquad B_3' = -p(p + 1)(p + 2)(p + 3)/32 , \\ &g_0' = \tfrac{1}{4}B_2's_2 + \tfrac{1}{3}s_3 + \tfrac{1}{32}s_2^2 + \tfrac{1}{2}\nu'\nu(\tfrac{1}{4}s_2 + B_2') + \tfrac{1}{2}t_1 + \tfrac{1}{8}t_0^2 , \\ &g_2' = -2g_0' + \tfrac{1}{6}s_3 , \qquad g_4' = g_0' - \tfrac{1}{6}s_3 . \end{aligned} \tag{3.5}$$

When $\gamma = \frac{1}{2}$, the limiting distribution of $-2 \log \lambda_2$ is the noncentral χ^2-distribution with f degrees of freedom and the noncentrality parameter $\delta^2 = \frac{1}{4}\,\mathrm{tr}\,\Theta^2 + \frac{1}{2}\nu'\nu$ and we have

$$\begin{aligned} &P_\gamma(-2 \log \lambda < \chi^2) = P(f, \delta^2) + \tfrac{1}{6}N^{-\frac{1}{2}}\{(s_3 + 3t_1)P(f + 4, \delta^2) \\ &\quad - (3s_3 + 6t_1)P(f + 2, \delta^2) + (2s_3 + 3t_1)P(f, \delta^2)\} \\ &\quad + N^{-1} \textstyle\sum_{\alpha=0}^{4} h'_{2\alpha}\, P(f + 2\alpha, \delta^2) + O(N^{-\frac{3}{2}}) , \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} h_0' &= -B_2' - \tfrac{3}{8}s_4 + \tfrac{1}{18}s_3^2 - \tfrac{1}{2}t_2 + \tfrac{1}{8}t_1^2 + \tfrac{1}{6}t_1s_3 , \\ h_2' &= B_2' + \tfrac{3}{4}s_4 - \tfrac{1}{6}s_3^2 + \tfrac{3}{2}t_2 - \tfrac{1}{2}t_1^2 - \tfrac{7}{12}t_1s_3 , \\ h_4' &= -\tfrac{1}{2}s_4 + \tfrac{13}{72}s_3^2 - \tfrac{3}{2}t_2 + \tfrac{3}{4}t_1^2 + \tfrac{3}{4}t_1s_3 , \\ h_6' &= \tfrac{1}{8}s_4 - \tfrac{1}{12}s_3^2 + \tfrac{1}{2}t_2 - \tfrac{1}{2}t_1^2 - \tfrac{5}{12}t_1s_3 , \\ h_8' &= \tfrac{1}{72}s_3^2 + \tfrac{1}{8}t_1^2 + \tfrac{1}{12}t_1s_3 . \end{aligned} \tag{3.7}$$

If we consider the more general sequence of alternatives

$$(3.8)\qquad K_{\gamma_1,\gamma_2}: \Sigma_0^{-\frac{1}{2}}(\mu - \mu_0) = N^{-\gamma_1}\nu\,, \qquad \Sigma_0^{-\frac{1}{2}}\Sigma\Sigma_0^{-\frac{1}{2}} = I + N^{-\gamma_2}\Theta\,,$$

the asymptotic distributions of $-2\log\lambda_2$ can easily be investigated; namely, they are normal, noncentral χ^2, and χ^2 according to $\min(\gamma_1, \gamma_2) < \frac{1}{2}, = \frac{1}{2}$, and $> \frac{1}{2}$, respectively. Exact descriptions of each of the parameters of the limiting distributions and normalizing factors are omitted here in order to save space.

4. Asymptotic expansion of the LR statistic for independence. The problem considered here is to test the hypothesis of independence between p_1 and p_2 sets of variates ($p_1 \leqq p_2$) in a p-variate ($p_1 + p_2 = p$) normal population, based on a random sample of side N. Put $m = \rho N$ for the correction factor $\rho = 1 - (p + 3)/2N$ and let λ_3 be the LR statistic for this problem. We can write the characteristic function of $-2\rho\log\lambda_3$ under the sequence of alternatives $K_\gamma: P = m^{-\frac{1}{2}}\Theta$ from the moments of λ_3 given in (2.5) in Sugiura and Fujikoshi [18] as

$$(4.1)\qquad \begin{aligned} &\Gamma_{p_1}(\tfrac{1}{2}m(1-2it) + \tfrac{1}{4}(p_1 - p_2 + 1)) \\ &\quad \times \Gamma_{p_1}(\tfrac{1}{2}m + \Delta)[\Gamma_{p_1}(\tfrac{1}{2}m + \tfrac{1}{4}(p_1 - p_2 + 1)) \\ &\qquad \times \Gamma_{p_1}(\tfrac{1}{2}m(1-2it) + \Delta)]^{-1} \cdot |I - m^{-1}\Theta^2|^{\frac{1}{2}m+\Delta} \\ &\qquad \times \textstyle\sum_{k=0}^{\infty}\sum_{(\kappa)}\left[\dfrac{\{(\frac{1}{2}m + \Delta)_\kappa\}^2}{m^k(\frac{1}{2}m(1-2it) + \Delta)_\kappa}\right]\dfrac{C_\kappa(\Theta^2)}{k!}\,, \end{aligned}$$

where $P = \operatorname{diag}(\rho_1, \cdots, \rho_{p_1})$ for the population canonical correlation ρ_j and $\Delta = (p_1 + p_2 + 1)/4$. The first factor gives the characteristic function of $-2\rho\log\lambda_3$ under the null hypothesis, which can be expanded using Box [2] or Anderson ([1], page 239), as

$$(4,2)\qquad \begin{aligned} &(1-2it)^{-f/2}[1 + \tfrac{1}{48}m^{-2}p_1p_2(p_1^2 + p_2^2 - 5) \\ &\qquad \times \{(1-2it)^{-2} - 1\} + O(m^{-4})]\,, \end{aligned}$$

where $f = p_1p_2$. The second factor in (4.1) can easily be expanded using (2.3)

$$(4.3)\qquad \begin{aligned} |I - m^{-1}\Theta^2|^{\frac{1}{2}m+\Delta} &= \operatorname{etr}(-\Theta^2/2)\cdot[1 - m^{-1}(\Delta s_2 + \tfrac{1}{4}s_4) \\ &\quad + m^{-2}\{\tfrac{1}{2}(\Delta s_2 + \tfrac{1}{4}s_4)^2 - \tfrac{1}{2}\Delta s_4 - \tfrac{1}{6}s_6\} + O(m^{-3})]\,, \end{aligned}$$

where $s_j = \operatorname{tr}\Theta^j$. Noting that

$$(4.4)\qquad \begin{aligned} (am + b)_\kappa &= (am)^k[1 + (am)^{-1}\{kb + \tfrac{1}{2}a_1(\kappa)\} \\ &\quad + \tfrac{1}{24}(am)^{-2}\{12b^2k(k-1) + 12(k-1)ba_1(\kappa) \\ &\quad + 3a_1(\kappa)^2 - a_2(\kappa) + k\} + O(m^{-3})]\,, \end{aligned}$$

and using a lemma in Sugiura and Fujikoshi [18], the third factor can be evaluated

$$(4.5)\qquad \begin{aligned} &\operatorname{etr}[\tfrac{1}{2}\Theta^2/(1-2it)]\cdot[1 + m^{-1}\{2\Delta s_2(1-2it)^{-1} \\ &\quad + \tfrac{1}{2}(s_4 - 2\Delta s_2)(1-2it)^{-2} - \tfrac{1}{4}s_4(1-2it)^{-3}\} \\ &\qquad + m^{-2}\textstyle\sum_{\alpha=1}^{6} g_{2\alpha}(1-2it)^{-\alpha} + O(m^{-3})]\,, \end{aligned}$$

where

$$
\begin{aligned}
g_2 &= 2\Delta^2 s_2\,,\\
g_4 &= (3\Delta + \tfrac{1}{4})s_4 + (2\Delta^2 + \tfrac{1}{4})s_2^{\,2} - 4\Delta^2 s_2\,,\\
g_6 &= \tfrac{5}{6}s_6 + \Delta s_2 s_4 - (4\Delta + \tfrac{1}{2})s_4 - (2\Delta^2 + \tfrac{1}{2})s_2^{\,2} + 2\Delta^2 s_2\,,\\
g_8 &= \tfrac{1}{8}s_4^{\,2} - s_6 - \Delta s_4 s_2 + (\tfrac{3}{2}\Delta + \tfrac{1}{4})s_4 + (\tfrac{1}{2}\Delta^2 + \tfrac{1}{4})s_2^{\,2}\,,\\
g_{10} &= -\tfrac{1}{8}s_4^{\,2} + \tfrac{1}{3}s_6 + \tfrac{1}{4}\Delta s_4 s_2\,,\\
g_{12} &= \tfrac{1}{32}s_4^{\,2}\,.
\end{aligned}
\tag{4.6}
$$

Multiplying these three factors and inversion of the characteristic function yields:

THEOREM 4.1. *Under K_γ: $P = m^{-\frac{1}{2}}\Theta$, the* LR *statistic $-2\rho \log \lambda_3$ for testing the independence between p_1 and p_2 sets of variates ($p_1 \leqq p_2$) has asymptotically noncentral χ^2 with $f = p_1 p_2$ degrees of freedom and noncentrality parameter $\delta^2 = \operatorname{tr}\Theta^2/2$ and*

$$
\begin{aligned}
P(-2\rho\log\lambda_3 < \chi^2) = P(f,\delta^2) &+ m^{-1}\{-(\tfrac{1}{4}s_4 + \Delta s_2)P(f,\delta^2)\\
&+ 2\Delta s_2 P(f+2,\delta^2) + \tfrac{1}{2}(s_4 - 2\Delta s_2)P(f+4,\delta^2)\\
&- \tfrac{1}{4}s_4 P(f+6,\delta^2)\} + m^{-2}[\tfrac{1}{48}f(p_1^{\,2} + p_2^{\,2} - 5)\\
&\times \{P(f+4,\delta^2) - P(f,\delta^2)\}\\
&+ \textstyle\sum_{\alpha=0}^{6} h_{2\alpha}P(f+2\alpha,\delta^2)] + O(m^{-3})\,,
\end{aligned}
\tag{4.7}
$$

where $s_j = \operatorname{tr}\Theta^j$ and $\Delta = (p+1)/4$ and

$$
\begin{aligned}
h_0 &= \tfrac{1}{2}(\tfrac{1}{4}s_4 + \Delta s_2)^2 - \tfrac{1}{6}s_6 - \tfrac{1}{2}\Delta s_4\,,\\
h_2 &= -\tfrac{1}{2}\Delta s_4 s_2 - 2\Delta^2 s_2^{\,2} + 2\Delta^2 s_2\,,\\
h_4 &= -\tfrac{1}{8}s_4^{\,2} - \tfrac{1}{4}\Delta s_4 s_2 + (3\Delta + \tfrac{1}{4})s_4 + (3\Delta^2 + \tfrac{1}{4})s_2^{\,2} - 4\Delta^2 s_2\,,\\
h_6 &= \tfrac{1}{16}s_4^{\,2} + \tfrac{5}{4}\Delta s_4 s_2 + \tfrac{5}{6}s_6 - (4\Delta + \tfrac{1}{2})s_4 - (2\Delta^2 + \tfrac{1}{2})s_2^{\,2} + 2\Delta^2 s_2\,,\\
h_\alpha &= g_\alpha \quad (\alpha = 8, 10, 12)\,.
\end{aligned}
\tag{4.8}
$$

Based on the moments given in Theorem 2.1 in Sugiura and Fujikoshi [18], we easily get:

THEOREM 4.2. *Under the sequence of alternatives K_γ: $P = m^{-\gamma}\Theta$, the statistic*

$$m^{\gamma-\frac{1}{2}}\{-2\rho\log\lambda_3 - m\log|I - P^2|\}$$

has asymptotically normal distribution with mean zero and variance $4\operatorname{tr}\Theta^2$, when $0 < \gamma < \frac{1}{2}$. For $\gamma > \frac{1}{2}$, the LR *statistic $-2\rho\log\lambda_3$ has asymptotically χ^2 distribution with $p_1 p_2$ degrees of freedom.*

Under K_γ, asymptotic expansions of Hotelling's statistic and Pillai's statistic for the same problem were obtained recently by Fujikoshi [6]. Noting that the asymptotic expansion of $P_H(-2\rho\log\lambda_3 < \chi^2)$ has the same form as in the case of the multivariate linear hypothesis given in Box [2] or Anderson ([1], page 208), we can use the asymptotic formula for the percentage point of the LR statistic for linear hypothesis given by (2.8) and (2.9) in Hill and Davis [7] for our present purpose, that is, only ν, p, q are replaced by p_1p_2, p_1, p_2 respectively in their formula.

EXAMPLE 4.1. When $N = 87$, $p_1 = 2$ and $p_2 = 3$, our asymptotic formula gives the approximate 5 % point of $-2\rho \log \lambda_3$ as 12.5932, the exact value of which can be obtained by Table 8 in Pillai and Jayachandran [14] (lower 5 % points of $\{W^{(2)}\}^{\frac{1}{2}}$, $m = 0$ and $n = 40$) as 12.59316. For the alternatives $K_1: \rho_1^2 = 0.001$, $\rho_2^2 = 0.05$, the asymptotic formula (4.7) gives the approximate power as 0.2921, the exact value of which is 0.2919 given in Table 9 in Pillai and Jayachandran [14]. For $K_1: \rho_1^2 = 0.05$, $\rho_2^2 = 0.1$, the asymptotic formula (2.12) in Sugiura and Fujikoshi [18] gives the approximate power as 0.806. The exact value is not tabulated.

5. Asymptotic expansion of the LR statistic for $\Sigma_1 = \Sigma_2$.

5.1. *Moments of the statistic under local alternatives.* Let λ_4^* be the modified LR statistic, as given in Sugiura and Nagao [19], for testing the equality of two covariance matrices, based on random samples of size $n_1 + 1$ and $n_2 + 1$ from p-variate normal distributions with covariance matrices Σ_1 and Σ_2. Unbiasedness of this modified LR criterion was proved by Sugiura and Nagao [19]. The limiting distribution of $-2 \log \lambda_4^*$ under the fixed alternative K was obtained by Sugiura [17], asymptotic expansion of which for the k sample case was obtained recently by Nagao [12], who used a different approach. We shall now consider the moments of the statistic λ_4^* under the sequence of alternatives $K_\gamma: \Sigma_2^{-\frac{1}{2}}\Sigma_1\Sigma_2^{-\frac{1}{2}} = I + m^{-1}\Theta$, where the matrix Θ is symmetric and $m = \rho n$, with the correction factor due to Box [2],

$$\rho = 1 - \frac{2p^2 + 3p - 1}{6(p+1)}\left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n}\right). \tag{5.1}$$

Without loss of generality, we may assume that $\Sigma_1 = \Gamma$ and $\Sigma_2 = I$, where $\Gamma = \Sigma_2^{-\frac{1}{2}}\Sigma_1\Sigma_2^{-\frac{1}{2}}$. We can express the hth moment of λ_4^* as

$$\begin{aligned}(n^n n_1^{-n_1} n_2^{-n_2})^{ph/2}\{2^{np/2}\Gamma_p(n_1/2)\Gamma_p(n_2/2)\}^{-1} \\ \times \int \textstyle\prod_{\alpha=1}^2 |S_\alpha|^{\{n_\alpha(1+h)-p-1\}/2} \cdot |\Gamma|^{-n_1/2}|S_1 + S_2|^{-nh/2} \\ \times \operatorname{etr}\{-\tfrac{1}{2}(S_1+S_2) + \tfrac{1}{2}(I - \Gamma^{-1})S_1\}\, dS_1\, dS_2\,,\end{aligned} \tag{5.2}$$

where the range of integration is such that two $p \times p$ symmetric matrices S_1 and S_2 are pd. We can expand the last part of $\operatorname{etr}\{(I - \Gamma^{-1})S_1/2\}$ in an infinite series by zonal polynomials

$$\textstyle\sum_{k=0}^\infty \sum_{(\kappa)} C_\kappa((I - \Gamma^{-1})S_1/2)/k!\,. \tag{5.3}$$

Transforming the variables (S_1, S_2) to (U_1, U_2) by $U_1 = S_1$ and $U_2 = U_1^{-\frac{1}{2}}S_2U_1^{-\frac{1}{2}}$ ($U_1^{\frac{1}{2}}$ is chosen pd) with the Jacobian $|\partial(S_1, S_2)/\partial(U_1, U_2)| = |U_1|^{(p+1)/2}$ and integrating out with respect to U_1 by the gamma type integral formula (12) in Constantine [3], we can write the integral in (5.2) as

$$\begin{aligned}2^{-np/2}\Gamma_p(\tfrac{1}{2}n)|\Gamma|^{-n_1/2} \textstyle\sum_{k=0}^\infty \sum_{(\kappa)} \{(\tfrac{1}{2}n)_\kappa/k!\} \\ \times \int |U_2|^{\{n_2(1+h)-p-1\}/2}|I + U_2|^{-n(1+h)/2}C_\kappa((I - \Gamma^{-1})(I + U_2)^{-1})\, dU_2\,.\end{aligned} \tag{5.4}$$

Putting $V = (I + U_2)^{-1}$ with the Jacobian $|\partial U_2/\partial V| = |V|^{-p-1}$ and integrating out with respect to V by the beta type integral formula (22) in Constantine [3], we can finally get the hth moment of λ_4^* as

$$(5.5)\quad \begin{aligned} &(n^n n_1^{-n_1} n_2^{-n_2})^{ph/2} \Gamma_p(\tfrac{1}{2}n)\{\Gamma_p(\tfrac{1}{2}n(1+h))\}^{-1} \\ &\quad \times \textstyle\prod_{\alpha=1}^2 \{\Gamma_p(\tfrac{1}{2}n_\alpha(1+h))/\Gamma_p(\tfrac{1}{2}n_\alpha)\} \\ &\quad \times |\Gamma|^{-n_1/2} {}_2F_1(\tfrac{1}{2}n, \tfrac{1}{2}n_1(1+h); \tfrac{1}{2}n(1+h); I - \Gamma^{-1}) . \end{aligned}$$

This expression can be obtained also from the joint distribution of the characteristic roots of $W = S_1(S_1 + S_2)^{-1}$ in Khatri [9]. However, for our purpose our derivation is more direct.

5.2. *Asymptotic distribution under* K_γ ($\gamma = 1$). Applying the Kummer transformation formula ${}_2F_1(a_1, a_2; b; Z) = |I - Z|^{-a_2} {}_2F_1(b - a_1, a_2; b; -Z(I - Z)^{-1})$ in James [8] to (5.5), we can write the characteristic function of $-2\rho \log \lambda_4^*$ under K_γ as

$$(5.6)\quad \begin{aligned} &(m^m m_1^{-m_1} m_2^{-m_2})^{-pit} \Gamma_p(\tfrac{1}{2}m + \Delta)\{\Gamma_p(\tfrac{1}{2}m(1-2it) + \Delta)\}^{-1} \\ &\quad \times \textstyle\prod_{\alpha=1}^2 \{\Gamma_p(\tfrac{1}{2}m_\alpha(1-2it) + \Delta_\alpha)/\Gamma_p(\tfrac{1}{2}m_\alpha + \Delta_\alpha)\} |I + m^{-1}\Theta|^{-m_1 it} \\ &\quad \times \textstyle\sum_{k=0}^\infty \sum_{(\kappa)} \left\{ \dfrac{(-mit)_\kappa (\tfrac{1}{2}m_1(1-2it) + \Delta_1)_\kappa}{(\tfrac{1}{2}m(1-2it) + \Delta)_\kappa} \right\} \dfrac{C_\kappa(-\theta)}{m^k k!} , \end{aligned}$$

where $m_\alpha = \rho n_\alpha$ and $\Delta_\alpha = (n_\alpha - m_\alpha)/2 = O(1)$ with $m = m_1 + m_2$, and $\Delta = \Delta_1 + \Delta_2$. Computation similar to that in the previous section yields:

THEOREM 5.1. *Under* K_γ: $\Sigma_2^{-\frac{1}{2}}\Sigma_1\Sigma_2^{-\frac{1}{2}} = I + m^{-1}\Theta$, *the distribution of the modified* LR *statistic* λ_4^* *for testing* $\Sigma_1 = \Sigma_2$, *has asymptotically*

$$(5.7)\quad \begin{aligned} P(-2\rho \log \lambda_4^* < \chi^2) = P(f) &+ \tfrac{1}{4}\rho_1\rho_2 m^{-1} \operatorname{tr} \Theta^2 \{P(f+2) - P(f)\} \\ &+ m^{-2}[\omega_2\{P(f+4) - P(f)\} \\ &+ \textstyle\sum_{\alpha=0}^2 g_{2\alpha} P(f + 2\alpha)] + O(m^{-3}) , \end{aligned}$$

where $f = p(p+1)/2$ *and for* $s_j = \operatorname{tr} \Theta^j$,

$$(5.8)\quad \begin{aligned} g_0 &= \rho_1\rho_2 \left\{ \frac{\rho_1\rho_2}{32} s_2^2 - \frac{\rho_1 - 2}{6} s_3 - \frac{\Delta}{2} s_2 \right\} , \\ g_2 &= \rho_1\rho_2 \left\{ -\frac{\rho_1\rho_2}{16} s_2^2 - \frac{\rho_2}{2} s_3 + (\Delta + \tfrac{1}{4}) s_2 + \tfrac{1}{4} s_1^2 \right\} , \\ g_4 &= \rho_1\rho_2 \left\{ \frac{\rho_1\rho_2}{32} s_2^2 + \frac{1 - 2\rho_1}{6} s_3 - (\tfrac{1}{2}\Delta + \tfrac{1}{4}) s_2 - \tfrac{1}{4} s_1^2 \right\} , \\ \omega_2 &= \tfrac{1}{48} p(p^2 - 1)(p+2)(\rho_1^{-2} + \rho_2^{-2} - 1) - \tfrac{1}{2} p(p+1)\Delta^2 . \end{aligned}$$

Under the null hypothesis H, we can get

$$(5.9)\quad \begin{aligned} &P(-2\rho \log \lambda_4^* < \chi^2) \\ &\quad = P(f) + m^{-2}\omega_2\{P(f+4) - P(f)\} + m^{-3}\omega_3\{P(f+6) - P(f)\} \\ &\quad\quad + m^{-4}[\omega_4\{P(f+8) - P(f)\} - \omega_2^2\{P(f+4) - P(f)\}] \\ &\quad\quad + O(m^{-5}) , \end{aligned}$$

where

$$\omega_3 = \tfrac{1}{720}p(6p^4 + 15p^3 - 10p^2 - 30p + 3)(\rho_1^{-3} + \rho_2^{-3} - 1) - \tfrac{1}{12}p(p^2 - 1)(p + 2)\Delta(\rho_1^{-2} + \rho_2^{-2} - 1) + \tfrac{4}{3}p(p + 1)\Delta^3,$$

$$\omega_4 = \tfrac{1}{2}\omega_2^2 + \tfrac{1}{480}p(p - 1)(2p^4 + 8p^3 + 3p^2 - 17p - 14)(\rho_1^{-4} + \rho_2^{-4} - 1) - \tfrac{1}{120}p(6p^4 + 15p^3 - 10p^2 - 30p + 3)(\rho_1^{-3} + \rho_2^{-3} - 1)\Delta + \tfrac{1}{4}p(p^2 - 1)(p + 2)\Delta^2(\rho_1^{-2} + \rho_2^{-2} - 1) - 3p(p + 1)\Delta^4. \tag{5.10}$$

The inverse formula can also be computed from (5.9) in Hill and Davis [7], in terms of u such that $P(\chi_f^2 > u) = \alpha$:

$$u + \frac{2\omega_2 u(u + f + 2)}{m^2 f(f + 2)} + \frac{2\omega_3 u}{m^3 f(f + 2)(f + 4)}\{u^2 + (f + 4)u + (f + 4)(f + 2)\} + m^{-4}\Big[\frac{2\omega_4 u}{f(f + 2)(f + 4)(f + 6)} \times \{u^3 + (f + 6)u^2 + (f + 6)(f + 4)u + (f + 6)(f + 4)(f + 2)\} - \frac{\omega_2^2 u}{f^2(f + 2)^2}\{u^3 + (f - 2)u^2 + (f + 2)(f - 6)u + (f + 2)^2(f - 2)\}\Big] + O(m^{-5}). \tag{5.11}$$

EXAMPLE 5.1. Specifying the alternatives K as $\Sigma_2^{-\frac{1}{2}}\Sigma_1\Sigma_2^{-\frac{1}{2}} = \text{diag}(\delta_1, \cdots, \delta_p)$, and using the 5 % points 3.801 for $p = 1$, $n_1 = 4$, $n_2 = 20$ in Sugiura and Nagao [20] and 7.82241 for $p = 2$, $n_1 = 13$, $n_2 = 63$ computed from (5.11), we can get the approximate powers 0.104 for $\delta_1 = 0.5$ in the first case and 0.05130 for $\delta_1 = \delta_2 = 1.05$ in the second case. The exact power in the first case can be found in Table 744a in Ramachandran [16] as 0.113. In the second case, we can see from Table 2 in Pillai and Jayachandran [15] ($m = 5$, $n = 30$) that the powers of the other tests, based on the largest root of $S_1S_2^{-1}$, tr $S_1S_2^{-1}$, tr $S_1(S_1 + S_2)^{-1}$ and $|I + S_1S_2^{-1}|$ are 0.0670, 0.0701, 0.0703 and 0.0703 respectively. However this does not mean that the modified LR criterion is worse, because the above four test criteria are to test H: $\Sigma_1 = \Sigma_2$ against K: $\delta_\alpha \geqq 1$ for $\alpha = 1, 2, \cdots, p$ and $\sum_{\alpha=1}^{p} \delta_\alpha > p$, which is an extension of the one-sided test, and our modified LR criterion is against all alternatives K: $\Sigma_1 \neq \Sigma_2$, which is an extension of the two-sided test.

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Added in proof. Theorem 4.1 was independently obtained by Yoong-Sin, Lee, "Distribution of the canonical correlations and asymptotic expansions for distributions of certain independence test statistics" [*Ann. Math. Statist.* **42** (1971) 526–537] and by R. J. Muirhead, "On the test of independence between two sets of variates" [*Ann. Math. Statist.* **43** (1972) 1491–1497]. The first paper was based on our result in [18] and the second paper was based on the system of differential equations for hypergeometric function ${}_2F_1$ of matrix argument.

Extension to independence between k-sets of variates from two sets of variates under fixed alternative was performed by H. Nagao, "Non-null distributions of the likelihood ratio criteria for independence and equality of mean vectors and covariance matrices" [*Ann. Inst. Statist. Math.* **24** (1972) 67–79].