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EVENTS WHICH ARE ALMOST INDEPENDENT[1]

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Often, the probability of the simultaneous occurrence of dependent events can be well approximated by assuming them to be independent. Here, we discuss bounds on the error in using this procedure and conditions when under (or over) estimates occur.

An inequality involving expectations of conditionally independent random variables is proven. Applications treat extreme values of exchangeable random variables and error probabilities for simultaneous inference.

Bivariate dependence concepts treated by Lehmann are generalized to the multivariate case in such a way that relations valid for the bivariate case continue to hold.

1. Introduction. Bonferroni bounds, which can be found for example in Feller (1957), have often been employed to approximate probabilities for simultaneous inference. If $A_1, \cdots, A_k$ are arbitrary events and $\bar{A}_i$ denotes the complement of A_i then

$$(1)\qquad 1 - \sum_{i=1}^k P(\bar{A}_i) \leqq P(\bigcap_{i=1}^k A_i) \leqq 1 - \sum_{i=1}^k P(\bar{A}_i) + \sum_{i<j} P(\bar{A}_i \cap \bar{A}_j)\,.$$

The lower bound is used as an approximation for $P(\bigcap_{i=1}^k A_i)$ and the upper bound establishes the error of this approximation. For examples see Lehmann (1966), Chew (1968), Thompson and Wilke (1963), and McDonald and Thompson (1967).

The Bonferroni lower bound is more accurate than one would suppose but frequently it is better to provisionally assume independence, using the approximation

$$(2)\qquad P(\bigcap_{i=1}^k A_i) \sim \prod_{i=1}^k P(A_i)\,.$$

There are two reasons for this.

First, if $A_1, \cdots, A_k$ are *pairwise negatively dependent*, that is if

$$P(A_i \cap A_j) \leqq P(A_i) \cdot P(A_j)\,; \qquad 1 \leqq i < j \leqq k\,,$$

then so are their complements and, writing $q_i = P(\bar{A}_i)$, we have from (1) that $P(\bigcap_{i=1}^k A_i)$ and $\prod_{i=1}^k (1 - q_i)$ are contained in the same interval of length $\sum_{i<j} q_i q_j$. But

$$\sum_{i<j} q_i q_j \leqq \frac{k-1}{2k} (\sum_{i=1}^k q_i)^2 \leqq \frac{k-1}{2k} [\log \prod_{i=1}^k (1 - q_i)]^2\,.$$

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Hence,

$$|P(\bigcap_{i=1}^k A_i) - \prod_{i=1}^k P(A_i)| \leqq \frac{k-1}{2k}[\log \prod_{i=1}^k P(A_i)]^2$$

and thus pairwise negatively dependent events with large probabilities are nearly independent. For example, if we provisionally assume independence to obtain a significance test at the .05 level then $\prod_i P(A_i) = .95$, $(\log .95)^2/2 = .0013$, and the true significance level is between .0513 and .0487.

Second, under various circumstances it can be proved that

$$P(\bigcap_{i=1}^k A_i) \geqq \prod_{i=1}^k P(A_i) . \tag{3}$$

When this inequality holds, the approximation (2) is more accurate than that obtained from the Bonferroni lower bound and admits the same error estimate. In fact we have from (1) and (3) that

$$\begin{aligned} 1 - \sum_{i=1}^k q_i < \prod_{i=1}^k (1 - q_i) &\leqq P(\bigcap_{i=1}^k A_i) \\ &\leqq 1 - \sum_{i=1}^k q_i + \sum_{i<j} P(\bar{A}_i \cap \bar{A}_j) . \end{aligned}$$

The study of conditions which imply (3) will form the bulk of our paper. Conditionally independent and identically distributed random variables (rv's) lead to an inequality of type (3); this, with applications, is the topic of Section 2.

Lehmann (1966) defines rv's X and Y to be *positively quadrant dependent* if

$$P(X \leqq x, Y \leqq y) \geqq P(X \leqq x)P(Y \leqq y)$$

for all x, y. This inequality is of type (3). In Section 3 we study positive quadrant dependence and its generalization to more variables.

Section 4 returns to conditional independence; it contains a theorem which does not require identically distributed rv's.

2. Conditionally i.i.d. random variables. $X_1, \cdots, X_k$ are called conditionally i.i.d. if there exists a probability measure μ on a class of one-dimensional distribution functions such that, for all k-dimensional Borel sets B,

$$P(B) = \int P_F(B)\, d\mu(F) .$$

Here, $P_F(B)$ is the probability of the event B computed under the assumption that the X's are i.i.d. with distribution function F. For a more precise statement of the foundations, see Section 4; here we are intentionally intuitive to make the applications easily accessible.

For the cylinder set $\{X_i \in A; i = 1, \cdots, k\}$ we have

$$\begin{aligned} P\{X_i \in A; i = 1, \cdots, k\} &= \int (\int_A dF)^k\, d\mu(F) \\ &= E(\int_A dF)^k \geqq (E \int_A dF)^k \\ &\geqq \prod_{i=1}^k P(X_i \in A) \end{aligned} \tag{4}$$

which is a special case of (3). We now treat two classes of applications.

a. *Exchangeable random variables.* Denumerably many rv's $X_1, X_2, \cdots,$ are called exchangeable, see Loève (1960) page 365, if the joint distribution of any subset of m of them does not depend upon which are included in the set but only on m, the size of the set. Exchangeability was introduced by De Finetti and his basic theorem is that it is equivalent to conditional independence with common distribution function. Hence, the inequality (4) will hold for exchangeable rv's. In particular, following Berman (1962), let $Z_n = \max(X_1, \cdots, X_n)$. It follows from (4) that

$$P(Z_n \leqq x) = P(X_1 \leqq x, \cdots, X_n \leqq x) \geqq P^n(X_1 \leqq x) .$$

The right-hand side of this inequality is the distribution of Z_n under the provisional assumption that the X's are independent. Hence if Z_n has some limiting distribution; that is if there exist sequences of constants $\{a_n\}$ and $\{b_n\}$, with $a_n > 0$, such that

$$\lim_{n\to\infty} P\left(\frac{Z_n - b_n}{a_n} \leqq x\right) = L(x)$$

where $L(x)$ is a df; and if in addition

$$\lim_{n\to\infty} P^n\left(\frac{X_1 - b_n}{a_n} \leqq x\right) = \Phi(x)$$

where Φ is necessarily one of the three extreme value distributions then $L(x) \geqq \Phi(x)$. Thus, with regard to limiting extreme value distributions, the classical case of independent rv's yields a lower bound for the case of exchangeable rv's provided both limits exist. Berman (1962) discusses the existence of such limits.

The analog of exchangeability for finitely many rv's is that they have a symmetric df, for the distribution of any subset of such variables depends only on the number of variables selected. It is tempting to conjecture, in analogy with De Finetti's theorem that rv's having symmetric df will be conditionally i.i.d. The df

$$H(x, y) = F(x)F(y)[1 - \tfrac{1}{2}(1 - F(x))(1 - F(y)] ,$$

given in Gumbel (1960), provides a counterexample. $H(x, y)$ is symmetric with marginal df's $F(x)$ and $F(y)$. But H is not conditionally i.i.d., for if it were then from (4) we would have $H(x, x) \geqq F^2(x)$, which is clearly false.

b. *Simultaneous prediction intervals.* Discussions of simultaneous prediction intervals can be found in Lieberman (1961), Chew (1968), Saunders (1968), and elsewhere. The inequality (4) can be used to construct conservative simultaneous prediction intervals.

Let $X_1, \cdots, X_n, X_{n+1}, \cdots, X_{n+k}$ denote $n + k$ i.i.d. random variables. Let ϕ denote a real-valued function defined on some appropriate subset of $n + 1$ dimensional Euclidean space. If we define k new random variables $Z_1, \cdots, Z_k$ by $Z_i = \phi(X_1, \cdots, X_n, X_{n+i})$, then these k random variables will be conditionally i.i.d. with respect to the σ-field generated by the random variables $X_1, \cdots, X_n$.

Thus, for any Borel set A it follows from (4) that

$$P(Z_i \in A;\ i = 1, \cdots, k) \geqq P^k(Z_1 \in A)\,. \tag{5}$$

(i) If the $n + k$ i.i.d. random variables are normally distributed, as in Chew (1968), the appropriate function ϕ to choose is

$$\phi(X_1, \cdots, X_n, X_{n+j}) = [n/(n+1)]^{\frac{1}{2}}(X_{n+j} - \bar{X}_n)/S_n$$

where

$$\bar{X}_n = \textstyle\sum_{i=1}^n X_i/n \quad \text{and} \quad S_n^{\,2} = \sum_{i=1}^n (X_i - \bar{X}_n)^2/(n-1)\,.$$

(ii) If the $n + k$ i.i.d. random variables have common density

$$\begin{aligned} f(x) &= (1/\theta)\exp[-(x-\eta)/\theta]\,, & x &\geqq \eta \\ &= 0\,, & x &< \eta \end{aligned}$$

where θ and η are unknown parameters, the appropriate function ϕ and variables $Z_1, \cdots, Z_k$ are given by

$$Z_j = \phi(X_1, \cdots, X_n, X_{n+j}) = (X_{n+j} - Y_1)/L$$

where

$$L = \textstyle\sum_{i=2}^r (Y_i - Y_1) + (n-r)(Y_r - Y_1)\,.$$

Here $Y_1 < Y_2 < \cdots < Y_n$ are the order statistics corresponding to $X_1, X_2, \cdots, X_n$. Thus, if the marginal distribution of Z_j is known, numbers c_1 and c_2 can be obtained such that

$$\gamma = P(c_1 \leqq Z_j \leqq c_2)\,; \qquad j = 1, \cdots, k$$

and thus

$$\gamma^k \leqq P\{c_1 \cdot L + Y_1 \leqq X_{n+j} \leqq c_2 \cdot L + Y_1;\ j = 1, \cdots, k\}\,.$$

Although only two specific illustrations have been given here it should be clear how (5) can be used in other situations.

Let us briefly consider the distribution of Z_j for case (ii). This distribution is independent of θ and η. Hence, to find it we take $\theta = 1$ and $\eta = 0$. It is well known that Y_1 and L are independent with gamma densities $\gamma(x; 1, 1/n)$ and $\gamma(x; r-1, 1)$ respectively. It can be shown that $X_{n+j} - Y_1$ has density $\gamma(x; 1, 1)$ given $X_{n+j} - Y_1 > 0$, and density $\gamma(-x; 1, 1/n)$ given $X_{n+j} - Y_1 < 0$. It follows in a straightforward manner that $(r-1)Z_j$ and $-n(r-1)Z_j$ have the F-density $F(x; 2, 2(r-1))$ given respectively that $Z_j > 0$ and $Z_j < 0$. Thus since $P(Z_j < 0) = 1/(n+1)$ and $P(Z_j > 0) = n/(n+1)$, $(r-1)Z_j$ has density $h(x)$ defined by

$$h(x) = (n/[n+1])F(x; 2, 2(r-1)) + (n/[n+1])F(-nx; 2, 2(r-1))\,.$$

Hence, with some ingenuity, one can employ the ordinary F-tables to obtain constants c_1 and c_2 such that

$$\gamma = P(c_1 \leqq Z_j \leqq c_2)\,.$$

3. Signed dependence. Directly generalizing Lehmann's concept we define $X_1, \cdots, X_n$ to be *positively orthant dependent* if

$$P(X_i \leqq x_i;\, i = 1, \cdots, n) \geqq \prod_{i=1}^{n} P(X_i \leqq x_i) \tag{6}$$

for every choice of $x_1, x_2, \cdots, x_n$.

Esary, Proschan, and Walkup (1967), while studying the relation of system to component reliability, introduce the concept of associated rv's and relate it to positive quadrant dependence. For the reliability application see Esary and Proschan (1970).

In the papers of Lehmann (1966) and Esary, Proschan, and Walkup (1967), it is shown for bivariate rv's that positive quadrant dependence, association, positive regression dependence, and positive likelihood ratio dependence are successivley stronger properties.

In a certain sense, this same structure is preserved when one discusses n random variables.

According to Esary *et al.* (1967), the rv's $X_1, \cdots, X_n$ are *associated* if $\operatorname{Cov}[f(X_1, \cdots, X_n), g(X_1, \cdots, X_n)] \geqq 0$ for all functions $f(x_1, \cdots, x_n)$ and $g(x_1, \cdots, x_n)$ which are non-decreasing in each argument, and for which the covariance is defined.

Elsewhere, Esary and Proschan (1968) define rv's $X_1, \cdots, X_n$ to be *stochastically increasing in sequence* if the conditional df of X_i given

$$X_{i-1} = x_{i-1}, \cdots, X_1 = x_1\,,$$

i.e. $F(x_i \mid x_{i-1}, \cdots, x_1)$, is non-increasing in $x_{i-1}, \cdots, x_1$ for $i = 2, \cdots, n$. This is a generalization of *positive regression dependence* as defined in Lehmann (1966). Esary and Proschan (1968) establish the fact that random variables that are stochastically increasing in sequence must also be associated. A simpler, more direct proof of this is given in the following theorem.

THEOREM 1. *If $X_1, \cdots, X_n$ are stochastically increasing in sequence and if $g_i(x_1, \cdots, x_n)$ are nonnegative and non-decreasing in each argument for $i = 1, 2, \cdots, k$ then*

$$E[\prod_{i=1}^{k} g_i(X_1, \cdots, X_n)] \geqq \prod_{i=1}^{k} E[g_i(X_1, \cdots, X_n]\,.$$

The same inequality is true if the g's are non-increasing instead of non-decreasing. If $k = 2$, and all expectations exist, we may omit the requirement that the g_i are nonnegative.

PROOF. It is true, see Kimball (1951) and Mitrinovic (1970, Section 2.5), that if $g_i(x) \geqq 0$, $i = 1, 2, \cdots, k$, are non-decreasing and X is a rv, then

$$E[\prod_{i=1}^{k} g_i(X)] \geqq \prod_{i=1}^{k} [Eg_i(X)]\,. \tag{7}$$

(If $k = 2$, the requirement that $g_i(x) \geqq 0$ may be omitted.)

It is also true that if $F_1 \geqq F_2$ are distribution functions, and if $g_1 \leqq g_2$ are both non-decreasing functions, then

$$\int g_1\,dF_1 \leqq \int g_2\,dF_1 \leqq \int g_2\,dF_2\,. \tag{8}$$

Then, by the definition of "stochastically increasing in sequence", (8), and the fact that $g_i(x_1, \cdots, x_n)$ is non-decreasing in all arguments, it follows that

$$\int g_i(x_1, \cdots, x_n)\,dF(x_1 \mid x_2, \cdots, x_n)$$

must be non-decreasing in $x_2, x_3, \cdots, x_n$. By precisely the same argument,

$$\int\int g_i(x_1, \cdots, x_n)\,dF(x_1 \mid x_2, \cdots, x_n)\,dF(x_2 \mid x_3, \cdots, x_n)$$

must be non-decreasing in $x_3, \cdots, x_n$, and in general,

$$\int \cdots \int g_i(x_1, \cdots, x_n)\,dF(x_1 \mid x_2, \cdots, x_n) \cdots dF(x_{j-1} \mid x_j, \cdots, x_n)$$

is non-decreasing in $x_j, x_{j+1}, \cdots, x_n$. Thus, using (7) repeatedly, we may say

$$\begin{aligned}
&E[\textstyle\prod_{i=1}^k g_i(X_1, \cdots, X_n)] \\
&\quad = \int \cdots \int \textstyle\prod_{i=1}^k g_i(x_1, \cdots, x_n)\,dF(x_1 \mid x_2, \cdots, x_n) \cdots dF(x_n) \\
&\quad \geqq \int \cdots \int \textstyle\prod_{i=1}^k \int g_i(x_1, \cdots, x_n)\,dF(x_1 \mid x_2, \cdots, x_n)\,dF(x_2 \mid x_3, \cdots, x_n) \cdots dF(x_n) \\
&\quad\ \vdots \\
&\quad \geqq \textstyle\prod_{i=1}^k \int \cdots \int g_i(x_1, \cdots, x_n)\,dF(x_1 \mid x_2, \cdots, x_n) \cdots dF(x_n) \\
&\quad = \textstyle\prod_{i=1}^k E[g_i(X_1, \cdots, X_n)]\,.
\end{aligned}$$

One generalization of Lehmann's positive likelihood ratio dependence to n rv's is the following. See Sarkar (1969) for another generalization.

DEFINITION. X_1 is *positively* (*negatively*) *likelihood ratio dependent* on $X_2, \cdots X_n$ if for $x_i' \geqq x_i$, $i = 1, \cdots, n$,

$$f(x_1, \cdots, x_n)f(x_1', x_2', \cdots, x_n') \geqq (\leqq) f(x_1', x_2, \cdots, x_n)f(x_1, x_2', \cdots, x_n')$$

where f denotes the joint density function.

The ordered *rv's are positively* (*negatively*) *likelihood ratio dependent* if X_i is positively (negatively) likelihood ratio dependent on $X_{i+1}, \cdots, X_n$ for $i = 1, \cdots, n - 1$.

With this definition, we can structure the multivariate concepts of dependence as follows:

THEOREM 2. *Each of the following properties implies the succeeding one*: (i) *positive likelihood ratio dependence* (ii) *stochastically increasing in sequence* (iii) *association*, *and* (iv) *positive orthant dependence*.

PROOF. That (i) implies (ii) is essentially the statement, extended to vector valued parameters, that rv's having monotone likelihood ratio are stochastically ordered; see Lehmann (1959, page 74). That (iii) follows from (ii) has been established earlier in this section. Esary *et al.* (1967) prove that association implies positive orthant dependence.

4. Conditionally independent rv's. In this section we provide a more precise foundation for Section 2 and we prove a theorem which can sometimes be used to show that $P(X_i \in A_i; i = 1, \cdots, k) \geqq \prod_{i=1}^k P(X_i \in A_i)$ where the A_i are not equal, and/or the X_i are not identically distributed. Note that this is a generalization of equation (4).

If $X_1, \cdots, X_k$ are conditionally independent (c.i.) with respect to a σ-algebra $\mathscr{B}$, see Loève (1960), then their joint df may be thought of as a mixture of distributions of independent rv's. Specifically, following Blum *et al.* (1958), if $\mathscr{F}$ is the class of one dimensional df's, let $\mathscr{F}(x, y) = \{F \in \mathscr{F}: F(x) \leqq y\}$ and $S(\mathscr{F})$ the σ-algebra generated by $\{\mathscr{F}(x, y): x, y \text{ real}\}$. Let $\mathscr{F}^k = \mathscr{F} \times \mathscr{F} \times \cdots \times \mathscr{F}$ and $[S(\mathscr{F})]^k$ be the usual product σ-algebra. If we let $F_i^{\mathscr{B}}(\omega, x_i)$ denote the conditional df of X_i for the appropriate σ-algebra $\mathscr{B}$, the mixing probability measure μ on $\mathscr{F}^k$ is determined by

$$\mu(F^*) = P\{\omega: (F_1^{\mathscr{B}}(\omega, x_1), \cdots, F_k^{\mathscr{B}}(\omega, x_k)) \in F^*\}$$

where $F^* \in [S(\mathscr{F})]^k$.

It then follows by a change of variable argument, see Loève (1960, page 342), that for Borel functions $g_1, g_2, \cdots, g_k$ such that $g_1(X_1), \cdots, g_k(X_k), \prod_{i=1}^k g_i(X_i)$ are all integrable,

$$E[\prod_{i=1}^k g_i(X_i)] = \int \prod_{i=1}^k E_{F_i}[g_i(X_i)] \, d\mu(F_1, \cdots, F_k)$$

where $E_{F_i}[g_i(X_i)]$ is the expected value of $g_i(X_i)$ if F_i were the df of X_i.

THEOREM 3. *If $X_1, \cdots, X_k$ are* c.i. *with mixing measure μ, and $g_1, g_2, \cdots, g_k$ are nonnegative Borel functions such that $g_1(X_1), \cdots, g_k(X_k), \prod_{i=1}^k g_i(X_i)$ are all integrable, then if there exist constants a_{ij} such that*

$$E_{F_i}[g_i(X_i)] \geqq E[g_i(X_i)]$$

iff

$$E_{F_j}[g_j(X_j)] \geqq a_{ij}\,, \qquad \text{a.s.} \quad (\mu)$$

$i = 1, \cdots, k - 1, j = i + 1, \cdots, k$, then

$$E[\prod_{i=1}^k g_i(X_i)] \geqq \prod_{i=1}^k E[g_i(X_i)]\,.$$

PROOF. Note first that

$$\int E_{F_1}[g_1(X_1)] \, d\mu = E[g_1(X_1)]\,.$$

Thus if

$$\mathscr{F}_1 = \{(F_1, \cdots, F_k) \in \mathscr{F}^k; E_{F_1}[g_1(X_1)] \leqq E[g_1(X_1)]\}\,, \qquad \text{and}$$
$$\mathscr{F}_2 = \{(F_1, \cdots, F_k) \in \mathscr{F}^k; E_{F_1}[g_1(X_1)] > E[g_1(X_1)]\}\,,$$

then

$$\begin{aligned} D &= E[\prod_{i=1}^k g_i(X_i)] - E[g_1(X_1)]E[\prod_{i=2}^k g_i(X_i)] \\ &= \int \{\prod_{i=2}^k E_{F_i}[g_i(X_i)]\}\{E_{F_1}[g_1(X_1)] - E[g_1(X_1)]\} \, d\mu \\ &= -\int_{\mathscr{F}_1} \{\prod_{i=2}^k E_{F_i}[g_i(X_i)]\}\{E[g_1(X_1)] - E_{F_1}[g_1(X_1)]\} \, d\mu \\ &\quad + \int_{\mathscr{F}_2} \{\prod_{i=2}^k E_{F_i}[g_i(X_i)]\}\{E_{F_1}[g_1(X_1)] - E[g_1(X_1)]\} \, d\mu\,. \end{aligned}$$

Now, since $\int \{E_{F_1}[g_1(X_1)] - E(g_1(X_1))]\} \, d\mu = 0$,

$$\begin{aligned} \int_{\mathscr{F}_1} E[g_1(X_1)] - E_{F_1}[g_1(X_1)] \, d\mu \\ = \int_{\mathscr{F}_2} E_{F_1}[g_1(X_1)] \dot{-} E[g_1(X_1)] \, d\mu \\ = C, \quad \text{say.} \end{aligned}$$

Thus

$$-\int_{\mathscr{F}_1} \prod_{i=2}^{k} E_{F_i}[g_i(X_i)]\{E[g_1(X_1)] - E_{F_1}[g_1(X_1)]\} \, d\mu \geqq -[\prod_{j=2}^{k} a_{1j}]C \, .$$

Similarly,

$$\int_{\mathscr{F}_2} \prod_{i=2}^{k} E_{F_i}[g_i(X_i)]\{E[g_i(X_i)] - E_{F_1}[g_1(X_1)]\} \, d\mu \geqq [\prod_{j=2}^{k} a_{1j}]C \, .$$

Adding these two inequalities gives us that D is nonnegative. Clearly we can repeat the same argument $k - 2$ times for the desired result.

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