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SOME LIMIT THEOREMS WITH APPLICATIONS IN SAMPLING THEORY

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From a finite population units are drawn with varying probabilities with replacement. There is a certain cost for observing a unit. In this paper samples are obtained partly by drawing a fixed number of times, and partly by drawing and observing units until the cost reaches a specified level. Let X_k be the number of times the kth unit has been drawn in either case. Consider for a given function $g(\cdot)$ the random variable $Z = \sum_k g(X_k, k)$. Under general conditions it is proved that Z is asymptotically normally distributed (actually a multidimensional generalization is considered). By appropriate choices of $g(\cdot)$ asymptotic distributions are obtained in successive sampling with varying probabilities without replacement and for the mean of the distinct units in a simple random sample with replacement. It is also investigated how heterogeneous catchability and effects of marking affect the "Petersen" estimator in capture-recapture theory.

1. Introduction. Suppose that we draw with replacement from a finite population $\{U_1, \cdots, U_N\}$. We take q independent samples $\Sigma_1, \cdots, \Sigma_q$. When obtaining Σ_s the probability of drawing U_k is p_{ks} in each drawing, and the drawings are independent. Furthermore, there is a cost $c_{ks} > 0$ associated with observing U_k in Σ_s, where (without loss of generality) $\sum_{k=1}^N c_{ks} = 1$.

First let Σ_s be obtained by drawing a fixed number of times $N \cdot t_s$. Let X_{ks} be the number of times U_k is drawn in Σ_s, and consider for a given function $g(\cdot)$ the random variable:

$$Z(t_1, \cdots, t_q) = \sum_{k=1}^N g(X_{k1}, \cdots, X_{kq}, k) . \tag{1.1}$$

In Section 3 it is proved that under general conditions this random variable is asymptotically normally distributed when $N \to \infty$. A generalization of a method given by Rényi (1962) to study the classical occupancy problem is used for the proof. By specializing the function $g(\cdot)$ various results on occupancy problems follow, see, e.g., Holst (1972) and the references given there.

Next we suppose that the sample Σ_s is obtained by drawing so many times $N \cdot T_s$, that the observation cost for the (different) units obtained for the first time reaches a level f_s, $0 < f_s < 1$. Let X_{ks} be the number of times U_k is drawn and consider for a given function $g(\cdot)$ the random variable

$$Z(T_1, \cdots, T_q) = \sum_{k=1}^N g(X_{k1}, \cdots, X_{kq}, k) . \tag{1.2}$$

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In Section 4 the asymptotic distribution of this random variable is derived under general conditions when $N \to \infty$.

In Sections 5 and 6 we suppose that to each unit in the population there is associated a real number. Using the result of Section 3 the asymptotic distribution of the mean of the distinct units in a simple random sample drawn with replacement is obtained in Section 5.

Suppose that the units are drawn one after the other without replacement so that in each drawing the probability of obtaining U_k is proportional to p_k if U_k has not been drawn before. The cost of drawing and observing U_k is $c_k > 0$. We keep on drawing units until the total cost reaches a specified level f. Using the result of Section 4 the asymptotic distribution for the sum of the real numbers corresponding to the units in the sample is derived in Section 6. The special case when the costs are equal, often called successive sampling, has been studied by Rosén (1972) by different methods. When as well the p's are equal we have simple random sampling without replacement. In applications in sampling theory one usually is interested in obtaining an unbiased estimator of the population total. For this purpose an "approximate" Horvitz–Thompson estimator of the population total is given.

In Section 7 we study how the large sample properties of the "Petersen estimator" in capture-recapture estimation for finding the size of a finite population are influenced, if the usual assumptions concerning simple random sampling and no effect of marking are not valid. The theorem in Section 4 is used to prove the asymptotic results.

2. Assumption and notation. To give precise formulations of the asymptotic results given below we will consider sequences of probabilities, costs, functions and so on. This will sometimes be indicated by an extra index N, but to facilitate the notation N will often be suppressed. We also introduce the following assumptions used below:

$$0 < C_1 \leqq Np_{ksN} \leqq C_2 < \infty , \quad \text{for } 1 \leqq k \leqq N,\ 1 \leqq s \leqq q, \text{ and all } N, \tag{2.1}$$

$$0 < C_3 \leqq Nc_{ksN} \leqq C_4 < \infty , \quad \textstyle\sum_{k=1}^{N} c_{ksN} = 1 , \quad \text{for } 1 \leqq k \leqq N,\ 1 \leqq s \leqq q, \text{ and all } N, \tag{2.2}$$

$$t_{sN} \to t_s , \quad 0 < t_s < \infty , \quad \text{when } N \to \infty, \text{ for } 1 \leqq s \leqq q , \tag{2.3}$$

$$f_{sN} \to f_s , \quad 0 < f_s < \infty , \quad \text{when } N \to \infty, \text{ for } 1 \leqq s \leqq q , \tag{2.4}$$

$$|g_N(x_1, \cdots, x_q, k)| \leqq |a_{kN}| \cdot \exp(O(x_1 + \cdots + x_q)) , \tag{2.5}$$

$$|g_N(x_1, \cdots, x_q, k)| \leqq |a_{kN}| , \tag{2.6}$$

$$\textstyle\sum_{k=1}^{N} a_{kN}^2 \leqq C_5 < \infty , \quad \text{for all } N , \tag{2.7}$$

$$\max_{1 \leqq k \leqq N} |a_{kN}| \to 0 , \quad \text{when } N \to \infty . \tag{2.8}$$

We denote by Po (m) a Poisson distribution with mean m, by Mult $(n, p_1, \cdots, p_N)$

a multinomial distribution, and by $N(m, \sigma^2)$ a normal distribution with mean m and variance σ^2. The notation, $X_N \in AsN(m_N, \sigma_N^2)$ when $N \to \infty$, means that the sequence $\{(X_N - m_N)/\sigma_N\}_N$ converges in law to $N(0, 1)$.

3. Fixed sample size. Let the sample Σ_s be obtained by drawing a fixed number of times Nt_s. In Σ_s the unit U_k is drawn X_{ks} times. Evidently

$$(X_{1s}, \cdots, X_{Ns}) \in \text{Mult}\,(Nt_s, p_{1s}, \cdots, p_{Ns})\,. \tag{3.1}$$

The samples $\Sigma_1, \cdots, \Sigma_q$ are independent. For convenience in notation we introduce independent random variables $\{\xi_{ks}\}$ where

$$\xi_{ks} \in \text{Po}\,(Np_{ks}t_s)\,, \tag{3.2}$$

and set

$$\xi_s = \xi_{1s} + \cdots + \xi_{Ns}\,. \tag{3.3}$$

We consider for a given function $g(\cdot)$ the random variables

$$Z = Z_N = Z(t_1, \cdots, t_q) = \sum_{k=1}^N g(X_{k1}, \cdots, X_{kq}, k)\,, \tag{3.4}$$

and

$$\zeta = \zeta_N = \zeta(t_1, \cdots, t_q) = \sum_{k=1}^N g(\xi_{k1}, \cdots, \xi_{kq}, k)\,. \tag{3.5}$$

Finally we set

$$\sigma^2 = \sigma_N^2 = \text{Var}\,(\zeta - \sum_{s=1}^q \beta_s \xi_s)\,, \tag{3.6}$$

where

$$\beta_s = \text{Cov}\,(\xi_s, \zeta)/Nt_s\,. \tag{3.7}$$

THEOREM 1. *If* (2.1), (2.3), (2.5), (2.7), *and* (2.8) *are satisfied and*

(A) *if*

$$\liminf_{N\to\infty} \sigma_N^2 > 0\,, \tag{3.8}$$

then

$$Z_N \in AsN(E\zeta_N, \sigma_N^2)\,, \qquad N \to \infty\,, \tag{3.9}$$

(B) *if*

$$\lim_{N\to\infty} \sigma_N^2 = 0\,, \tag{3.10}$$

then

$$Z_N - E\zeta_N \to 0\,, \quad \textit{in law}\,, \qquad N \to \infty\,. \tag{3.11}$$

REMARK. From the conditions it follows that

$$\limsup_{N\to\infty} \sigma_N^2 < \infty\,. \tag{3.12}$$

PROOF. In Holst (1972) the characteristic function of Z is derived in a special case. We obtain in an analogous way:

$$
\begin{aligned}
E(\exp(iuZ)) &= (1+o(1)) \cdot \textstyle\prod_{s=1}^{q} (Nt_s/2\pi)^{\frac{1}{2}} \\
&\quad \times \textstyle\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp\{\sum_{s=1}^{q} Nt_s(\exp(i\theta_s) - 1 - i\theta_s)\} \\
&\quad \times \textstyle\prod_{k=1}^{N} [1 + \sum_{x_1,\cdots,x_q=0}^{\infty} P(\xi_{ks} = x_s, 1 \leqq s \leqq q) \\
&\quad \times \exp\{\textstyle\sum_{s=1}^{q} (i\theta_s x_s - Np_{ks} t_s(\exp(i\theta_s) - 1))\} \qquad (3.13) \\
&\quad \times \exp(iug(x_1, \cdots, x_q, k)) - 1\}]\, d\theta_1 \cdots d\theta_q \\
&= (1+o(1)) \cdot \textstyle\prod_{s=1}^{q} (Nt_s/2\pi)^{\frac{1}{2}} \\
&\quad \times \textstyle\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} A_N(u, \theta_1, \cdots, \theta_q)\, d\theta_1 \cdots d\theta_q .
\end{aligned}
$$

We will study this integral when $N \to \infty$ as in Holst (1972), following a method by Rényi (1962).

LEMMA 3.1. *For fixed u and $\delta > 0$ we have when $N \to \infty$ that*

$$N^{q/2} \int \cdots \int_A A_N(u, \theta_1, \cdots, \theta_q)\, d\theta_1 \cdots d\theta_q \to 0 , \qquad (3.14)$$

where

$$A = \{|\theta_s| \leqq \pi, 1 \leqq s \leqq q\} \backslash \{|\theta_s| \leqq \delta, 1 \leqq s \leqq q\} . \qquad (3.15)$$

PROOF. In the region A we have for some s and $K_\delta > 0$ that

$$|\exp\{Nt_s(\exp(i\theta_s) - 1 - i\theta_s)\}| = \exp\{Nt_s(\cos\theta_s - 1)\} \leqq \exp(-K_\delta N) . \qquad (3.16)$$

Using this estimate and the conditions of the theorem it follows that

$$
\begin{aligned}
&|N^{q/2} \int \cdots \int_A A_N(\cdots)\, d\theta_1 \cdots d\theta_q| \\
&\leqq N^{q/2} \cdot \exp(-K_\delta N) \cdot \textstyle\prod_{k=1}^{N} \{1 + \sum_{x_1,\cdots,x_q=0}^{\infty} P(\xi_{kr} = x_r, 1 \leqq r \leqq q) \qquad (3.17) \\
&\qquad \times O(g(x_1, \cdots, x_q, k))\} \\
&\leqq N^{q/2} \cdot \exp(-K_\delta N + o(N)) .
\end{aligned}
$$

As this estimate converges to 0 when $N \to \infty$ the lemma is proved. □

LEMMA 3.2. *For fixed u and $\delta > 0$ we have when $N \to \infty$ that*

$$N^{q/2} \int \cdots \int_B A_N(u, \theta_1, \cdots, \theta_q)\, d\theta_1 \cdots d\theta_q \to 0 , \qquad (3.18)$$

where

$$B = \{|\theta_s| \leqq \delta, 1 \leqq s \leqq q\} \backslash \{|\theta_s| \leqq (Nt_s)^{\frac{1}{6}-\frac{1}{2}}, 1 \leqq s \leqq q\} . \qquad (3.19)$$

PROOF. For some $K_\delta > 0$ we have in the region B that

$$\exp\{Nt_s(\cos\theta_s - 1)\} \leqq \exp(-Nt_s K_\delta \theta_s^2) . \qquad (3.20)$$

By expanding the logarithm of $\prod_{k=1}^{N} \{\cdots\}$ in the integrand and using the assumptions we find that (3.18) can be majorized by

$$
\begin{aligned}
&N^{q/2} \int \cdots \int_B \exp(-\textstyle\sum_1^q Nt_s K_\delta \theta_s^2) \cdot |\prod_1^N \{\cdots\}|\, d\theta_1 \cdots d\theta_q \\
&\leqq N^{q/2} \int \cdots \int_B \exp\{-\textstyle\sum_1^q Nt_s K_\delta \theta_s^2 \qquad (3.21) \\
&\qquad + \textstyle\sum_1^q O(N^{\frac{1}{2}}\theta_s) + O(1)\}\, d\theta_1 \cdots d\theta_q \\
&= \int \cdots \int_{B'} \exp\{-K_\delta \textstyle\sum_1^q \psi_s^2 + \sum_1^q O(\psi_s) + O(1)\}\, d\psi_1 \cdots d\psi_q ,
\end{aligned}
$$

where

$$B' = \{|\psi_s| \leqq (Nt_s)^{\frac{1}{2}}\delta,\ 1 \leqq s \leqq q\} \backslash \{|\psi_s| \leqq (Nt_s)^{\frac{1}{6}},\ 1 \leqq s \leqq q\}. \tag{3.22}$$

As the estimate in (3.21) converges to 0 when $N \to \infty$, the lemma is proved. □

LEMMA 3.3. *For fixed u we have when $N \to \infty$ that*

$$\exp(-iuE\zeta) \cdot \prod_{s=1}^{q} (Nt_s/2\pi)^{\frac{1}{2}} \cdot \int \cdots \int_C A_N(u, \theta_1, \cdots, \theta_q)\, d\theta_1 \cdots d\theta_q = \exp(-\sigma_N^2 u^2/2 + o(1)), \tag{3.23}$$

where

$$C = \{|\theta_s| \leqq (Nt_s)^{\frac{1}{6}-\frac{1}{2}},\ 1 \leqq s \leqq q\}. \tag{3.24}$$

PROOF. By expanding $\ln A_N$ and using the assumptions we find that the left-hand side of (3.23) can be written:

$$\begin{aligned} &\prod_{s=1}^{q} (Nt_s/2\pi)^{\frac{1}{2}} \cdot \int \cdots \int_C \exp\Big[-\sum_{s=1}^{q} \sum_{k=1}^{N} \sum_{x_1,\cdots,x_q=0}^{\infty} P(\xi_{kr} = x_r,\ 1 \leqq r \leqq q) \\ &\quad \times g(x_1, \cdots, x_q, k)(x_s - Np_{ks}t_s)\theta_s u \\ &\quad - \sum_{k=1}^{N} \sum_{x_1,\cdots,x_q=0}^{\infty} P(\xi_{kr} = x_r,\ 1 \leqq r \leqq q)(g(x_1, \cdots, x_q, k))^2 \cdot u^2/2 \\ &\quad + \sum_{k=1}^{N} \Big\{\sum_{x_1,\cdots,x_q=0}^{\infty} P(\xi_{kr} = x_r,\ 1 \leqq r \leqq q) g(x_1, \cdots, x_q, k)\Big\}^2 \cdot u^2/2 \\ &\quad - \sum_{s=1}^{q} Nt_s\theta_s^2/2 + \sum_{s=1}^{q} O(N\theta_s^3) + o(1)\Big]\, d\theta_1 \cdots d\theta_q \\ &\qquad = \exp(-\sigma_N^2 u^2/2) \cdot \int \cdots \int_{C'} (2\pi)^{-q/2} \cdot \exp\Big[-\sum_{s=1}^{q} \{(\psi_s + O(1))^2 \\ &\qquad\quad + O(\psi_s^3/N^{\frac{1}{2}}) + o(1)\}/2\Big]\, d\psi_1 \cdots d\psi_q, \end{aligned} \tag{3.25}$$

where

$$C' = \{|\psi_s| \leqq (Nt_s)^{\frac{1}{6}},\ 1 \leqq s \leqq q\}. \tag{3.26}$$

As the last integral in (3.25) converges to 1 when $N \to \infty$, the assertion follows. □

Combining the lemmas gives:

$$E\{\exp(iuZ_N)\} = \exp\{iuE\zeta_N - \sigma_N^2 u^2/2 + o(1)\}. \tag{3.27}$$

By the continuity theorem for characteristic functions the theorem follows. □

4. Fixed sample cost. Let us consider the random variable studied in the previous section

$$Z(t_1, \cdots, t_q) = \sum_{k=1}^{N} g(X_{k1}(t_1), \cdots, X_{kq}(t_q), k), \tag{4.1}$$

as a function of $(t_1, \cdots, t_q)$. In this way we define the random process $Z(\cdot)$. In (4.1) $X_{ks}(t_s)$ denote the number of times U_k is obtained after Nt_s drawings. Let us also define

$$e_{ks}(t_s) = \min(1, X_{ks}(t_s)), \tag{4.2}$$

the random variable indicating if U_k has been obtained after Nt_s drawings. The total cost associated with the first Nt_s drawings in the sth sample is:

$$Z_s(t_s) = \sum_{k=1}^{N} c_{ks} e_{ks}(t_s). \tag{4.3}$$

For notational convenience we introduce independent random variables $\{\xi_{ks}(t_s)\}$ where

$$\xi_{ks}(t_s) \in \text{Po}\,(Np_{ks}\,t_s)\ . \tag{4.4}$$

Furthermore, we set

$$\zeta(t_1, \cdots, t_q) = \textstyle\sum_{k=1}^{N} g(\xi_{k1}(t_1), \cdots, \xi_{kq}(t_q), k), \tag{4.5}$$

and

$$\zeta_s(t_s) = \textstyle\sum_{k=1}^{N} c_{ks}\,\varepsilon_{ks}(t_s) = \sum_{k=1}^{N} c_{ks} \min\,(1, \xi_{ks}(t_s))\ . \tag{4.6}$$

We define a function $\mu(\cdot)$ by

$$\mu(t_1, \cdots, t_q) = E\zeta(t_1, \cdots, t_q)\ , \tag{4.7}$$

and functions $\mu_s(\cdot)$ $(1 \leqq s \leqq q)$ by

$$\mu_s(t_s) = E\zeta_s(t_s)\ . \tag{4.8}$$

Now we suppose that the sample Σ_s is obtained by drawing units until the total cost for the first time is at least f_s. If (2.1), (2.2), and (2.4) are satisfied then it is easily seen, that the drawing when this happens

$$N \cdot T_s = \min\,\{Nt_s;\ Z_s(t_s) \geqq f_s\}\ , \tag{4.5$'$}$$

is a proper random variable, and that the equation

$$\mu_s(t_s) = \textstyle\sum_{k=1}^{N} c_{ks} \cdot \{1 - \exp(-Np_{ks}\,t_s)\} = f_s\ , \tag{4.6$'$}$$

has a unique solution $t_s = \tau_s$, $0 < \tau_s < \infty$. To simplify notation we set

$$\xi_{ks} = \xi_{ks}(\tau_s)\ , \qquad \varepsilon_{ks} = \varepsilon_{ks}(\tau_s)\ , \qquad \zeta_s = \zeta_s(\tau_s)\ , \qquad \zeta = \zeta(\tau_1, \cdots, \tau_q)\ , \tag{4.7$'$}$$

$$\xi_s = \textstyle\sum_{k=1}^{N} \xi_{ks}(\tau_s)\ , \tag{4.8$'$}$$

$$\sigma^2 = \text{Var}\,(\zeta - \textstyle\sum_{s=1}^{q} \gamma_s \zeta_s)\ , \tag{4.9}$$

$$\gamma_s = \text{Cov}\,(\xi_s, \zeta)/\text{Cov}\,(\xi_s, \zeta_s)\ . \tag{4.10}$$

In the following theorem we consider $Z(T_1, \cdots, T_q)$, the random process $Z(\cdot)$ at the stopping time $(T_1, \cdots, T_q)$. We will study the asymptotic behavior of $Z(T_1, \cdots, T_q)$, or more precisely, the limiting distribution of the sequence $\{Z_N(T_{1N}, \cdots, T_{qN})\}_N$ when $N \to \infty$. But to facilitate the notation the index N will often be suppressed.

THEOREM 2. *If* (2.1), (2.2), (2.4), (2.6), (2.7), *and* (2.8) *are satisfied and*

(A) *if*

$$\liminf_{N\to\infty} \sigma_N^2 > 0\ , \tag{4.11}$$

then

$$Z_N \in AsN(\mu(\tau_1, \cdots, \tau_q), \sigma_N^2)\ , \qquad N \to \infty\ , \tag{4.12}$$

(B) *if*

$$\liminf_{N\to\infty} \sigma_N^2 = 0\ , \tag{4.13}$$

then

$$Z_N - \mu(\tau_1, \cdots, \tau_q) \to 0 , \quad \textit{in law} , \qquad N \to \infty . \tag{4.14}$$

REMARK. From the conditions it follows that

$$\limsup_{N\to\infty} \sigma_N^2 < \infty . \tag{4.15}$$

PROOF. Let us write

$$\begin{aligned} Z(T_1, \cdots, T_q) - \mu(\tau_1, \cdots, \tau_q) = \{Z(T_1, \cdots, T_q) - \mu(T_1, \cdots, T_q)\} \\ + \{\mu(T_1, \cdots, T_q) - \mu(\tau_1, \cdots, \tau_q)\} . \end{aligned} \tag{4.16}$$

In Lemma 4.2 the second part of the right-hand side of (4.16) is studied; the first part is investigated in Lemmas 4.3—4.6. But first we consider the T's.

LEMMA 4.1. *We have*

$$V_s = N^{\frac{1}{2}}(T_s - \tau_s) = -N^{\frac{1}{2}}(Z_s(\tau_s) - f_s)/A_{sN} + o_p(1) , \tag{4.17}$$

where, for some real numbers K_1, K_2,

$$A_{sN} = \operatorname{Cov}(\zeta_s, \xi_s)/\tau_s , \qquad 0 < K_1 \leqq A_{sN} \leqq K_2 < \infty . \tag{4.18}$$

PROOF. From the definitions of Z_s and T_s we have for a fixed real number v_s that

$$Z_s(\tau_s + v_s/N^{\frac{1}{2}}) \geqq f_s \Leftrightarrow T_s \leqq \tau_s + v_s/N^{\frac{1}{2}} \Leftrightarrow V_s \leqq v_s . \tag{4.19}$$

Hence

$$\begin{aligned} P(V_s \leqq v_s) = P(N^{\frac{1}{2}}\{Z_s(\tau_s + v_s/N^{\frac{1}{2}}) - \mu_s(\tau_s + v_s/N^{\frac{1}{2}})\} \\ \geqq N^{\frac{1}{2}}\{f_s - \mu_s(\tau_s + v_s/N^{\frac{1}{2}})\}) . \end{aligned} \tag{4.20}$$

By expanding $\mu_s(\tau_s + v_s/N^{\frac{1}{2}})$ we find that (4.20) can be written:

$$\begin{aligned} P(V_s \leqq v_s) = P(N^{\frac{1}{2}}\{Z_s(\tau_s + v_s/N^{\frac{1}{2}}) - \mu_s(\tau_s + v_s/N^{\frac{1}{2}})\} \\ \geqq -A_{sN} v_s + o(1)) . \end{aligned} \tag{4.21}$$

Hence V_s and $-N^{\frac{1}{2}} \cdot \{Z(\tau_s + v_s/N^{\frac{1}{2}}) - \mu_s(\tau_s + v_s/N^{\frac{1}{2}})\}$ have the same asymptotic distribution, and by using Theorem 1 we see that $V_s \in AsN(0, O(1))$.

After some straightforward calculation we find

$$N \cdot \operatorname{Var}(Z_s(\tau_s + v_s/N^{\frac{1}{2}}) - \mu_s(\tau_s + v_s/N^{\frac{1}{2}}) - Z_s(\tau_s) + f_s) \to 0 , \qquad N \to \infty . \tag{4.22}$$

Hence we have

$$N^{\frac{1}{2}}(Z_s(\tau_s + v_s/N^{\frac{1}{2}}) - \mu_s(\tau_s + v_s/N^{\frac{1}{2}})) = N^{\frac{1}{2}}(Z_s(\tau_s) - f_s) + o_p(1) . \tag{4.23}$$

Combining these results proves the lemma. □

LEMMA 4.2. *When* $N \to \infty$ *we have*

$$\mu(T_1, \cdots, T_q) - \mu(\tau_1, \cdots, \tau_q) = \textstyle\sum_{s=1}^q B_{sN} V_s + o_p(1) , \tag{4.24}$$

where for some real number K,

$$B_{sN} = \operatorname{Cov}(\xi_s, \zeta)/N^{\frac{1}{2}}\tau_s , \qquad |B_{sN}| \leqq K < \infty . \tag{4.25}$$

PROOF. Let $v_1, \cdots, v_q$ be fixed real numbers. By expanding μ we find

$$\mu(\tau_1 + v_1/N^{\frac{1}{2}}, \cdots, \tau_q + v_q/N^{\frac{1}{2}}) - \mu(\tau_1, \cdots, \tau_q) = \sum_{s=1}^{q} B_{sN} v_s + o(1) . \tag{4.26}$$

Let $\varepsilon > 0$ and $\delta > 0$ be fixed. Because $V_s \in AsN(0, O(1))$ there exists a K_δ such that

$$P(|V_s| \leqq K_\delta, 1 \leqq s \leqq q) > 1 - \delta . \tag{4.27}$$

From (4.26) it follows that

$$\begin{aligned} &P(|\mu(\tau_1 + V_1/N^{\frac{1}{2}}, \cdots, \tau_q + V_q/N^{\frac{1}{2}}) \\ &\quad - \mu(\tau_1, \cdots, \tau_q) - \textstyle\sum_{s=1}^{q} B_{sN} V_s| > \varepsilon) \\ &\quad = P(|\cdots| > \varepsilon, |V_s| \leqq K_\delta \text{ for all } s) \\ &\qquad + P(|\cdots| > \varepsilon, |V_s| > K_\delta \text{ for some } s) \\ &\quad \leqq P(\max_{|v_s| \leqq K_\delta} |\mu(\tau_1 + v_1/N^{\frac{1}{2}}, \cdots, \tau_q + v_q/N^{\frac{1}{2}}) \\ &\qquad - \mu(\tau_1, \cdots, \tau_q) - \textstyle\sum_{s=1}^{q} B_{sN} v_s| > \varepsilon) + \delta = \delta , \end{aligned} \tag{4.28}$$

for N sufficiently great. This proves the lemma. □

In Lemmas 4.3, 4.4, and 4.5 we only consider the case $q = 1$. In these lemmas we suppress the index $s = 1$.

LEMMA 4.3. *When $N \to \infty$ we have*

$$Z(T) - \mu(T) = Z(\tau) - \mu(\tau) + o_p(1) . \tag{4.29}$$

PROOF. Let $\varepsilon > 0$ and $\delta > 0$ be fixed, and choose K_δ as in Lemma 4.2. We have

$$\begin{aligned} &P(|Z(T) - \mu(T) - Z(\tau) - \mu(\tau))| > \varepsilon) \\ &\quad = P(|\cdots| > \varepsilon, |V| \leqq K_\delta) + P(|\cdots| > \varepsilon, |V| > K_\delta) \\ &\quad \leqq P(\max_{|v| \leqq K_\delta} |Z(\tau + v/N^{\frac{1}{2}}) \\ &\qquad - \mu(\tau + v/N^{\frac{1}{2}}) - (Z(\tau) - \mu(\tau))| > \varepsilon) + \delta . \end{aligned} \tag{4.30}$$

Combining (4.30) and the next lemma it follows that it is sufficient to prove that

$$\begin{aligned} &P(\max_{|v| \leqq K_\delta} |Z(\tau + v/N^{\frac{1}{2}}) - EZ(\tau + v/N^{\frac{1}{2}}) \\ &\quad - (Z(\tau) - EZ(\tau))| > \varepsilon) \to 0 , \qquad N \to \infty . \end{aligned} \tag{4.31}$$

As the stopping time $N \cdot T$ is an integer-valued variable we have

$$\begin{aligned} &P(\max_{|v| \leqq K_\delta} |\cdots| > \varepsilon) \leqq 2K_\delta N^{\frac{1}{2}} \max_{|v| \leqq K_\delta} P(|Z(\tau + v/N^{\frac{1}{2}}) \\ &\quad - EZ(\tau + v/N^{\frac{1}{2}}) - (Z(\tau) - EZ(\tau))| > \varepsilon) \\ &\quad \leqq 2K_\delta N^{\frac{1}{2}} \max_{|v| \leqq K_\delta} E|\cdots|^4/\varepsilon^4 . \end{aligned} \tag{4.32}$$

In Lemma 4.5 it is proved that $N^{\frac{1}{2}}E|\cdots|^4 \to 0$ uniformly for $|v| \leqq K_\delta$. From this the lemma follows. □

LEMMA 4.4. *When $N \to \infty$ we have*

$$\mu(\tau + v/N^{\frac{1}{2}}) - \mu(\tau) = EZ(\tau + v/N^{\frac{1}{2}}) - EZ(\tau) + o(1) . \tag{4.33}$$

PROOF. Suppose that $v > 0$ ($v < 0$ can be treated in an analogous way). Consider independent random variables such that $X_k \in \text{Bin}(N\tau, p_k)$, $Y_k \in \text{Bin}(N^{\frac{1}{2}}v, p_k)$, $\xi_k \in \text{Po}(N\tau p_k)$, and $\eta_k \in \text{Po}(N^{\frac{1}{2}}vp_k)$. (4.33) can be written

$$\sum_{k=1}^N E\{g(X_k + Y_k, k) - g(X_k, k) - g(\xi_k + \eta_k, k) + g(\xi_k, k)\} = o(1) . \tag{4.34}$$

Using the Poisson approximation of the binomial-distribution it follows that the left-hand side in (4.34) can be estimated in the following manner

$$|\sum_{k=1}^N E\{\cdots\}| \leqq \sum_{k=1}^N |a_k| o(N^{-\frac{1}{2}}) \leqq (\sum_1^N a_k^2)^{\frac{1}{2}} \cdot o(1) = o(1) . \ \square \tag{4.35}$$

LEMMA 4.5. *When $N \to \infty$ we have*

$$N^{\frac{1}{2}} E|Z(\tau + v/N^{\frac{1}{2}}) - EZ(\tau + v/N^{\frac{1}{2}}) - Z(\tau) + EZ(\tau)|^4 \to 0 . \tag{4.36}$$

PROOF. We only consider the case $v > 0$ ($v < 0$ can be treated in an analogous way). Let $(X_1, \cdots, X_N) \in \text{Mult}(N\tau, p_1, \cdots, p_N)$, and $(Y_1, \cdots, Y_N) \in \text{Mult}(N^{\frac{1}{2}}v, p_1, \cdots, p_N)$ be independent random vectors. With

$$\Delta_k = \{g(X_k + Y_k, k) - g(X_k, k) - Eg(X_k + Y_k, k) + Eg(X_k, k)\}/a_k , \tag{4.37}$$

(if $a_k = 0$ set $\Delta_k = 0$) we can write

$$\begin{aligned} E|\cdots|^4 &= E|\textstyle\sum_{k=1}^N a_k \Delta_k|^4 \\ &= \textstyle\sum_{k=1}^N a_k^4 E\Delta_k^4 + 4 \sum_{i \neq k} a_i a_k^3 E\Delta_i \Delta_k^3 \\ &\quad + 3 \textstyle\sum_{i \neq k} a_i^2 a_k^2 E\Delta_i^2 \Delta_k^2 + 6 \sum_{i \neq k \neq m \neq i} a_i^2 a_k a_m E\Delta_i^2 \Delta_k \Delta_m \\ &\quad + 24 \textstyle\sum_{i<k<m<p} a_i a_k a_m a_p E\Delta_i \Delta_k \Delta_m \Delta_p . \end{aligned} \tag{4.38}$$

Using the assumptions we find after some elementary but cumbersome calculations the following estimates:

$$E\Delta_k^4 = O(N^{-\frac{1}{2}}) , \tag{4.39}$$

$$E\Delta_i \Delta_k^3 = O(N^{-\frac{3}{2}}) , \tag{4.40}$$

$$E\Delta_i^2 \Delta_k^2 = O(N^{-1}) , \tag{4.41}$$

$$E\Delta_i^2 \Delta_k \Delta_m = O(N^{-2}) , \tag{4.42}$$

$$E\Delta_i \Delta_k \Delta_m \Delta_p = O(N^{-3}) . \tag{4.43}$$

Hence, when $N \to \infty$,

$$N^{\frac{1}{2}} \sum a_k^4 E\Delta_k^4 \leqq K_1 \sum a_k^4 \leqq K_2 \max a_k^2 \to 0 , \tag{4.44}$$

$$|N^{\frac{1}{2}} \sum a_i a_k^3 E\Delta_i \Delta_k^3| \leqq K_3 N^{-1} \sum |a_i|\,|a_k|^3 \leqq K_4 \max a_k^2 \to 0 , \tag{4.45}$$

$$N^{\frac{1}{2}} \sum a_i^2 a_k^2 E\Delta_i^2 \Delta_k^2 \leqq K_5 N^{-\frac{1}{2}} (\sum a_k^2)^2 \leqq K_6 \cdot N^{-\frac{1}{2}} \to 0 , \tag{4.46}$$

$$|N^{\frac{1}{2}} \sum a_i^2 a_k a_m E\Delta_i^2 \Delta_k \Delta_m| \leqq K_7 N^{-\frac{3}{2}} \sum a_i^2 |a_k|\,|a_m| \leqq K_8 \max |a_k| \to 0 , \tag{4.47}$$

$$\begin{aligned} &|N^{-\frac{1}{2}} \textstyle\sum a_i a_k a_m a_p E\Delta_i \Delta_k \Delta_m \Delta_p| \\ &\quad \leqq K_9 N^{-\frac{5}{2}} \textstyle\sum |a_i a_k a_m a_p| \leqq K_9 (\sum |a_k|/N^{\frac{1}{2}})^4 \cdot N^{-\frac{1}{2}} \leqq K_{10} \cdot N^{-\frac{1}{2}} \to 0 . \end{aligned} \tag{4.48}$$

Using these estimates in (4.38) the lemma follows. $\square$

LEMMA 4.6. *When $N \to \infty$ we have*

$$Z(T_1, \cdots, T_q) - \mu(T_1, \cdots, T_q) = Z(\tau_1, \cdots, \tau_q) - \mu(\tau_1, \cdots, \tau_q) + o_p(1) . \tag{4.49}$$

PROOF. It is sufficient to prove that for $r = 1, 2, \cdots, q$ we have

$$\begin{aligned} Y_r &= Z(\tau_1, \cdots, \tau_{r-1}, T_r, \cdots, T_q) - \mu(\tau_1, \cdots, \tau_{r-1}, T_r, \cdots, T_q) \\ &= Z(\tau_1, \cdots, \tau_r, T_{r+1}, \cdots, T_q) - \mu(\tau_1, \cdots, \tau_r, T_{r+1}, \cdots, T_q) + o_p(1) \\ &= Y_{r+1} + o_p(1) . \end{aligned} \tag{4.50}$$

For fixed $\varepsilon > 0$ we have (summation over all x's)

$$\begin{aligned} P(|Y_r - Y_{r+1}| > \varepsilon) = \sum & P(X_{ks}(\tau_s) = x_{ks}, 1 \leqq k \leqq N, 1 \leqq s < r) \\ & \times P(X_{ks}(T_s) = x_{ks}, 1 \leqq k \leqq N, r < s \leqq q) \\ & \times P(|Y_r - Y_{r+1}| > \varepsilon \mid \{X_{ks} = x_{ks}\}_{k,s\neq r}) . \end{aligned} \tag{4.51}$$

Because the random variables defined for different values of s are independent, we can consider $\{Y_r - Y_{r+1}$ given $\{Z_{ks} = x_{ks}\}_{k,s\neq r}\}$ as a random variable defined for the case $q = 1$. By Lemma 4.3 we have

$$P(|Y_r - Y_{r-1}| > \varepsilon \mid \{Z_{ks} = x_{ks}\}_{k,s\neq r}) = o(1) , \tag{4.52}$$

where $o(1)$ is uniform in $\{x_{ks}\}$ because of the condition (2.6). Hence we have

$$P(|Y_r - Y_{r+1}| > \varepsilon) = [\sum P(\{X_{ks} = x_{ks}\}_{k,s\neq r})] \cdot o(1) = o(1) . \ \square \tag{4.53}$$

Combining the results of Lemmas 4.2 and 4.6 we have proved that

$$\begin{aligned} Z(T_1, \cdots, T_q) &- \mu(\tau_1, \cdots, \tau_q) \\ &= Z(\tau_1, \cdots, \tau_q) - \mu(\tau_1, \cdots, \tau_q) \\ &\quad - \textstyle\sum_{s=1}^q \{\mathrm{Cov}\,(\xi_s \zeta)/\mathrm{Cov}\,(\xi_s, \zeta_s)\}(Z_s(\tau_s) - f_s) + o_p(1) \\ &= [Z(\tau_1, \cdots, \tau_q) - \textstyle\sum_{s=1}^q \{\mathrm{Cov}\,(\xi_s, \zeta)/\mathrm{Cov}\,(\xi_s, \zeta_s)\} Z_s(\tau_s)] \\ &\quad - [\mu(\tau_1, \cdots, \tau_q) - \textstyle\sum_{s=1}^q \{\mathrm{Cov}\,(\xi_s, \zeta)/\mathrm{Cov}\,(\xi_s, \zeta_s)\} f_s] + o_p(1) . \end{aligned} \tag{4.54}$$

Using Theorem 1 on (4.54) proves Theorem 2. □

REMARK. The condition (2.5) is sufficient for proving Lemmas 4.1—4.5. So for the case $q = 1$ the stronger condition (2.6) can be replaced by the weaker (2.5) in the theorem.

5. Mean of the distinct units. Suppose that to each unit in the population is attached a real number, say a_k for the unit U_k. The population mean is denoted by m_a and the population variance by σ_a^2. For estimating m_a a simple random sample with replacement of size n is taken. The sample mean is an unbiased estimator, but a "better" estimator is the mean of the distinct units in the sample; see Lanke (1972) and the references given there for a discussion of this estimator. By defining

$$\begin{aligned} e_k &= 1 \quad \text{if } U_k \text{ is obtained,} \\ &= 0 \quad \text{otherwise,} \end{aligned} \tag{5.1}$$

the mean of distinct units in the sample can be written

(5.2) $$m^* = \sum_{k=1}^N a_k e_k / \sum_{k=1}^N e_k \,.$$

THEOREM 3. *If, when* $N \to \infty$,

(5.3) $$n/N \to t \,, \qquad 0 < t < \infty \,,$$

(5.4) $$\max_{1 \leq k \leq N} (a_k - m_a)^2 / N\sigma_a{}^2 \to 0 \,,$$

then

(5.5) $$m^* \in AsN(m_a, \sigma_a{}^2/(e^t - 1) \cdot N) \,.$$

PROOF. We have

(5.6) $$N^{\frac{1}{2}}(\sum_1^N a_k e_k / \sum_1^N e_k - m_a)/\sigma_a = (\sum_1^N (a_k - m_a) e_k / N^{\frac{1}{2}} \sigma_a)/(\sum_1^N e_k/N) \,.$$

From Thorem 1 it follows that

(5.7) $$\sum_1^N e_k/N = e^{-t} + o_p(1) \,,$$

and

(5.8) $$\sum_1^N (a_k - m_a) e_k / N^{\frac{1}{2}} \sigma_a e^{-t} \in AsN(0, 1/(e^t - 1)) \,.$$

Combining (5.6) and (5.7) it follows by the Cramér–Slutsky theorem that (5.6) and (5.8) have the same asymptotic distribution. □

REMARK. The sample mean is $AsN(m_a, \sigma_a{}^2/n)$. Thus, the ratio of the asymptotic variances is $t/(e^t - 1) < 1$. Lanke (1972) has proved that the same limit is found for the limit of the ratio of the variances.

6. Successive sampling. As in Section 5 we consider the finite population with real numbers attached to the units. We also suppose that it costs $c_k > 0$ for observing U_k, $\sum_1^N c_k = 1$. We draw units without replacement one after the other, so that the probability of drawing U_k is proportional to p_k if U_k has not been obtained before, $\sum_1^N p_k = 1$. Keep on observing units until the total cost for the first time reaches the level f, $0 < f < 1$. Let

(6.1) $$e_k = 1 \quad \text{if} \quad U_k \quad \text{is observed,}$$
$$= 0 \quad \text{otherwise.}$$

The sum of the a's corresponding to the observed units can be written

(6.2) $$\sum_{k=1}^N a_k e_k \,.$$

Using

(6.3) $$g(x, k) = \min(1, x) a_k / (\sum_1^N a_k{}^2)^{\frac{1}{2}}$$

in Theorem 2 we immediately have:

THEOREM 4. *If* (2.1), (2.2), (2.4) *are satified and*

(6.4) $$\max_{1 \leq k \leq N} (a_k{}^2 / \sum_1^N a_k{}^2) \to 0 \,, \qquad N \to \infty \,,$$

then

$$\sum_1^N a_k e_k \in AsN(\sum_1^N a_k \pi_k, \sum_1^N c_k^2 \pi_k(1-\pi_k)((a_k/c_k) - \sum_1^N (a_j/c_j)w_j)^2), \quad N \to \infty, \tag{6.5}$$

where

$$\pi_k = 1 - \exp(-Np_k \tau), \tag{6.6}$$

and τ *is defined by*

$$\sum_1^N (1 - \exp(-Np_k\tau))c_k = f, \tag{6.7}$$

and

$$w_j = c_j(1-\pi_j)\ln(1-\pi_j)/\sum_1^N c_k(1-\pi_k)\ln(1-\pi_k). \tag{6.8}$$

It is easily seen that we get the same theorem if the last unit drawn is not observed, so that the total sampling cost does not exceed f.

To obtain an unbiased estimator of the population total one can use the Horvitz–Thompson estimator

$$(HT) = \sum_1^N a_k e_k / P(e_k = 1). \tag{6.9}$$

But we have no simple expression for the inclusion probability $P(e_k = 1)$, so let us instead consider an approximation of (6.9)

$$(AHT) = \sum_1^N a_k e_k/\pi_k. \tag{6.10}$$

If in Theorem 4 a_k is replaced by a_k/π_k, $k = 1, \cdots, N$, then we obtain the asymptotic distribution of (AHT).

There is an important special case of the sampling procedure above when the costs are equal. In this case a fixed number Nf of different units are drawn, so called *successive sampling*. Rosén (1972) has investigated this case using methods quite different from ours.

THEOREM 5. *If* (2.1) *and* (2.4) *are satisfied, and if*

$$\max_{1 \le k \le N} (a_k - m_a)^2/N\sigma_a^2 \to 0, \quad N \to \infty, \tag{6.11}$$

then we have for successive sampling that

$$\sum_1^N a_k e_k \in AsN(\sum_1^N a_k \pi_k, \sum_1^N \pi_k(1-\pi_k)(a_k - \sum_1^N a_j w_j)^2), \quad N \to \infty, \tag{6.12}$$

where the π'*s and* w'*s are defined in Theorem* 4 *using equal* c'*s.*

PROOF. We have

$$\sum_1^N a_k e_k = N^{\frac{1}{2}}\sigma_a \sum_1^N (a_k - m_a)e_k/N^{\frac{1}{2}}\sigma_a + f\sum_1^N a_k. \tag{6.13}$$

Using Theorem 2 with $c_k = 1/N$ and

$$g(x, k) = \min(1, x) \cdot (a_k - m_a)/N^{\frac{1}{2}}\sigma_a, \tag{6.14}$$

we find the asymptotic distribution of $\sum_1^N (a_k - m_a)e_k/N^{\frac{1}{2}}\sigma_a$. Now the theorem follows from (6.13). □

REMARK 1. (6.11) is often called Noether's condition. Rosén (1972) needed a somewhat stronger condition.

REMARK 2. By replacing a_k by a_k/π_k we obtain a theorem concerning an approximate Horvitz–Thompson estimator.

REMARK 3. If $c_k = 1/N$, $p_k = 1/N$, $1 \leqq k \leqq N$, then we have simple random sampling without replacement. In this case Hájek (1960) showed that Noether's condition is necessary and sufficient.

7. Capture-recapture estimation. Suppose that the finite population has unknown size N, e.g., the population can consist of an unknown number of some sort of animals. To estimate N, methods based on capture-recapture are sometimes used. The most simple of these methods is: take a sample of size M, mark it, replace it into the population, take a new sample of size n. On the basis of the number of marked units, m, in the second sample, N is often estimated by the "Petersen estimator"

$$N^* = Mn/m \,. \tag{7.1}$$

For a discussion of the logical grounds for capture-recapture estimates and for further references on the subject, see Cormack (1972). Here we will only indicate how Theorem 2 can be used to study the large sample properties of the Petersen estimator, if the samples are not simple random due to heterogeneous catching probabilities or effects of marking.

Sampling procedure 1. The two samples are obtained by successive sampling of M respectively n units (cf. Section 6).

In practice the sampling scheme should be planned so that the samples are simple random (i.e., all the p's equal in successive sampling). But due to, e.g., differences in the age distribution or the location of the units, heterogeneous catchability can occur. The object of Theorem 6 is to elucidate how such heterogeneity affects the properties of the Petersen estimator.

THEOREM 6. *If two samples of size M and n are obtained by sampling procedure* 1, *the condition* (2.1) *is satisfied and if, when $N \to \infty$,*

$$M/N \to p\,, \qquad 0 < p < 1\,, \tag{7.2}$$

$$n/N \to f\,, \qquad 0 < f < 1\,, \tag{7.3}$$

then

$$N^* = Mn/m \in AsN(Mn/\textstyle\sum_1^N \pi_{k1}\pi_{k2},\, (Mn\sigma_N)^2/(\sum_1^N \pi_{k1}\pi_{k2})^4)\,, \tag{7.4}$$

where the π's are defined as in Theorem 5, *and*

$$\sigma_N^2 = \textstyle\sum_1^N \{\pi_{k1}(1-\pi_{k1})\pi_{k2}(1-\pi_{k2}) + \pi_{k1}(1-\pi_{k1})(\pi_{k2}-\gamma_2)^2 + \pi_{k2}(1-\pi_{k2})(\pi_{k1}-\gamma_1)^2\}\,, \tag{7.5}$$

$$\gamma_1 = \textstyle\sum_1^N \pi_{k1}(1-\pi_{k2})\ln(1-\pi_{k2})/\sum_1^N (1-\pi_{k2})\ln(1-\pi_{k2})\,, \tag{7.6}$$

$$\gamma_2 = \textstyle\sum_1^N \pi_{k2}(1-\pi_{k1})\ln(1-\pi_{k1})/\sum_1^N (1-\pi_{k1})\ln(1-\pi_{k1})\,. \tag{7.7}$$

PROOF. Let us consider Theorem 2 with $q = 2$, equal c's and

$$g(x_1, x_2, k) = 1 \quad x_1 \geqq 1\,, \quad x_2 \geqq 1\,, \qquad = 0 \quad \text{otherwise.} \tag{7.8}$$

For this case the random variable $Z(T_1, T_2)$ in Theorem 2 is the number of units which belongs to both samples, or the number of marked units in the second sample, i.e., m. Using Theorem 2 it follows that

$$m \in AsN(\textstyle\sum_1^N \pi_{k1} \cdot \pi_{k2}, \sigma_N^2)\,, \qquad N \to \infty\,. \tag{7.9}$$

Using the Cramér–Slutsky theorem the assertion follows. □

REMARK. If the samples are simple random then m has a hypergeometric distribution. In this case (7.9) gives

$$m \in AsN(np, (1-f)np(1-p))\,, \qquad N \to \infty\,, \tag{7.10}$$

the normal approximation of the hypergeometric distribution. We also get

$$N^* \in AsN(N, N \cdot (1-f)(1-p)/fp)\,. \tag{7.11}$$

Sampling procedure 2. The units $1, \cdots, M$ are marked. A sample of size n is obtained by successive sampling.

This sampling scheme can occur in practice, e.g., if a random sample of M units is marked, but due to the marking procedure the catchability is changed.

THEOREM 7. *If a sample is obtained by the sampling procedure* 2, *and if the conditions* (2.1), (7.2), (7.3) *are satisfied, then*

$$N^* = Mn/m \in AsN(Mn/\textstyle\sum_1^M \pi_k, (Mn\sigma_N)^2/(\sum_1^M \pi_k)^4)\,, \tag{7.12}$$

where the π's are defined in Theorem 5, *and*

$$\sigma_N^2 = \textstyle\sum_1^M \pi_k(1-\pi_k)(1-\gamma)^2 + \sum_{M+1}^N \pi_k(1-\pi_k)\gamma^2\,, \tag{7.13}$$

$$\gamma = \textstyle\sum_1^M (1-\pi_k) ln(1-\pi_k) / \sum_1^N (1-\pi_k) ln(1-\pi_k)\,. \tag{7.14}$$

PROOF. Consider Theorem 2 with $q = 1$, the c's equal, and

$$g(x, k) = 1 \quad \text{if} \quad 1 \leqq k \leqq M \quad \text{and} \quad x \geqq 1\,, \qquad = 0 \quad \text{otherwise.} \tag{7.15}$$

In this case $Z(T)$ is the number of units in the sample among $U_1, \cdots, U_M$, or the number of marked units in the sample, i.e., m. Using Theorem 2 it follows that

$$m \in AsN(\textstyle\sum_1^M \pi_k, \sigma_N^2)\,, \qquad N \to \infty\,. \tag{7.16}$$

Using the Cramér–Slutsky theorem the assertion follows. □

REMARK. For simple random sampling we obtain the same result as in the previous remark.

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