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APPROXIMATING CIRCULANT QUADRATIC FORMS IN JOINTLY STATIONARY GAUSSIAN TIME SERIES[1]

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Let $\{X(t), t = 0, \pm 1, \pm 2, \cdots\}$ be a p-dimensional, zero mean, stationary Gaussian time series with matrix-valued covariance function $R(u) = (r_{jk}(u))$, $u = 0, \pm 1, \pm 2, \cdots; j, k = 1, 2, \cdots, p$. Let $F(\omega)$ be the spectral density matrix of the time series (if $F(\omega)$ exists), and assume that $F(\omega)$ is positive definite for all $\omega \in (0, 2\pi]$. Let $\hat{F}_X(\omega_l)$ be an estimator of $F(\omega_l)$ formed by averaging $(2n + 1)$ periodogram ordinates centered and equally spaced around ω_l, $l = 1, 2, \cdots, M$, where the ω_l's themselves are equally spaced on $(0, \pi]$, and where all periodogram ordinates are based on the same record of length T, $(2n + 1)M \leqq T/2$, taken from the time series $\{X(t)\}$. Wahba (1968) has shown that if $\sum_j \sum_k \sum_u |ur_{jk}(u)| < \infty$ and $\log_2 M \leqq n$, then it is possible to construct M independent complex Wishart matrices $W_T(\omega_l)$, $l = 1, 2, \cdots, M$, such that $\{\hat{F}_X(\omega_l), l = 1, 2, \cdots, M\}$ converge simultaneously in mean square to $\{W_T(\omega_l), l = 1, 2, \cdots, M\}$ as n, M (and thus T) get large. In the present paper, it is shown that Wahba's result holds under the less restrictive condition that $\sum_j \sum_k \sum_u |u|^{\frac{1}{2}} |r_{jk}(u)| < \infty$, and without our needing to assume that $\log_2 M \leqq n$. In consequence, a form of weak convergence of the averaged periodogram to a certain matrix-valued Wishart stochastic process is demonstrated (something, by the way, that Wahba (1968) cannot show because of her restriction that $\log_2 M \leqq n$). This result is a consequence of some general conclusions concerning the approximation of circulant quadratic forms in the time series $\{X(t)\}$.

1. Introduction and summary.

Let $\{X(t), t = 0, \pm 1, \pm 2, \cdots\}$ be a p-dimensional, zero mean, stationary Gaussian time series with matrix-valued covariance function:

$$R(u) = E(X(t)X'(t + u)) = (r_{jk}(u)) : p \times p \tag{1.1}$$

for $t, u = 0, \pm 1, \pm 2, \cdots; j, k = 1, 2, \cdots, p$. Let

$$F(\omega) = (2\pi)^{-1} \sum_{u=-\infty}^{\infty} e^{-i\omega u} R(u) , \qquad 0 < \omega \leqq 2\pi , \tag{1.2}$$

be the spectral density matrix of the time series (conditions are given shortly that guarantee the existence of $F(\omega)$ for all $\omega \in (0, 2\pi]$). We assume that $F(\omega)$ is a positive definite matrix for all $\omega \in (0, 2\pi]$.

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Wahba (1968) has shown that if

$$\sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{u=-\infty}^{\infty} |u| \, |r_{jk}(u)| < \infty \,, \tag{1.3}$$

and if $\hat{F}_X(\omega_l)$, $l = 1, 2, \cdots, M$, are estimators of the spectral density matrix, each consisting of averages of $(2n + 1)$ periodogram ordinates based on a record of length T, with the ω_l equally spaced of the form $\omega_l = 2\pi l/T$ for integer l, and $(2n + 1)M \leqq T/2$, then provided $\log_2 M \leqq n$ it is possible to construct, on the same sample space as $\{X(t), t = 0, \pm 1, \pm 2, \cdots\}$, M independent complex Wishart matrices $W_T(\omega_l)$, $l = 1, 2, \cdots, M$, such that $\{\hat{F}_X(\omega_l), l = 1, \cdots, M\}$ converge simultaneously in mean square to $\{W_T(\omega_l), l = 1, 2, \cdots, M\}$ as n, M (and thus T) get large. For some applications of this result see Wahba (1968). In the present paper, we prove Wahba's result under the less restrictive condition

$$\sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{u=-\infty}^{\infty} |u|^{\frac{1}{2}} \, |r_{jk}(u)| < \infty \,, \tag{1.4}$$

and also dispense with her requirement that $\log_2 M \leqq n$. In consequence, a form of weak convergence of the averaged periodogram to a certain matrix-valued Wishart stochastic process is demonstrated (something, by the way, that Wahba (1968) cannot show because of her restriction that $\log_2 M \leqq n$).

In Section 2, we discuss Condition (1.4) and some of its consequences. Section 3 contains some needed notation, plus a lemma of general use in proving results about approximating quadratic forms. Finally, Section 4 contains the proof of the main result of this paper.

2. Condition (1.4) and its consequences. Since

$$p^{-2}\Big(\sum_{j=1}^{p} \sum_{k=1}^{p} |r_{jk}(u)|\Big)^2 \leqq \sum_{j=1}^{p} \sum_{k=1}^{p} r_{jk}^2(u) \leqq \Big(\sum_{j=1}^{p} \sum_{k=1}^{p} |r_{jk}(u)|\Big)^2 \,, \tag{2.1}$$

and since $\operatorname{tr} R(u)R(-u) = \sum_{j=1}^{p} \sum_{k=1}^{p} r_{jk}^2(u)$, all $u = 0, \pm 1, \pm 2, \cdots$, it follows that Condition (1.4) is equivalent to the condition that

$$\sum_{u=-\infty}^{\infty} |u|^{\frac{1}{2}} (\operatorname{tr} R(u)R(-u))^{\frac{1}{2}} = C < \infty \,. \tag{2.2}$$

A comparison of (1.4) with (1.3), shows that Wahba's Condition (1.3) implies (1.4). On the other hand, the example of a one-dimensional time series $\{X(t), t = 0, \pm 1, \cdots\}$ with covariance function $r(u)$ proportional to $(1 + u^2)^{-1}$ shows that (1.4) (or equivalently (2.2)) may hold in contexts where (1.3) is false. The above remarks thus establish the assertion made in Section 1 that Condition (1.4) is less restrictive than Wahba's (1968) Condition (1.3).

The following lemma gives further consequences of Condition (1.4), and is of interest in its own right.

LEMMA 2.1. *Assume that*

$$\sum_{u=-\infty}^{\infty} |u|^{\alpha} (\operatorname{tr} R(u)R(-u))^{\frac{1}{2}} < \infty \tag{2.3}$$

for some α, $0 < \alpha \leqq 1$. *Then*

(i) $F(\omega) = (f_{jk}(\omega))$ *exists for all* $\omega \in (0, 2\pi]$;

(ii) *there exists a constant* Λ, $0 < \Lambda < \infty$, *such that* $\Lambda I_p - F(\omega)$ *is positive semi-definite* (p.s.d.) *for all* $\omega \in (0, 2\pi]$, *where* I_p *is the* $p \times p$ *identity matrix;*

(iii) *each component* $f_{jk}(\omega)$ *of* $F(\omega)$ *is* Lip (α)*;*

(iv) *if* $F(\omega)$ *is positive definite for all* $\omega \in (0, 2\pi]$, *there exists* $\lambda > 0$ *such that* $F(\omega) - \lambda I_p$ *is* p.s.d. *for all* $\omega \in (0, 2\pi]$.

PROOF. Parts (i) and (ii) follow since for any $p \times 1$ vector c, $c'c = 1$, we have

$$(2.4) \qquad \begin{aligned} 2\pi c'F(\omega)c &= \textstyle\sum_{u=-\infty}^{\infty} e^{-i\omega u} c'R(u)c \leqq \sum_{u=-\infty}^{\infty} |c'R(u)c| \\ &\leqq (\operatorname{tr} R^2(0))^{\frac{1}{2}} + \textstyle\sum_{u=-\infty}^{\infty} |u|^\alpha (\operatorname{tr} R(u)R(-u))^{\frac{1}{2}} < \infty \end{aligned}$$

by the given (2.3). Hence, for all $\omega \in (0, 2\pi]$, the maximum latent root of $F(\omega)$ is bounded above by a finite constant, thus proving parts (i) and (ii). Part (iv) follows from part (iii) since $f_{jk}(\omega)$ being Lip (α) for $\alpha > 0$ implies, since the roots of a matrix are continuous functions of the entries, that the minimum latent root of $F(\omega)$ is continuous in ω over $(0, 2\pi]$, and (since $F(\omega)$ is everywhere positive definite and is periodic of period 2π) is bounded below by a positive constant λ. It therefore remains to prove part (iii).

Let $\omega_1, \omega_2 \in (0, 2\pi]$ and let b be the largest integer less than or equal to $(2\pi/|\omega_1 - \omega_2|)$. Then

$$\begin{aligned} 2\pi\, |f_{jk}(\omega_1) - f_{jk}(\omega_2)| &= |\textstyle\sum_{u=-\infty}^{\infty} (e^{-i\omega_1 u} - e^{-i\omega_2 u}) r_{jk}(u)| \\ &\leqq |\omega_1 - \omega_2| \textstyle\sum_{|u| \leqq b} |u|\, |r_{jk}(u)| + 2\pi \sum_{|u| > b} |r_{jk}(u)| \\ &\leqq b^{1-\alpha} |\omega_1 - \omega_2| \textstyle\sum_{|u| \leqq b} |u|^\alpha |r_{jk}(u)| + 2\pi b^{-\alpha} \sum_{|u| > b} |u|^\alpha |r_{jk}(u)| \\ &\leqq |\omega_1 - \omega_2|^\alpha (2\pi)^{1-\alpha} \textstyle\sum_{u=-\infty}^{\infty} |u|^\alpha |r_{jk}(u)| \,. \end{aligned}$$

This result, together with (2.3), establishes that $f_{jk}(\omega)$ is Lip (α). □

From Lemma 2.1 it follows that under either (1.3) or (1.4), $F(\omega)$ exists and there exist constants $\lambda, \Lambda, 0 < \lambda \leqq \Lambda < \infty$, such that $\Lambda I_p - F(\omega)$ and $F(\omega) - \lambda I_p$ are p.s.d., all $\omega \in (0, 2\pi]$. However, we see from Lemma 2.1 (iii) that (1.3) implies that each $f_{jk}(\omega)$ is Lip (1), and hence must have bounded left- and right-hand derivatives, $j, k = 1, 2, \cdots, p$. On the other hand, Condition (1.4) implies only that each $f_{jk}(\omega)$ is Lip $(\frac{1}{2})$, and Lip (1) is a proper subset of Lip $(\frac{1}{2})$.

3. Notation and some useful lemmas. Assume that the time series $\{X(t), t = 0, \pm 1, \pm 2, \cdots\}$ is observed at the T consecutive points $1, 2, \cdots, T$. Let $X(1), X(2), \cdots, X(T)$ be the record of observations, where

$$X(t) = (x_1(t), x_2(t), \cdots, x_p(t))' : p \times 1 \,.$$

Let

$$(3.1) \qquad y = (x_1(1), x_1(2), \cdots, x_1(T), x_2(1), x_2(2), \cdots, x_2(T), \cdots, x_p(1), \cdots, x_p(T))' \,,$$

$$(3.2) \qquad Z = (X(1), X(2), \cdots, X(T)) \,.$$

Here, y is $pT \times 1$ and Z is $p \times T$. Because we have assumed that the time series $\{X(t),\ t = 0, \pm 1, \pm 2, \cdots\}$ is Gaussian, y has a pT-variate normal distribution with mean vector equal to zero and a covariance matrix Σ which we can represent as follows.

First, let us define the $T \times T$ shift matrices $S(0), S(1), S(-1), \cdots$, in the following manner: $S(0) = I_T$ is the $T \times T$ identity matrix, and

$$S(1) = (s_{jk}), \quad \text{where } s_{jk} = 1, \quad k = j+1 \\ = 0, \quad \text{otherwise,} \tag{3.3}$$

$$S(u) = (S(1))^u, \quad \text{if } u > 0 \\ = (S'(1))^{|u|}, \quad \text{if } u < 0. \tag{3.4}$$

Straightforward calculation shows that $S(u) = 0$ for all u for which $|u| \geqq T$.

In terms of the shift matrices $S(0), S(1), S(-1), \cdots$, defined above, it is not hard to show that

$$\Sigma = \textstyle\sum_{u=1-T}^{T-1} R(u) \times S(u), \tag{3.5}$$

where $A \times B$ denotes the $pT \times pT$ Kronecker product of $A: p \times p$ and $B: T \times T$.

It has been noted by Wahba (1968) that the estimator $\hat{F}_X(\omega_l)$, which is the average of $(2n+1)$ periodogram ordinates $I(\omega_l - T^{-1}2\pi n)$, $I(\omega_l - T^{-1}2\pi(n-1))$, $\cdots$, $I(\omega_l + T^{-1}2\pi(n-1))$, $I(\omega_l + T^{-1}2\pi n)$ centered around $\omega_l = T^{-1}2\pi j_l$, for integer j_l, can be written as a generalized quadratic form;

$$\hat{F}_X(\omega_l) = ZQ(\omega_l)Z', \tag{3.6}$$

where Z is defined by (3.2) and $Q(\omega_l)$ is a certain $T \times T$ circulant Hermitian matrix, $l = 1, 2, \cdots, M$. ($Q(\omega_l)$ is explicitly defined in Section 4.) It is easily seen that Z, and thus $\hat{F}_X(\omega_l)$, is also a function of y. Let $\Theta(w)$ be a function mapping $pT \times 1$ vectors w into $p \times T$ matrices in such a way that the (j, k)th element of $\Theta(w)$ is the $[T(j-1)+k]$th element of w, $j = 1, 2, \cdots, p$; $k = 1, 2, \cdots, T$. Then, $Z = \Theta(y)$ and $\hat{F}_X(\omega_l) = \Theta(y)Q(\omega_l)\Theta'(y)$.

To approximate $\hat{F}_X(\omega_l)$, we transform y by a sequence of linear transformations $y \to A^{\frac{1}{2}}\Sigma^{-\frac{1}{2}}y$, where A is a $pT \times pT$ p.s.d. matrix and $A^{\frac{1}{2}}$, $\Sigma^{\frac{1}{2}}$ are any p.s.d. square roots of A, Σ respectively. From the transformed data vector $A^{\frac{1}{2}}\Sigma^{-\frac{1}{2}}y$, we then construct the new approximating quadratic forms

$$\hat{F}_{X,A}(\omega_l) = \Theta(A^{\frac{1}{2}}\Sigma^{-\frac{1}{2}}y)Q(\omega_l)\Theta'(A^{\frac{1}{2}}\Sigma^{-\frac{1}{2}}y). \tag{3.7}$$

If A is chosen appropriately, the new quadratic forms $\hat{F}_{X,A}(\omega_l)$, $l = 1, 2, \cdots, M$, have a more convenient joint distribution than $\hat{F}_X(\omega_l)$, $l = 1, 2, \cdots, M$, and yet the new forms are "close" to the original estimators in the sense that

$$\Delta(\omega_1, \omega_2, \cdots, \omega_M; A, n) = \textstyle\sum_{l=1}^{M} \mathscr{E} \operatorname{tr} (\hat{F}_X(\omega_l) - \hat{F}_{X,A}(\omega_l))^2 \tag{3.8}$$

is small for large enough n and M.

We now present a lemma which is of use in establishing rates of convergence for $\Delta(\omega_1, \cdots, \omega_M; A, n)$, and which also is of more general use. For $j, k =$

$1, 2, \cdots, p$, let

$E_{jk}, j \neq k$, be the $p \times p$ matrix whose (j, k)th and (k, j)th coordinates are $\frac{1}{2}$, and all of whose other coordinates are 0,

(3.9)

E_{jj} be the $p \times p$ diagonal matrix with jth diagonal element equal to 1 and all other elements equal to 0.

LEMMA 3.1. *Let y be a pT-dimensional normally distributed vector with mean vector equal to 0 and covariance matrix I_{pT}. Let A and B be any two real $pT \times pT$* p.s.d. *matrices and let $Q_1, Q_2, \cdots, Q_M$ be $MT \times T$ Hermitian matrices. Then*

$$\sum_{l=1}^{M} \mathscr{E} \operatorname{tr} [\Theta(A^{\frac{1}{2}}y)Q_l\Theta'(A^{\frac{1}{2}}y) - \Theta(B^{\frac{1}{2}}y)Q_l\Theta'(B^{\frac{1}{2}}y)]^2$$
$$\leqq (p+1)\left[\frac{\text{max root}\,(A+B)}{\text{min root}\,(A+B)}\right][\text{max root} \textstyle\sum_{l=1}^{M} Q_l^2] \operatorname{tr}(A-B)^2$$
$$+ 2 \sum_{l=1}^{M} \sum_{j=1}^{p} \sum_{k=1}^{p} [\operatorname{tr}(A-B)(E_{jk} \times Q_l)]^2 . \tag{3.10}$$

PROOF. Using an idea of Wahba's ((1968) page 1857), we note that

$$\mathscr{E} \operatorname{tr} [\Theta(A^{\frac{1}{2}}y)Q_l\Theta'(A^{\frac{1}{2}}y) - \Theta(B^{\frac{1}{2}}y)Q_l\Theta'(B^{\frac{1}{2}}y)]^2$$
$$= \sum_{j=1}^{p} \sum_{k=1}^{p} \mathscr{E}\{y'[A^{\frac{1}{2}}(E_{jk} \times Q_l)A^{\frac{1}{2}} - B^{\frac{1}{2}}(E_{jk} \times Q_l)B^{\frac{1}{2}}]y\}^2 , \tag{3.11}$$

$l = 1, 2, \cdots, M$. Further, from the proof of Lemma 5 in Wahba's (1968) paper,

$$\mathscr{E}\{y'[A^{\frac{1}{2}}(E_{jk} \times Q_l)A^{\frac{1}{2}} - B^{\frac{1}{2}}(E_{jk} \times Q_l)B^{\frac{1}{2}}]y\}^2$$
$$= \operatorname{tr} [A^{\frac{1}{2}}(E_{jk} \times Q_l)A^{\frac{1}{2}} - B^{\frac{1}{2}}(E_{jk} \times Q_l)B^{\frac{1}{2}}]^2$$
$$+ 2[\operatorname{tr}(A-B)(E_{jk} \times Q_l)]^2 , \tag{3.12}$$

$j, k = 1, 2, \cdots, p$; $l = 1, 2, \cdots, M$. Now note that

$$A^{\frac{1}{2}}(E_{jk} \times Q_l)A^{\frac{1}{2}} - B^{\frac{1}{2}}(E_{jk} \times Q_l)B^{\frac{1}{2}}$$
$$= (A^{\frac{1}{2}} - B^{\frac{1}{2}})(E_{jk} \times Q_l)A^{\frac{1}{2}} + B^{\frac{1}{2}}(E_{jk} \times Q_l)(A^{\frac{1}{2}} - B^{\frac{1}{2}}) , \tag{3.13}$$

and thus (from Lemma A.1 of Wahba (1968)),

$$\operatorname{tr} [A^{\frac{1}{2}}(E_{jk} \times Q_l)A^{\frac{1}{2}} - B^{\frac{1}{2}}(E_{jk} \times Q_l)B^{\frac{1}{2}}]^2$$
$$\leqq 2 \operatorname{tr} [(A^{\frac{1}{2}} - B^{\frac{1}{2}})(E_{jk} \times Q_l)(A+B)(E_{jk} \times Q_l)(A^{\frac{1}{2}} - B^{\frac{1}{2}})] , \tag{3.14}$$

$j, k = 1, 2, \cdots, p$; $l = 1, 2, \cdots, M$. Since for E p.s.d. and any matrix C for which CEC' is defined,

$$\operatorname{tr} CEC' \leqq [\text{max root}\, E] \operatorname{tr} CC' , \tag{3.15}$$

it follows from (3.14) that

$$\sum_{l=1}^{M} \sum_{j=1}^{p} \sum_{k=1}^{p} \operatorname{tr} [A^{\frac{1}{2}}(E_{jk} \times Q_l)A^{\frac{1}{2}} - B^{\frac{1}{2}}(E_{jk} \times Q_l)B^{\frac{1}{2}}]^2$$
$$\leqq 2[\text{max root}\,(A+B)]$$
$$\times \sum_{l=1}^{M} \sum_{j=1}^{p} \sum_{k=1}^{p} \operatorname{tr}(A^{\frac{1}{2}} - B^{\frac{1}{2}})(E_{jk} \times Q_l)^2(A^{\frac{1}{2}} - B^{\frac{1}{2}})$$
$$= 2[\text{max root}\,(A+B)] \operatorname{tr}(A^{\frac{1}{2}} - B^{\frac{1}{2}})(\tfrac{1}{2}(p+1)I_p \times \textstyle\sum_{l=1}^{M} Q_l^2)(A^{\frac{1}{2}} - B^{\frac{1}{2}})$$
$$\leqq (p+1)[\text{max root}\,(A+B)][\text{max root}\,(I_p \times \textstyle\sum_{l=1}^{M} Q_l^2)] \operatorname{tr}(A^{\frac{1}{2}} - B^{\frac{1}{2}})^2 . \tag{3.16}$$

Finally, from the proof of Lemma A.2 of Wahba (1968),

$$\text{(3.17)}\qquad \begin{aligned}\operatorname{tr}(A^{\frac{1}{2}} - B^{\frac{1}{2}})^2 &\leqq [\text{min root}\,(A^{\frac{1}{2}} + B^{\frac{1}{2}})]^{-2} \operatorname{tr}(A - B)^2 \\ &\leqq [\text{min root}\,(A + B)]^{-1} \operatorname{tr}(A - B)^2 ,\end{aligned}$$

and from known facts about the roots of Kronecker products,

$$\text{(3.18)}\qquad \text{max root}\,(I_p \times \textstyle\sum_{l=1}^{M} Q_l^2) = [\text{max root}\,(I_p)][\text{max root}\,(\sum_{l=1}^{M} Q_l^2)] .$$

Combining (3.11), (3.12), (3.16), (3.17), and (3.18) gives us (3.10) and completes the proof. □

4. Proof of the main result. Using Lemma 3.1, we now successively approximate the quadratic forms $\hat{F}_X(\omega_l)$, $l = 1, 2, \cdots, M$. Use of the explicit form of the matrices $Q(\omega_l)$ of the quadratic forms $\hat{F}_X(\omega_l)$, $l = 1, 2, \cdots, M$, is delayed until knowledge of the exact forms is needed, so as to make our results as general as we can.

4.1 *A block circulant approximation to* Σ. Let H_{rr} be the $T \times T$ diagonal matrix having 1 in the rth diagonal place, and 0 elsewhere. Define the Fejér polynomial of degree T,

$$\text{(4.1)}\qquad K_T(\omega) = (2\pi T)^{-1} \textstyle\sum_{u=1-T}^{T-1} (T - |u|)e^{-i\omega u} R(u) , \qquad 0 < \omega \leqq 2\pi .$$

Let W be the $T \times T$ unitary matrix whose (r, s)th element w_{rs} is given by

$$\text{(4.2)}\qquad w_{rs} = T^{-\frac{1}{2}} e^{-i2\pi rs/T} , \qquad r, s = 1, 2, \cdots, T .$$

Let

$$\text{(4.3)}\qquad \Gamma = I_p \times W ,$$

and

$$\text{(4.4)}\qquad D_K = \textstyle\sum_{r=1}^{T} K_T\left(\frac{2\pi r}{T}\right) \times H_{rr} .$$

The matrix

$$\text{(4.5)}\qquad \Sigma_K = 2\pi \Gamma D_K \Gamma^{*\prime} ,$$

where $C^{*\prime}$ denotes the conjugate transpose of the complex matrix C, can be shown (Lemma 4.1(i)) to be composed of $T \times T$ blocks each of which is a circulant matrix. Hence, the arguments used by Wahba (1968) in her Lemma 3 can now be utilized to obtain a convenient distributional representation for generalized quadratic forms $\Theta(\Sigma_K^{\frac{1}{2}}\Sigma^{-\frac{1}{2}}y)Q\Theta'(\Sigma_K^{\frac{1}{2}}\Sigma^{-\frac{1}{2}}y)$ where Q is a circulant symmetric matrix.

LEMMA 4.1. *Let* Σ_K *be defined by* (4.5). *Then*

(i) $\Sigma_K = T^{-1} \sum_{u=1-T}^{T-1} \{(T - |u|)R(u) + |u|\, R(\delta_u(T - |u|))\} \times S(u)$,

where $\delta_u = -1$ *if* $u \geqq 0$, 1 *if* $u < 0$,

(ii) $\operatorname{tr}(\Sigma - \Sigma_K)^2$

$$= (2T)^{-1} \textstyle\sum_{u=1-T}^{T-1} \sum_{j,k=1}^{p} |u|\, (T - |u|)(r_{jk}(u) - r_{jk}(\delta_u(T - |u|)))^2 .$$

(iii) *For any circulant Hermitian matrix* $Q: T \times T$,

$$\operatorname{tr}(\Sigma - \Sigma_K)(E_{jk} \times Q) = 0\,, \qquad \text{all } j, k = 1, 2, \cdots, p\,.$$

(iv) *For any circulant Hermitian matrix* $Q: T \times T$, *the random matrix*

$$\Theta(\Sigma_K^{\frac{1}{2}}\Sigma^{-\frac{1}{2}}y)Q\Theta'(\Sigma_K^{\frac{1}{2}}\Sigma^{-\frac{1}{2}}y)$$

has the distribution of $\sum_{r=1}^{T} \alpha_r z_r z_r^{*\prime}$, *where* $\alpha_1, \alpha_2, \cdots, \alpha_T$ *are the (real) roots of* Q, *and for each* r, z_r *is a* p*-dimensional complex normal random column vector with complex covariance*

$$\mathscr{E} z_r z_r^{*\prime} = 2\pi K_T\left(\frac{2\pi r}{T}\right), \tag{4.6}$$

with $z_r \equiv z_{T-r}^*$ *and* z_r, z_s *independent for* $s \neq r$ *or* $T - r$.

(v) *If* $Q_1, Q_2, \cdots, Q_M$ *are all* $T \times T$ *circulant Hermitian matrices, then*

$$\begin{aligned} &\sum_{l=1}^{M} \mathscr{E} \operatorname{tr}[\Theta(y)Q_l\Theta'(y) - \Theta(\Sigma_K^{\frac{1}{2}}\Sigma^{-\frac{1}{2}}y)Q_l\Theta'(\Sigma_K^{\frac{1}{2}}\Sigma^{-\frac{1}{2}}y)]^2 \\ &\quad \leqq 2(p+1)\frac{\Lambda}{\lambda}[\max \text{root}\,(\textstyle\sum_{l=1}^{M} Q_l^2)](\textstyle\sum_{u=1-T}^{T-1} |u|^{\frac{1}{2}}(\operatorname{tr} R(u)R(-u))^{\frac{1}{2}})^2\,. \end{aligned} \tag{4.7}$$

PROOF. Part (i) follows in a straightforward manner once one notices that

$$\begin{aligned} \sum_{r=1}^{T} e^{-i2\pi ru/T} WH_{rr}W^{*\prime} &= S(u) + S'(T - |u|)\,, \qquad & u \geqq 0 \\ &= S(u) + S(T - |u|)\,, & u < 0\,. \end{aligned} \tag{4.8}$$

Part (ii) follows from (3.5) and part (i) by using the facts that $\operatorname{tr}(A \times B) = (\operatorname{tr} A)(\operatorname{tr} B)$ and $\operatorname{tr} S(u)S(v) = 0$ unless $u = -v$. To prove part (iii), first note that if Q is circulant Hermitian, then there exist complex numbers $q_{1-T}, q_{2-T}, \cdots, q_{T-1}$, such that

$$Q = \sum_{u=1-T}^{T-1} q_u S(u)\,, \tag{4.9}$$

where for $u > 0$, $q_u = q_{-u}^* = q_{T-u}$. From (4.9), $\operatorname{tr} S(v)Q = (T - |v|)q_{-v}$. Thus since $R(u) = R'(-u)$, and from (3.9),

$$\begin{aligned} &\operatorname{tr}(\Sigma - \Sigma_K)(E_{jk} \times Q) \\ &\quad = T^{-1} \sum_{u=1-T}^{T-1} |u|(T - |u|)q_{-u} \operatorname{tr}\{R(u) - R(\delta_u(T - |u|))\}E_{jk} \\ &\quad = 0\,. \end{aligned}$$

This proves part (iii). Part (iv) is a direct consequence of the construction (4.5) of Σ_K, and the fact (Wahba (1968), Lemma 1) that Q is $T \times T$ circulant Hermitian if and only if there exist real numbers $\alpha_1, \alpha_2, \cdots, \alpha_T$, such that

$$Q = W(\sum_{r=1}^{T} \alpha_r H_{rr})W^{*\prime}\,. \tag{4.10}$$

The method of proof of part (iv) now parallels the proof of Wahba's (1968) Lemma 3. Note that (4.10) implies that $\alpha_1, \alpha_2, \cdots, \alpha_T$ are the latent roots of Q.

We finally have left the proof of (v). This result is a direct consequence of parts (ii) and (iii), Lemma 3.1, the easily verified facts that

$$\text{(4.11)} \qquad \begin{aligned} &\max \text{root}\,(\Sigma + \Sigma_K) \leqq \max \text{root}\,\Sigma + \max \text{root}\,\Sigma_K \leqq 2\Lambda \\ &\min \text{root}\,(\Sigma + \Sigma_K) \geqq \min \text{root}\,\Sigma + \min \text{root}\,\Sigma_K \geqq 2\lambda\,, \end{aligned}$$

and the fact that

$$\begin{aligned} &(2T)^{-1} \textstyle\sum_{u=1-T}^{T-1} \sum_{j,k=1}^{p} |u|\,(T - |u|)\{r_{jk}(u) - r_{jk}(\delta_u(T - |u|))\}^2 \\ &\qquad \leqq 2 \textstyle\sum_{u=1-T}^{T-1} |u| \operatorname{tr} R(u)R(-u) \leqq 2\{\sum_{u=1-T}^{T-1} |u|^{\frac{1}{2}} (\operatorname{tr} R(u)R(-u))^{\frac{1}{2}}\}^2\,. \end{aligned}$$

This completes the proof of the lemma. □

Let us now introduce the particular circulant Hermitian forms $Q(\omega_l)$, $l = 1, 2, \cdots, M$, implicitly defined in (3.6). Choose $M, n > 0$ so that $2M(2n + 1) \leqq T$. Let $j_1, j_2, \cdots, j_M$ be integers such that

$$\text{(4.12)} \qquad 0 < j_1 - n < j_1 + n < j_2 - n < j_2 + n < \cdots < j_M - n < j_M + n \leqq T/2\,.$$

Let $\omega_l = 2\pi j_l/T$. Then

$$\text{(4.13)} \qquad Q(\omega_l) = (2n + 1)^{-1} W(\textstyle\sum_{r=j_l-n}^{j_l+n} H_{rr}) W^{*\prime}\,,$$

$l = 1, 2, \cdots, M$. Since $Q(\omega_l)$ is of the form (4.10), $Q(\omega_l)$ is circulant Hermitian, $l = 1, 2, \cdots, M$. Wahba ((1968) page 1854) verifies that with the $Q(\omega_l)$'s defined by (4.13), the $p \times p$ matrices $\hat{F}_X(\omega_l)$ defined by (3.6) are indeed the averages of $(2n + 1)$ periodogram ordinates $I(\omega_l + 2\pi T^{-1}(j - n - 1))$, $j = 1, 2, \cdots, 2n + 1$, centered at $\omega_l = 2\pi j_l/T$, $l = 1, 2, \cdots, M$.

COROLLARY 4.1. *Let Σ_K be given by* (4.5) *and let $\hat{F}_{X,\Sigma_K}(\omega_l)$ be given by* (3.7) *with $A = \Sigma_K$. Then*

(i) *the random matrices $\hat{F}_{X,\Sigma_K}(\omega_l)$ are mutually stochastically independent, $l = 1, 2, \cdots, M$,*

(ii) *$\hat{F}_{X,\Sigma_K}(\omega_l)$ has the distribution of $(2n + 1)^{-1} \sum_{r=j_l-n}^{j_l+n} z_r z_r^{*\prime}$, where the complex random vectors z_r are defined in Lemma* 4.1, $l = 1, 2, \cdots, M$,

$$\begin{aligned} \text{(iii)} \quad &\textstyle\sum_{l=1}^{M} \mathscr{E} \operatorname{tr} (\hat{F}_X(\omega_l) - \hat{F}_{X,\Sigma_K}(\omega_l))^2 \\ &\qquad \leqq 2(p + 1) \frac{\Lambda}{\lambda} (2n + 1)^{-2} (\textstyle\sum_{u=1-T}^{T-1} |u|^{\frac{1}{2}} (\operatorname{tr} R(u)R(-u))^{\frac{1}{2}})^2\,, \end{aligned}$$

so that if (1.4) *holds, $\sum_{l=1}^{M} \mathscr{E} \operatorname{tr} (\hat{F}_X(\omega_l) - \hat{F}_{X,\Sigma_K}(\omega_l))^2$ is $O((2n + 1)^{-2})$ as $n \to \infty$.*

PROOF. Part (i) follows since $Q(\omega_r)Q(\omega_s) = 0$, all $r \neq s$. Parts (ii) and (iii) are immediate corollaries of Lemma 4.1. □

4.2 *A second block circulant approximation to Σ.* $\hat{F}_{X,\Sigma_K}(\cdot)$ is not a satisfactory approximation to $\hat{F}_X(\cdot)$ since the distribution in Corollary 4.1 (ii) is not a pleasant one with which to work. The reason for introducing $F_{X,\Sigma_K}(\cdot)$ is analytical. It proves to be an intermediary between $\hat{F}_X(\cdot)$ and $\hat{F}_{X,\Sigma_L}(\cdot)$, which we now define, whose distribution is straightforward. Let $M, n > 0$ and let

$j_1, j_2, \cdots, j_M$ be chosen as in (4.12). Further, let

$$
\begin{aligned}
L\left(\frac{2\pi r}{T}\right) &= (2n+1)^{-1} \textstyle\sum_{s=j_l-n}^{j_l+n} K_T\left(\frac{2\pi s}{T}\right), \qquad \text{if } |r - j_l| \leqq n, \\
&= K_T\left(\frac{2\pi r}{T}\right), \qquad \text{if } 0 < r \leqq T/2 \text{ and } j_l + n < r < j_{l+1} - n \text{ for some } l = 1, 2, \cdots, M, \\
&= L\left(\frac{2\pi(T-r)}{T}\right), \qquad \text{if } r > T/2,
\end{aligned} \tag{4.14}
$$

and let

$$D_L = \textstyle\sum_{r=1}^{T} L\left(\frac{2\pi r}{T}\right) \times H_{rr}, \tag{4.15}$$

$$\Sigma_L = 2\pi \Gamma D_L \Gamma^{*\prime}. \tag{4.16}$$

Using the characterization (4.10) of circulant matrices, it is straightforward to show that Σ_L is a block circulant matrix.

LEMMA 4.2. *Let Σ_L be defined by* (4.16) *and let $\hat{F}_{X,\Sigma_L}(\omega_l)$, $l = 1, 2, \cdots, M$, be defined by* (3.7). *Then*

(i) *the random matrices $\hat{F}_{X,\Sigma_L}(\omega_l)$ are mutually stochastically independent, $l = 1, 2, \cdots, M$,*

(ii) *$\hat{F}_{X,\Sigma_L}(\omega_l)$ has a complex Wishart distribution with $(2n+1)$ degrees of freedom and expected value $2\pi \underline{K}(\omega_l)$ where*

$$\underline{K}(\omega_l) \equiv (2n+1)^{-1} \textstyle\sum_{r=j_l-n}^{j_l+n} K_T\left(\frac{2\pi r}{T}\right), \tag{4.17}$$

(4.18) (iii) $\sum_{l=1}^{M} \mathscr{E}\, \mathrm{tr}\,(\hat{F}_X(\omega_l) - \hat{F}_{X,\Sigma_L}(\omega_l))^2$

$$\leqq 2(p+1)\frac{\Lambda}{\lambda}(2n+1)^{-2}(2+8\pi^2 n)\left[\textstyle\sum_{u=1-T}^{T-1} |u|^{\frac{1}{2}}\, (\mathrm{tr}\, R(u)R(-u))^{\frac{1}{2}}\right]^2$$

so that if Condition (1.4) *holds, then as $n \to \infty$,*

$$\textstyle\sum_{l=1}^{M} \mathscr{E}\, \mathrm{tr}\,(\hat{F}_X(\omega_l) - \hat{F}_{X,\Sigma_L}(\omega_l))^2 = O(1/(2n+1)). \tag{4.19}$$

PROOF. Parts (i) and (ii) follow from the definitions of Σ_L and $Q(\omega_l)$, $l = 1, 2, \cdots, M$, using a proof parallel to that of Lemma 3 of Wahba (1968). The independence follows since $Q(\omega_r)Q(\omega_s) = 0$, $r \neq s$. It remains to prove part (iii). Now

$$
\begin{aligned}
(2n+1)\, &\mathrm{tr}\,(\Sigma_K - \Sigma_L)(E_{jk} \times Q(\omega_l)) \\
&= 2\pi\, \mathrm{tr}\,(D_K - D_L)(E_{jk} \times \textstyle\sum_{r=j_l-n}^{j_l+n} H_{rr}) = 0,
\end{aligned}
$$

which together with Lemma 4.1 (iii), lets us conclude that

$$\mathrm{tr}\,(\Sigma - \Sigma_L)(E_{jk} \times Q(\omega_l)) = 0, \tag{4.20}$$

$j, k = 1, 2, \cdots, p$; $l = 1, 2, \cdots, M$. Also

$$
\begin{aligned}
\mathrm{tr}\,(\Sigma_K - \Sigma_L)^2 &= (2\pi)^2\, \mathrm{tr}\,(D_K - D_L)^2 \\
&= 8\pi^2 \textstyle\sum_{l=1}^{M} \sum_{r=j_l-n}^{j_l+n} \mathrm{tr}\left\{K_T\left(\frac{2\pi r}{T}\right) - L\left(\frac{2\pi r}{T}\right)\right\}^2 \\
&= \frac{4\pi^2}{2n+1} \textstyle\sum_{l=1}^{M} \sum_{r,s=j_l-n}^{j_l+n} \mathrm{tr}\left\{K_T\left(\frac{2\pi r}{T}\right) - K_T\left(\frac{2\pi s}{T}\right)\right\}^2.
\end{aligned} \tag{4.21}
$$

Let $J_l = \{r : j_l - n \leqq r \leqq j_l + n\}$. Then, from (4.21)

$$\begin{aligned}
&\operatorname{tr}(\Sigma_K - \Sigma_L)^2 \\
&\leqq 8\pi^2 n \sum_{l=1}^{M} \max_{r,s\in J_l} \operatorname{tr}\left\{K_T\left(\frac{2\pi r}{T}\right) - K_T\left(\frac{2\pi s}{T}\right)\right\}^2 \\
&= 2n \sum_{l=1}^{M} \max_{r,s\in J_l} \operatorname{tr}\{\textstyle\sum_{u=1-T}^{T-1} T^{-1}(T - |u|)R(u)(e^{-i2\pi ru/T} - e^{-i2\pi su/T})\}^2 \\
&\leqq 8\pi^2 n \sum_{l=1}^{M} \max_{r,s\in J_l} \left\{\textstyle\sum_{u=1-T}^{T-1} (\operatorname{tr} R(u)R(-u))^{\frac{1}{2}} \min\left(\frac{|u|\,|r-s|}{T}, 1\right)\right\}^2 \qquad (4.22)\\
&\leqq 8\pi^2 nM\left\{\textstyle\sum_{u=1}^{[T/2n]} \frac{2n}{T}|u|(\operatorname{tr} R(u)R(-u))^{\frac{1}{2}} + \sum_{u=[T/2n]+1}^{T-1} (\operatorname{tr} R(u)R(-u))^{\frac{1}{2}}\right\}^2 \\
&\leqq \frac{16\pi^2 n^2 M}{T}\{\textstyle\sum_{u=1-T}^{T-1} |u|^{\frac{1}{2}} (\operatorname{tr} R(u)R(-u))^{\frac{1}{2}}\}^2 \\
&< 8\pi^2 n\{\textstyle\sum_{u=1-T}^{T-1} |u|^{\frac{1}{2}} (\operatorname{tr} R(u)R(-u))^{\frac{1}{2}}\}^2 .
\end{aligned}$$

It is not hard to show that

$$\max \operatorname{root}(\Sigma_K + \Sigma_L) \leqq 2\Lambda , \qquad \min \operatorname{root}(\Sigma_K + \Sigma_L) \geqq 2\lambda , \tag{4.23}$$

and since (Lemma A.1, Wahba (1968))

$$\begin{aligned}
&\mathscr{E} \operatorname{tr}(\hat{F}_X(\omega_l) - \hat{F}_{X,\Sigma_L}(\omega_l))^2 \\
&\qquad \leqq 2[\mathscr{E} \operatorname{tr}(\hat{F}_X(\omega_l) - \hat{F}_{X,\Sigma_K}(\omega_l))^2 + \mathscr{E} \operatorname{tr}(\hat{F}_{X,\Sigma_K}(\omega_l) - \hat{F}_{X,\Sigma_L}(\omega_l))^2] ,
\end{aligned} \tag{4.24}$$

part (iii) now follows from (4.20), Lemma 3.1, (4.22), and (4.24). The result (4.19) is a direct consequence of part (iii). □

With the proof of Lemma 4.2, we have actually accomplished the proof of the assertion made in Section 1. However, although the random matrices $\hat{F}_{X,\Sigma_L}(\omega_l)$ do have complex Wishart distributions and are mutually stochastically independent, they are not the random matrices constructed by Wahba (1968) to approximate $\hat{F}_X(\omega_l)$, $l = 1, 2, \cdots, M$. Therefore, we now demonstrate that under Condition (1.4), the random matrices constructed by Wahba (1968) also converge in expected mean square to the estimators $\hat{F}_X(\omega_l)$, $l = 1, 2, \cdots, M$.

4.3 *A third block circulant approximation to* Σ : *Proof of the main theorem.* Let

$$\begin{aligned}
G\left(\frac{2\pi r}{T}\right) &= (2n+1)^{-1} \textstyle\sum_{s=j_l-n}^{j_l+n} F\left(\frac{2\pi s}{T}\right), \qquad \text{if } |r - j_l| \leqq n , \\
&= F\left(\frac{2\pi r}{T}\right), \qquad \text{if } 0 < r \leqq T/2 \text{ and } j_l + n < r < j_{l+1} - n \text{ for some } l = 1, 2, \cdots, M, \\
&= G\left(\frac{2\pi(T-r)}{T}\right), \qquad \text{if } r > T/2 ,
\end{aligned} \tag{4.25}$$

and let

$$D_G = \textstyle\sum_{r=1}^{T} G\left(\frac{2\pi r}{T}\right) \times H_{rr} , \tag{4.26}$$

$$\Sigma_G = 2\pi\Gamma D_G \Gamma^{*\prime} . \tag{4.27}$$

Again, Σ_G is a block circulant matrix.

MAIN THEOREM. *Let Σ_G be defined by* (4.27) *and let* $\hat{F}_{X,\Sigma_G}(\omega_l)$, $l = 1, 2, \cdots, M$, *be defined by* (3.7). *We note that* $\hat{F}_{X,\Sigma_G}(\omega_l)$ *is Wahba's* (1968) $\hat{F}_{\bar{\bar{X}}}(\omega_l)$. *Then*

(i) *the random matrices* $\hat{F}_{X,\Sigma_G}(\omega_l)$ *are mutually stochastically independent*, $l = 1, 2, \cdots, M$;

(ii) $\hat{F}_{X,\Sigma_G}(\omega_l)$ *has a complex Wishart distribution with* $(2n + 1)$ *degrees of freedom and expected value* $2\pi \underline{F}(\omega_l)$ *where*

$$\underline{F}(\omega_l) \equiv (2n + 1)^{-1} \textstyle\sum_{r=j_l-n}^{j_l+n} F\left(\frac{2\pi r}{T}\right); \tag{4.28}$$

(iii) *under Condition* (1.4), *as* $n \to \infty$ (*regardless of whether* M *stays fixed, or* $M \to \infty$),

$$\textstyle\sum_{l=1}^{M} \mathscr{E} \operatorname{tr} (\hat{F}_X(\omega_l) - \hat{F}_{X,\Sigma_G}(\omega_l))^2 = O(1/(2n + 1)) . \tag{4.29}$$

PROOF. Parts (i) and (ii) have been proven by Wahba (1968). We turn now to the proof of part (iii). From Lemma A.1 of Wahba (1968) and Lemma 4.2,

$$\begin{aligned}\textstyle\sum_{l=1}^{M} \mathscr{E} \operatorname{tr} [\hat{F}_X(\omega_l) - \hat{F}_{X,\Sigma_G}(\omega_l)]^2 &\leqq 2\{\textstyle\sum_{l=1}^{M} \mathscr{E} \operatorname{tr} [\hat{F}_X(\omega_l) - \hat{F}_{X,\Sigma_L}(\omega_l)]^2 \\ &\quad + \textstyle\sum_{l=1}^{M} \mathscr{E} \operatorname{tr} [\hat{F}_{X,\Sigma_L}(\omega_l) - \hat{F}_{X,\Sigma_G}(\omega_l)]^2\} ,\end{aligned} \tag{4.30}$$

and since we have a bound for the first term from Lemma 4.2 (iii), we need concentrate only on the quantity $\sum_{l=1}^{M} \mathscr{E} \operatorname{tr} [\hat{F}_{X,\Sigma_L}(\omega_l) - \hat{F}_{X,\Sigma_G}(\omega_l)]^2$. It is straightforward to prove that the latent roots of Σ_G are bounded, above by Λ and below by λ. Thus, from Lemma 3.1,

$$\begin{aligned}\textstyle\sum_{l=1}^{M} E \operatorname{tr} [\hat{F}_{X,\Sigma_L}(\omega_l) - \hat{F}_{X,\Sigma_G}(\omega_l)]^2 &\leqq (p + 1)\frac{\Lambda}{\lambda}(2n + 1)^{-2} \operatorname{tr} (\Sigma_L - \Sigma_G)^2 \\ &\quad + 2 \textstyle\sum_{l=1}^{M} \sum_{j=1}^{p} \sum_{k=1}^{p} [\operatorname{tr} (\Sigma_L - \Sigma_G)(E_{jk} \times Q(\omega_l))]^2 .\end{aligned} \tag{4.31}$$

Now

$$\begin{aligned}\operatorname{tr} (\Sigma_L - \Sigma_G)^2 &= 2 \textstyle\sum_{l=1}^{M} \sum_{r \in J_l} (2\pi)^2 \operatorname{tr}\left(L\left(\frac{2\pi r}{T}\right) - G\left(\frac{2\pi r}{T}\right)\right)^2 \\ &\leqq 2 \textstyle\sum_{l=1}^{M} (2n + 1)^{-1} \sum_{r \in J_l} (2\pi)^2 \operatorname{tr}\left(K_T\left(\frac{2\pi r}{T}\right) - F\left(\frac{2\pi r}{T}\right)\right)^2 \\ &\leqq 2 \textstyle\sum_{l=1}^{M} \max_{r \in J_{l'}} \operatorname{tr} [\sum_{u=1-T}^{T-1} T^{-1} |u| R(u) e^{-i2\pi r u/T} \\ &\quad + \textstyle\sum_{|u| \geqq T} R(u) e^{-i2\pi r u/T}]^2 \\ &\leqq 2MT^{-1}C^2 .\end{aligned} \tag{4.32}$$

Also,

$$\begin{aligned}&[\operatorname{tr} (\Sigma_L - \Sigma_G)(E_{jk} \times Q(\omega_l))]^2 \\ &\quad = 2(2\pi)^2(2n + 1)^{-2}\left(\textstyle\sum_{r \in J_l} \operatorname{tr}\left(K_T\left(\frac{2\pi r}{T}\right) - F\left(\frac{2\pi r}{T}\right)\right) E_{jk}\right)^2 \\ &\quad \leqq 2[\textstyle\sum_{u=1-T}^{T-1} T^{-1} |u| |\operatorname{tr} R(u) E_{jk}| + \sum_{|u| \geqq T} |\operatorname{tr} R(u) E_{jk}|]^2 \leqq 2T^{-1}C^2 ,\end{aligned} \tag{4.33}$$

and thus

$$(4.34) \qquad \sum_{l=1}^{M} \sum_{j=1}^{p} \sum_{k=1}^{p} [\mathrm{tr}\,(\Sigma_L - \Sigma_G)(E_{jk} \times Q(\omega_l))]^2$$
$$\leqq 2MT^{-1} \sum_{j=1}^{p} \sum_{k=1}^{p} (\sum_{u=-\infty}^{\infty} |u|^{\frac{1}{2}} |\mathrm{tr}\,(R(u)E_{jk}|)^2 \leqq 2MT^{-1}p^2C^2 .$$

Assuming that Condition (1.4) holds, (4.29) is now a direct consequence of (4.30), (4.31), (4.32), and (4.34). This completes the proof of the theorem. □

5. Concluding remarks. Although in this paper, we use a somewhat different series of block circulant approximations to Σ than does Wahba (1968), this difference does not account for the difference in results (i.e., the relaxing of Wahba's conditions). Using Lemma 3.1, we could have used the series of block circulant approximations to Σ used by Wahba (1968) and still have come up with the same conclusions. Our method of successive approximation was chosen to reduce the amount of work we had to do with infinite sums (note that infinite sums only appear in Section 4.3) and to exhibit an interesting set of alternative approximations (the results in Sections 4.1 and 4.2) to $\hat{F}_X(\omega_l)$, $l = 1, 2, \cdots, M$. The key devices that allow us to relax Wahba's conditions are: (i) Lemma 3.1, which provides a useful generalization to Wahba's Lemma 5 and enables us to remove her condition that $\log_2 M \leqq n$, and (ii) elimination of consideration of the metric $\varphi(A - B)$, introduced by Wahba on page 1852 of her (1968) paper, in favor of a more direct analysis of the quantities involved (this enables us to replace Condition (1.3) by Condition (1.4)). It should be noted that the methods of analysis of the present paper can be used to show that Condition (1.3) implies

$$(5.1) \qquad \sum_{l=1}^{M} \mathscr{E}\,\mathrm{tr}\,(\hat{F}_X(\omega_l) - \hat{F}_{X,\Sigma_G}(\omega_l))^2$$
$$= O\left(\frac{1}{\min\{T, (2n+1)^2\}}\right) = O\left(\frac{1}{2n+1}\right),$$

as $n, M \to \infty$.

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