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MULTIVARIATE LOGISTIC DISTRIBUTIONS[1]

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In this paper a multivariate analogue of the logistic distribution is considered. We suggest two families of multivariate logistic distributions with the property that marginal distributions are of univariate form and discuss some distributional properties of the multivariate distributions.

1. Introduction. The logistic distribution with density

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}\,, \qquad -\infty < x < \infty\,, \tag{1.1}$$

has been widely used by Berkson [2], Berkson and Hodges [3] as a model for analyzing bioassay and other experiments involving quantal response, by Pearl and Reed [8] in studies pertaining to population growth, and by Plackett [9] in connection with problems involving censored data.

Gumbel [5], [6] suggested two bivariate exponential and two bivariate logistic distributions with exponential and logistic margins respectively. Marshall and Olkin [7] derived two multivariate exponential distributions and a multivariate Weibull distribution. In this paper a multivariate analogue of the logistic distribution is considered, with the property that marginal distributions are of univariate form and discuss some distributional properties of the multivariate distributions.

2. A multivariate logistic distribution. Dubey [4] has proved that the logistic distribution defined by equation (1.1) is a compound extreme value distribution with an exponential distribution as a compounder. We use this fact to derive a p-variate logistic distribution.

Let $X_1, X_2, \cdots, X_p$, given α, be independent random variables with the conditional distribution function

$$F(x_k \mid \alpha) = \exp\{-\alpha \exp(-x_k)\}\,, \qquad \alpha > 0,\ -\infty < x_k < \infty,\ k = 1, 2, \cdots, p\,, \tag{2.1}$$

where α has the density function

$$g(\alpha) = \exp(-\alpha)\,, \qquad \alpha > 0\,. \tag{2.2}$$

The conditional density and moment generating functions corresponding to $F(x_k \mid \alpha)$ are

$$f(x_k \mid \alpha) = \alpha \exp(-x_k)\exp\{-\alpha \exp(-x_k)\}\,, \qquad \alpha > 0,\ -\infty < x_k < \infty\,, \tag{2.3}$$

$$M(t_k \mid \alpha) = \alpha^{t_k}\Gamma(1 - t_k)\,. \tag{2.4}$$

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The joint distribution function of $X_1, X_2, \cdots, X_p$ is

(2.5) $$F(x_1, \cdots, x_p) = \int_0^\infty \prod_{i=1}^p F(x_i \mid \alpha) g(\alpha) d\alpha = \{1 + \sum_{k=1}^p \exp(-x_k)\}^{-1},$$

with density function

(2.6) $$f(x_1, \cdots, x_p) = p! \exp\{-\sum_{k=1}^p x_k\}[1 + \sum_{k=1}^p \exp(-x_k)]^{-(p+1)}$$
$$-\infty < x_k < \infty, k = 1, 2, \cdots, p.$$

We define the density function (2.6) as the p-variate multivariate logistic density. When $p = 2$ we obtain the bivariate logistic distribution obtained by Gumbel [6].

THEOREM 2.1. *If the p-dimensional random variable* $(x_1, \cdots, x_p)$ *has the multivariate logistic distribution* (2.5) *then the joint marginal distribution function of* $(X_1, \cdots, X_s)$ $(s < p)$ *is that of an s-variate logistic distribution.*

Note that the conditional density $f(x_1, \cdots, x_s \mid x_{s+1}, \cdots, x_p)$ of $X_1, \cdots, X_s$ given $X_{s+1} = x_{s+1}, \cdots, X_p = x_p$ is not a multivariate logistic distribution.

The moment generating function of the p-dimensional random variable $(X_1, \cdots, X_p)$ having the p-variate logistic distribution is given by

$$M(t_1, \cdots, t_p) = \int_0^\infty \prod_{i=1}^p M(t_i \mid \alpha) g(\alpha) d\alpha = (1 + \sum_{k=1}^p t_k) \prod_{k=1}^p \Gamma(1 - t_k).$$

THEOREM 2.2. *For* $i = 1, 2, \cdots, p$, *let* $X_{n \cdot i}$ *denote the largest* ith *component in a sequence of n observations on the vector* $(X_1, X_2, \cdots, X_p)$ *having a p-variate logistic distribution then* $X_{n\cdot 1}, X_{n\cdot 2}, \cdots, X_{n\cdot p}$ *are asymptotically independent.*

PROOF. The joint probability function of $X_{n\cdot 1}, X_{n\cdot 2}, \cdots, X_{n\cdot p}$ is

(2.7) $$F^n(x_1, x_2, \cdots, x_p) = \{1 + \exp(-x_1) + \cdots + \exp(-x_p)\}^{-n}.$$

It is known, Gumbel [6], that the most probable largest values $U_{n\cdot 1}, U_{n\cdot 2}, \cdots, U_{n\cdot p}$ in the univariate cases are $U_{n\cdot 1} = U_{n\cdot 2} = \cdots = U_{n\cdot p} = \log_e n$. Hence the p-variate probability function for the reduced largest values $x_1 - \log_e n, \cdots, x_p - \log_e n$ is

$$F^n(x_1 + \log_e n, \cdots, x_p + \log_e n).$$

Now passing to the limit the result follows.

THEOREM 2.3. *For* $i = 1, 2, \cdots, p$, *let* $X_{1 \cdot i}$ *denote the smallest* ith *component in a sequence of n observations on the vector* $(X_1, X_2, \cdots, X_p)$ *with a multivariate logistic distribution; then* $X_{1\cdot 1}, X_{1\cdot 2}, \cdots, X_{1\cdot p}$ *are not asymptotically independent.*

3. Multivariate logistic distribution of type II. Gumbel's [7] bivariate logistic distribution of type II may also be extended to a p-variate case. A p-variate distribution with logistic marginal cdf's $F(x_1), \cdots, F(x_p)$ may be obtained from a system

$$F(x) = \prod F(x_i)[1 + \sum b_{12} \bar{F}(s_1)\bar{F}(s_2) + \sum b_{123} \bar{F}(s_1)\bar{F}(s_2)\bar{F}(s_3) + \cdots$$
$$+ b_{123\cdots p} \bar{F}(x_1)\bar{F}(x_2) \cdots \bar{F}(x_p)]$$

where $\bar{F} = 1 - F$ and $\sum b_{12} = \sum_{i<j} b_{ij}$, etc.

The following theorems easily follow.

THEOREM 3.1. *If the p-dimensional random variable* $(X_1, \cdots, X_p)$ *has the p-variate logistic distribution of type* II *then the joint marginal distribution function of the s-dimensional random variable* $(X_1, \cdots, X_s)$ $(s < p)$ *is that of an s-variate logistic distribution of type* II.

THEOREM 3.2. *For* $i = 1, 2, \cdots, p$, *let* $X_{n \cdot i}$ *denote the largest* ith *component in a sequence of n observations on the vector* $(X_1, X_2, \cdots, X_p)$ *with a p-variate logistic distribution of type* II *then* $X_{n \cdot 1}, X_{n \cdot 2}, \cdots, X_{n \cdot p}$ *are asymptotically independent.*

THEOREM 3.3. *For* $i = 1, 2, \cdots, p$, *let* $X_{1 \cdot i}$ *denote the smallest* ith *component in a sequence of n observations on the vector* $(X_1, X_2, \cdots, X_p)$ *with a p-variate logistic distribution of type* II *then* $X_{1 \cdot 1}, X_{1 \cdot 2}, \cdots, X_{1 \cdot p}$ *are asymptotically independent.*

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