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PROJECTED TESTS FOR ORDER RESTRICTED ALTERNATIVES

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Consider the model where X_{ij}, $i = 1, 2, \dots, k$, $j = 1, 2, \dots, n$, are independent random variables distributed according to a one-parameter exponential family, with natural parameter θ_i. We test H_0: $\theta_1 = \cdots = \theta_k$ versus H_1: $\boldsymbol{\theta} \in \mathcal{C} - \{\boldsymbol{\theta}: \boldsymbol{\theta} \in H_0\}$, where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)'$ and $\mathcal{C}$ is a cone determined by $A\boldsymbol{\theta} \geq 0$, where the rows of A are contrasts with two nonzero elements. We offer a method of generating "good" tests for H_0 versus H_1. The method is to take a "good" test for H_0 versus H_2: not H_0, and apply the test to projected sample points, where the projection is onto $\mathcal{C}$. "Good" tests for H_0 versus H_2 are tests that are Schur convex. "Good" tests for H_0 versus H_1 are tests which are monotone with respect to a cone order. We demonstrate that if the test function for H_0 versus H_2 is a constant size Schur convex test, then the resulting projected test is monotone.

1. Introduction and summary. Consider the model where X_{ij}, $i = 1, 2, \dots, k$, $j = 1, 2, \dots, n$, are independent random variables distributed according to a one-parameter exponential family, with natural parameter θ_i; that is, the density of X_{ij} is

$$f_{X_{ij}}\left(x_{ij} \mid \theta_i\right) = \exp\left[-M(\theta_i) + x_{ij}\theta_i\right] h(x_{ij}). \tag{1.1}$$

The dominating measure for each X_{ij} is Lebesgue measure on $(-\infty, \infty)$ for the continuous case, and counting measure for the case where the X_{ij} are integer valued. Let $S_i = \sum_{j=1}^{n} X_{ij}$ and $\mathbf{S}' = (S_1, \dots, S_k)$. Clearly $\mathbf{S}$ is a sufficient statistic for the model and each S_i is exponential family with natural parameter θ_i. The joint density of $\mathbf{S}$ is

$$f_{\mathbf{S}}\left(\mathbf{s} \mid \boldsymbol{\theta}\right) = \exp\left[-M^*(\boldsymbol{\theta}) + \mathbf{s}'\boldsymbol{\theta}\right] h^*(\mathbf{s}), \tag{1.2}$$

where $\boldsymbol{\theta}' = (\theta_1, \theta_2, \dots, \theta_k)$.

We consider the null hypothesis H_0: $\theta_1 = \theta_2 = \cdots = \theta_k$ and alternatives H_1: $\boldsymbol{\theta} \in \mathcal{C} - \{\boldsymbol{\theta}: \boldsymbol{\theta} \in H_0\}$, where $\mathcal{C} = \{\boldsymbol{\theta}: A\boldsymbol{\theta} \geq 0\}$ and A is a $r \times k$ matrix whose rows are contrasts. The set $\mathcal{C}$ is a polyhedral cone. Specific matrices A describe the simple order alternative $\{\theta_1 \geq \theta_2 \geq \cdots \geq \theta_k\}$, the simple tree order $\{\theta_i \geq \theta_k, i = 1, 2, \dots, k-1\}$ and umbrella alternatives, for example, $\{\theta_1 \leq \cdots \leq \theta_j \geq \theta_{j+1} \geq \cdots \geq \theta_k\}$. We will also refer to another alternative, H_2: not H_0.

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We offer a method of deriving "good" tests for H_0 versus H_1 when the rows of A are pairwise contrasts such that no row is a nonnegative linear combination of the $(r-1)$ remaining rows. This includes the cases of simple order, tree order and umbrella order. Our method of deriving "good" tests entails consideration of a family of tests for H_0 versus H_2. Tests of constant size α which are Schur convex represent a complete class of tests among the class of constant size α permutation invariant tests. That is, any constant size α permutation invariant test which is not Schur convex is inadmissible. Furthermore, if $h(x_{ij})$ in (1.1) is log-concave, then the Schur convex tests are unbiased for testing H_0 versus H_2 [see Cohen and Sackrowitz (1987)]. If $\varphi(\mathbf{s})$ is a constant size Schur convex test, we suggest

$$\varphi^*(\mathbf{s}) = \varphi(P\mathbf{s} \mid \mathcal{C}) \tag{1.3}$$

as a test for H_0 versus H_1, where $P\mathbf{s} \mid \mathcal{C}$ is the projection of $\mathbf{s}$ onto the cone $\mathcal{C}$. We demonstrate that if $\mathcal{C}$ is such that the rows of A have exactly two nonzero elements, then the projected test is monotone with respect to the cone order induced by $\mathcal{C}^*$, the dual of $\mathcal{C}$.

In many cases the projected test is unbiased and admissible for testing H_0 versus H_1. The idea of using a projected test first appears in Cohen, Perlman and Sackrowitz (1991) for the simple order case. The problem of testing H_0 versus H_1 has received considerable attention in the literature. The book by Robertson, Wright and Dykstra (hereafter, RWD) (1988) discusses this problem extensively and offers many references. Most often the likelihood ratio test is studied although other test procedures are studied, for example, in Conaway, Pillers, Robertson and Sconing (1991) and Mukerjee, Robertson and Wright (1987). Lee (1987) studies the multinomial distribution when considering H_0 versus H_1. He also proposes test procedures that relate to statistics that are relevant for testing H_0 versus H_2.

In addition to having the desirable monotonicity property, the projected tests recommended here are often easy to compute and perform very well in simulation studies.

In deriving the result for projected tests we develop a lemma and a converse to the lemma that has independent interest. Roughly the lemma asserts that if $\mathcal{C}$ is such that the rows of A are all needed (i.e., no row is a positive linear combination of other rows) and each row has exactly two nonzero elements, then points ordered according to the cone $\mathcal{C}^*$, say, $\mathbf{u} \le \mathbf{v}$, project into $\mathcal{C}$ in such a way that $P\mathbf{u} \mid \mathcal{C}$ is majorized by $P\mathbf{v} \mid \mathcal{C}$. (Definitions will be given in Section 2.) We also prove the converse, namely, that if projections are majorized as above, then $\mathcal{C}$ is such that each of its rows has exactly two nonzero elements. This converse then enhances our results since it demonstrates that projected tests can be shown by these methods to have the desirable properties *only* under the assumption we make about $\mathcal{C}$.

In the next section we give preliminaries. In Section 3 we discuss projected tests.

2. Preliminaries. In this section we give some definitions and some known results concerned with polyhedral cones.

Let $\mathbf{a}_i'$, $i = 1, 2, \ldots, r$, denote the rows of A. Then the cone $\mathcal{C} = \{\boldsymbol{\theta} \in \mathbb{R}^k : \mathbf{a}_i'\boldsymbol{\theta} \geq 0,\ i = 1, \ldots, r\}$. Next let Γ denote the space spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_r$ and define

$$\Gamma_+ = \left\{ \boldsymbol{\theta} \in \mathbb{R}^k : \boldsymbol{\theta} = \sum_{i=1}^{r} \lambda_i \mathbf{a}_i \text{ where } \lambda_i \geq 0,\ i = 1, 2, \ldots, k-1 \right\}.$$

The dual of $\mathcal{C}$ is defined to be $\mathcal{C}^* = \{\boldsymbol{\nu} \in \mathbb{R}^k : \boldsymbol{\nu}'\boldsymbol{\theta} \geq 0 \text{ for all } \boldsymbol{\theta} \in \mathcal{C}\}$. It is known that $\mathcal{C}^* = \Gamma_+$ [see Van Tiel (1984)]. Described in other words, $\mathbf{a}_i$ are the generators of the dual of $\mathcal{C}$.

A cone $\mathcal{K}$ induces a cone ordering $(\leq)$ on a set $\mathcal{A} \subseteq \mathbb{R}^k$ as follows:
For $\mathbf{u} \in \mathcal{A}$ and $\mathbf{v} \in \mathcal{A}$,

$$\mathbf{u} \leq \mathbf{v}[\mathcal{K}] \quad \text{iff } \mathbf{v} - \mathbf{u} \in \mathcal{K}. \tag{2.1}$$

In other words, in the case of a polyhedral cone, $\mathbf{u} \leq \mathbf{v}[\mathcal{K}]$ if $\mathbf{v} - \mathbf{u} = \sum_{i=1}^{r} b_i \mathbf{z}_i$, where $b_i \geq 0$ and $\mathbf{z}_i$ are the generators of $\mathcal{K}$.

From here on, unless stated otherwise, $\mathcal{A} = \mathbb{R}^k$.

DEFINITION 2.1. A function f on $\mathbb{R}^k$ is said to be monotone with respect to the cone ordering $(\leq)$ if, whenever $\mathbf{u} \leq \mathbf{v}[\mathcal{K}]$,

$$f(\mathbf{u}) \leq f(\mathbf{v}). \tag{2.2}$$

DEFINITION 2.2. In $\mathbb{R}^k$, $\mathbf{u} \prec \mathbf{v}$ ($\mathbf{v}$ majorizes $\mathbf{u}$) if $\sum_{i=1}^{j} u_{(i)} \leq \sum_{i=1}^{j} v_{(i)}$, $j = 1, 2, \ldots, k-1$, and $\sum_{i=1}^{k} u_{(i)} = \sum_{i=1}^{k} v_{(i)}$, where $u_{(1)} \geq u_{(2)} \geq \cdots \geq u_{(k)}$ are the ordered components of $\mathbf{u}$ [see Marshall and Olkin (1979)].

DEFINITION 2.3. A function f is said to be Schur convex if, whenever $\mathbf{u} \prec \mathbf{v}$,

$$f(\mathbf{v}) \geq f(\mathbf{u}). \tag{2.3}$$

DEFINITION 2.4. Let B be a closed convex set in $\mathbb{R}^k$. The unique closest point of B to a point $\mathbf{u} \in \mathbb{R}^k$ is said to be the projection of $\mathbf{u}$ onto B and is denoted by $P\mathbf{u} \mid B$.

3. Projected tests. To test H_0 versus H_1 we are recommending the projected tests defined in (1.3). We prove that such tests are monotone with respect to the cone ordering induced by $\mathcal{C}^*$. To start we have the following definition.

DEFINITION 3.1. For $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)'$ and for any permutation of the coordinates we define a simple order cone as $\{\boldsymbol{\theta} : \theta_{i_1} \leq \theta_{i_2} \leq \cdots \leq \theta_{i_k}\}$. Note that there are $k!$ possible simple order cones.

It may be helpful to the reader to realize that when we write the rows of A are contrasts with two nonzero elements, then without loss of generality this means that each row has one element $+1$, one element -1 and all other elements are 0.

Next we prove the following lemma.

LEMMA 3.2. *Assume that the rows of A have exactly two nonzero elements. Assume $\mathcal{C}$ is nondegenerate. Then $\mathcal{C}$ is a closed convex cone which is the union of simple order cones.*

PROOF. The conditions on the rows of A imply that $\mathcal{C} = \cap_{i=1}^{r} H_i$, where $H_i = \{\boldsymbol{\theta}: \mathbf{a}_i'\boldsymbol{\theta} \geq 0\}$. Since each H_i is convex, $\mathcal{C}$ is convex. Furthermore, since $\mathbf{a}_i$ is a pairwise contrast each half-space H_i is a union of simple order cones which do not overlap, except on their boundaries. Then $\mathcal{C}$ is a union of those simple cones that lie in all of the half-spaces. □

Since $\mathcal{C}$ is a convex union of simple cones, express $\mathcal{C} = \cup_{i=1}^{m} \mathcal{A}_i$, where $\mathcal{A}_i$ is some simple order cone. Note that $\mathcal{C}^* = \cap_{i=1}^{m} \mathcal{A}_i^*$. We need a series of lemmas in order to prove the main result of this section. Toward this end let $\mathcal{W}$ be a polyhedral cone with generators $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_\rho$. For every set of integers $1 \leq i_1 < i_2 < \cdots < i_q \leq \rho$, where $q \geq 1$, define

$$\mathcal{W}(i_1, \ldots, i_q) = \left\{ \mathbf{w} \in \mathcal{W}: \mathbf{w} = \sum_{j=1}^{q} c_{i_j} \mathbf{w}_{i_j}, c_{i_j} > 0,\ j = 1, 2, \ldots, q \right\}. \tag{3.1}$$

Also let $\mathcal{W}(\phi) = \{\mathbf{0}\}$. Then $\mathcal{W} = \{0\} \cup \mathcal{W}(1) \cup \cdots \cup \mathcal{W}(1, 2, \ldots, \rho)$. The notation $\langle \cdot, \cdot \rangle$ denotes inner product.

LEMMA 3.3. *Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be such that their projections $\mathbf{v}_j^* = P\mathbf{v}_j \mid \mathcal{W} \in \mathcal{W}(i_1, \ldots, i_q)$ for $j = 1, 2$ and some set $i_1, \ldots, i_q$. Then, for all $\mathbf{w} \in \mathcal{W}$,*

$$\langle \mathbf{v}_i - \mathbf{v}_i^*, \mathbf{v}_j^* - \mathbf{w} \rangle \geq 0, \qquad i = 1, 2;\ j = 1, 2. \tag{3.2}$$

PROOF. First let $i = 1$. By RWD [(1988), Theorem 8.2.2, page 375],

$$\langle \mathbf{v}_1 - \mathbf{v}_1^*, \mathbf{v}_1^* - \mathbf{w} \rangle \geq 0 \quad \text{for all } \mathbf{w} \in \mathcal{W}. \tag{3.3}$$

Define $\mathbf{w}(\pm\varepsilon) = \mathbf{v}_1^* \pm \varepsilon(\mathbf{v}_2^* - \mathbf{v}_1^*)$. Since $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$ are in $\mathcal{W}(i_1, \ldots, i_q)$ and the c_{i_j}'s in (3.1) are strictly positive, there exists $\varepsilon > 0$ sufficiently small that $\mathbf{w}(\pm\varepsilon) \in \mathcal{W}(i_1, \ldots, i_q) \subseteq \mathcal{W}$. Thus from (3.3) it follows that

$$0 \leq \langle \mathbf{v}_1 - \mathbf{v}_1^*, \mathbf{v}_1^* - \mathbf{w}(\pm\varepsilon) \rangle = \left\langle \mathbf{v}_1 - \mathbf{v}_1^*, \mp\varepsilon\left(\mathbf{v}_2^* - \mathbf{v}_1^*\right) \right\rangle,$$

which implies

$$\langle \mathbf{v}_1 - \mathbf{v}_1^*, \mathbf{v}_2^* - \mathbf{v}_1^* \rangle = 0. \tag{3.4}$$

Next note that, by (3.4) and (3.3),

$$\langle \mathbf{v}_1 - \mathbf{v}_1^*, \mathbf{v}_2^* - \mathbf{w} \rangle = \langle \mathbf{v}_1 - \mathbf{v}_1^*, \mathbf{v}_2^* - \mathbf{v}_1^* + \mathbf{v}_1^* - \mathbf{w} \rangle \geq 0.$$

This proves the lemma for $i = 1$. For $i = 2$, the proof is the same. □

LEMMA 3.4. *If* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are such that, for* $j = 1, 2$, $\mathbf{v}_j^* = P\mathbf{v}_j \mid \mathcal{W} \in \mathcal{W}(i_1, \dots, i_q)$ *for some set* $(i_1, \dots, i_q)$, *then* $\gamma \mathbf{v}_1^* + (1-\gamma)\mathbf{v}_2^* = P(\gamma \mathbf{v}_1 + (1-\gamma)\mathbf{v}_2) \mid \mathcal{W}$, $0 \leq \gamma \leq 1$.

PROOF. For any $\mathbf{w} \in \mathcal{W}$, consider

$$\begin{aligned} &\Big\langle \big(\gamma \mathbf{v}_1 + (1-\gamma)\mathbf{v}_2\big) - \big(\gamma \mathbf{v}_1^* + (1-\gamma)\mathbf{v}_2^*\big), \big(\gamma \mathbf{v}_1^* + (1-\gamma)\mathbf{v}_2^*\big) - \mathbf{w} \Big\rangle \\ &\quad = \Big\langle \gamma\big(\mathbf{v}_1 - \mathbf{v}_1^*\big) + (1-\gamma)\big(\mathbf{v}_2 - \mathbf{v}_2^*\big), \gamma\big(\mathbf{v}_1^* - \mathbf{w}\big) + (1-\gamma)\big(\mathbf{v}_2^* - \mathbf{w}\big) \Big\rangle \geq 0 \end{aligned}$$

by Lemma 3.3. The result now follows from RWD [(1988), Theorem 8.2.2, page 375]. □

Next we apply Lemma 3.4 to $\mathcal{C} = \cup_{i=1}^m \mathcal{A}_i$.

LEMMA 3.5. *Assume that the rows of A have exactly two nonzero elements. Fix* $\mathbf{s}_1$ *and* $\mathbf{s}_2$. *There exist* $N < \infty$ *and a set* $0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_N \leq 1$ *such that*

$$P\big(\gamma \mathbf{s}_1 + (1-\gamma)\mathbf{s}_2\big) \mid \mathcal{C} = P\big(\gamma \mathbf{s}_1 + (1-\gamma)\mathbf{s}_2\big) \mid \mathcal{A}_{i_j},$$

for all $\gamma_j \leq \gamma \leq \gamma_{j+1}$ *and some* $\mathcal{A}_{i_j}$.

PROOF. Assume the statement is false. Then there must exist a sequence $0 < \beta_1 < \beta_2 < \cdots < 1$ and a simple order cone $\mathcal{A}_{i^*}$ such that

$$P\big(\beta_j \mathbf{s}_1 + (1-\beta_j)\mathbf{s}_2\big) \mid \mathcal{C} \in \mathcal{A}_{i^*} \quad \text{iff } j \text{ is odd.} \tag{3.5}$$

Since A_{i^*} is a polyhedral cone it can be partitioned into subsets $\mathcal{W}(i_1, \dots, i_q)$ in a manner similar to (3.1). Since there are only a finite number of such subsets, there must exist at least two odd integers j^* and j^{**} such that $P(\beta_j \mathbf{s}_1 + (1 - \beta_j)\mathbf{s}_2) \mid \mathcal{C}$ is in the same subset for both $j = j^*$ and $j = j^{**}$. By Lemma 3.4,

$$P\big(\beta \mathbf{s}_1 + (1-\beta)\mathbf{s}_2\big) \big| \mathcal{C} = \frac{\beta_{j^{**}} - \beta}{\beta_{j^{**}} - \beta_{j^*}} Ps^* \mid \mathcal{C} + \frac{\beta - \beta_{j^*}}{\beta_{j^{**}} - \beta_{j^*}} Ps^{**} \mid \mathcal{C}, \tag{3.6}$$

where $s^* = \beta_{j^*} s_1 + (1 - \beta_{j^*}) s_2$ and $s^{**} = \beta_{j^{**}} s_1 + (1 - \beta_{j^{**}}) s_2$, is also in the same subset for all $\beta_{j^*} \leq \beta \leq \beta_{j^{**}}$. This is a contradiction for the β_j where $j^* < j < j^{**}$ and j is even. □

LEMMA 3.6. *Assume that the rows of A have exactly two nonzero elements. If* $\mathbf{s}'' \geq \mathbf{s}'[\mathcal{C}^*]$, *then* $P\mathbf{s}''|\mathcal{C} \succ P\mathbf{s}'|\mathcal{C}$.

PROOF. Marshall and Olkin [(1979), page 426] note that, on $\mathcal{D} = \{\mathbf{u}: u_1 \geq \cdots \geq u_k\}$, majorization is a cone ordering and the generators of the cone inducing the ordering are $(1,-1,0,\ldots,0),(0,1,-1,0,\ldots,0),\ldots,(0,\ldots,0,1,-1)$. In fact this cone is

$$\mathcal{M} = \left\{\mathbf{x}: \sum_{i=1}^{j} x_i \geq 0,\ j = 1,\ldots,k-1, \sum_{i=1}^{k} x_i = 0\right\}.$$

A Schur convex function is interpreted as a monotone function with respect to majorization. Now note that the cone order induced by a simple cone $\mathcal{A}_j$ is the same as the cone order induced by the majorization cone on $\mathcal{A}_j$.

Next suppose $\mathbf{s}'' \geq \mathbf{s}'[\mathcal{C}^*]$. This implies we may write $\mathbf{s}'' = \mathbf{s}' + \lambda\mathbf{a}$, where $\lambda > 0$ and $\lambda\mathbf{a} \in \mathcal{C}^*$. However, $\mathcal{C}^* = \cap\mathcal{A}_j^*$ and so $\lambda\mathbf{a}$ lies in every $\mathcal{A}_j^*, j = 1,2,\ldots,m$. By the remark above concerning the cone ordering on $\mathcal{A}_j$, $\mathbf{a} \in \mathcal{A}_j^*$ implies $\mathbf{s}'' \succ \mathbf{s}'$. Now if $P\mathbf{s}'' \,|\, \mathcal{C}$ and $P\mathbf{s}' \,|\, \mathcal{C}$ both lie in the same simple cone $\mathcal{A}_j$, the result of the lemma follows from Robertson and Wright [(1982), Corollary 2.3]. (There ISO* on $\mathcal{A}_j$ and majorization are equivalent.) If $P\mathbf{s}'' \,|\, \mathcal{C}$ and $P\mathbf{s}' \,|\, \mathcal{C}$ are not in the same simple cone, we need to use Lemma 3.5, which asserts that there is a line segment connecting $\mathbf{s}'$ to $\mathbf{s}' + \lambda_1\mathbf{a}$, say, such that the projections $P(\mathbf{s} + \lambda\mathbf{a}) \,|\, \mathcal{C}$ are into the same simple cone (say, $\mathcal{A}_1$) for all λ, $0 \leq \lambda \leq \lambda_1$. All such points preserve the majorization order by virtue of the argument above, that is, $P(\mathbf{s}' + \lambda_1\mathbf{a}) \,|\, \mathcal{C} > P(\mathbf{s}' + \lambda\mathbf{a}) \,|\, \mathcal{C}$ for $0 \leq \lambda \leq \lambda_1$. Furthermore, for $\lambda > \lambda_1$, $P(\mathbf{s}' + \lambda\mathbf{a}) \,|\, \mathcal{C}$ passes into another simple cone, say, $\mathcal{A}_2$. By continuity of the projection operator [see RWD (1988), Theorem 8.2.5, page 376], the projection $P(\mathbf{s}' + \lambda_1\mathbf{a}) \,|\, \mathcal{C}$ must lie on the boundary of $\mathcal{A}_1 \cap \mathcal{A}_2$. Next take the segment of the line $\mathbf{s}' + \lambda\mathbf{a}$ that projects into $\mathcal{A}_2$ and use a similar argument to note that the majorization ordering preserves monotonicity among points along the segment as λ increases. Since Lemma 3.5 states there are a finite number of such line segments, proceeding stepwise yields the conclusion of the lemma. □

A converse to the lemma, which enhances the results of the paper, will now be stated.

Let ∂H_i and $\partial\mathcal{C}$ denote the boundary of H_i and $\mathcal{C}$, respectively.

LEMMA 3.7. *Assume $\partial H_i \cap \partial\mathcal{C}$ contains a nonempty set U_i which is open in the relative topology of $\partial H_i, i = 1,2,\ldots,k-1$. If $\mathbf{s}'' \geq \mathbf{s}'$ implies $P\mathbf{s}''|\mathcal{C} \succ P\mathbf{s}'|\mathcal{C}$, then each row of A has exactly two nonzero elements.*

The proof is omitted.

Before stating the main result of this section some clarification of the notion of our projected tests should prove helpful. The size α Schur convex test $\varphi(\mathbf{s})$ can be performed conditionally given $T = t$, and because of the Neyman structure of the test the conditional size is also α. This is also true for the projected tests of (1.3) $\varphi^*(\mathbf{s})$ for testing H_0 versus H_1. In other words, critical values are determined for each fixed $T = t$ in actually carrying out a test. Further, note that if the test φ has size α for testing H_0 versus H_2, the projected test φ^* will not have size α for testing H_0 versus H_1. However, this is not a problem in achieving size α for the projected tests since conditional critical values yielding size α are appropriate and obtainable.

THEOREM 3.8. *Assume that each row of A has exactly two nonzero elements. Let $\varphi(\mathbf{s})$ be a Schur convex test function of constant size. Then the projected test of size α is monotone with respect to the cone ordering induced by $\mathcal{C}^*$.*

PROOF. Let $\mathbf{s}'' \geq \mathbf{s}'[\mathcal{C}^*]$. From Lemma 3.6 we have

$$Ps''|\mathcal{C} \succ Ps'|\mathcal{C}. \tag{3.7}$$

Now consider the projected test of size α, $\varphi^*(\mathbf{s}) = \varphi(P\mathbf{s} \mid \mathcal{C})$, where φ is a Schur convex function. We have from (1.3) and (3.7) that

$$\varphi^*(\mathbf{s}') = \varphi(P\mathbf{s}' \mid \mathcal{C}) \leq \varphi(P\mathbf{s}'' \mid \mathcal{C}) = \varphi^*(\mathbf{s}''). \tag{3.8}$$

Thus φ^* is monotone with respect to the cone ordering induced by $\mathcal{C}^*$. □

Some discussion regarding the implementation of projected tests is in order. Since the test is performed conditionally on $T = t$, simulation can be used to determine a conditional p-value. For the observed value of $T = t$, the conditional distribution of $\mathbf{S} \mid T$ under H_0 is used to generate samples via Monte Carlo methods. For each sample the projected statistic is calculated and the percentage of times the statistic exceeds the observed value of the statistic yields the p-value of the observed statistic.

It would also be possible to use Monte Carlo methods to compute some conditional power function values for some points in H_1.

We conclude this section with some remarks on specific alternatives. When $\mathcal{C}$ is the simple order cone, the projected tests are monotone and are also unbiased [see Cohen, Perlman and Sackrowitz (1991)]. When $\mathcal{C}$ is the simple tree cone, unbiasedness again follows. To see this we need to refer to the result of Cohen and Sackrowitz [(1990), Theorem 3.1]. That theorem claims that the projected test $\varphi^*(\mathbf{s})$ is unbiased of size α if it is of size α, if it is monotone with respect to the cone ordering induced by $\mathcal{C}^*$ and if it is Schur convex in $s_1, \ldots, s_{k-1}$, for fixed t and s_k, where s_k represents the statistic corresponding to the control population. In the proof of Theorem 3.8 we indicated that φ^* is monotone with respect to $\mathcal{C}^*$. Hence to prove unbiasedness requires showing the Schur convex property for fixed t and s_k. We claim that this last property follows using the properties of projection onto the simple tree cone and properties of majorization. We omit the details of a proof.

We note that the umbrella alternative or unimodal restriction alternative represents another cone that can be treated with the results of this paper [see RWD (1988), page 85].

In many special cases admissibility of the projected tests can be shown. This is especially true in the discrete cases and often for tests in the normal case by using a theorem of Stein (1956).

For the cone representing a convex alternative [one row would be $(1, -2, 1, 0, \dots, 0)$] or a cone representing the star shape ordering [one row would be $(1, 1, -2, 0, \dots, 0)$], the results of this paper cannot be used and in fact some projected tests are not monotone.

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