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A REVERSE SUBMARTINGALE PROPERTY OF THE RANGE

BY J. S. HUANG[1] AND W. J. HUANG[2]

University of Guelph and National Sun Yat-sen University

If R_n is the range from an exchangeable sequence of random variables, then $\{R_n/n\}$ forms a reverse submartingale sequence.

1. Introduction. It is known that $\left\{\binom{n}{2}^{-1}R_n\right\}$ forms a reverse submartingale (RSM) sequence, where R_n is the range of size n from an exchangeable sequence of random variables. Something stronger is true: $\{n^{-1}R_n\}$ is RSM. A tighter bound for the moments of R_n follows as an immediate consequence.

2. The results. Let $X_1, X_2, \dots$ be an exchangeable sequence of random variables; $X_{1,n} \le \cdots \le X_{n,n}$ be the order statistics based on $\{X_1, \dots, X_n\}$ and $R_n = X_{n,n} - X_{1,n}$ be the range. Bhattacharyya (1970) shows that

(i) if $E|X_1| < \infty$, then $\left\{\binom{n}{2}^{-1}R_n\right\}$ is RSM, and

(ii) if $E|X_1|^k < \infty$, then

$$E(R_j^k) \le \left[\frac{j(j-1)}{i(i-1)}\right]^k E(R_i^k), \qquad i \le j. \tag{1}$$

Result (ii) is a consequence of the RSM property of (i), and (i) is based on the following inequality:

LEMMA 1. (Bhattacharyya). *Given $n+m$ numbers $x_1 \le \cdots \le x_{n+m}$, let $x_1^{(i)} \le \cdots \le x_n^{(i)}$, $i = 1, 2, \dots, \binom{n+m}{n}$, be the set of all possible subsets of n-tuples that can be formed from the $n+m$ x's, and $R_n^{(i)}$ be the range of the i-th subset. For $n \ge 2$,*

$$\binom{n+m}{n}\binom{n}{2}R_{m+n} \le \binom{n+m}{2}\sum_i R_n^{(i)}. \tag{2}$$

The sum of subsample ranges in the rhs of (2) has a convenient representation in terms of quasiranges:

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LEMMA 1a. *For $n \geq 2$,*

$$\sum_i R_n^{(i)} = \sum_{k=0}^{m} \binom{n+m-1-k}{n-1} (x_{n+m-k} - x_{1+k}). \tag{3}$$

PROOF. For any (i,j) with $1 \leq i < j \leq n+m$ and $j - i \geq n-1$, there are exactly $\binom{j-i-1}{n-2}$ subsets such that $R_n = x_j - x_i$. Thus

$$\begin{aligned}
\sum_i R_n^{(i)} &= \sum_{j=n}^{n+m} \sum_{i=1}^{j-n+1} \binom{j-i-1}{n-2} (x_j - x_i) \\
&= \sum_j x_j \sum_i \binom{j-i-1}{n-2} - \sum_{i=1}^{m+1} x_i \sum_{j=i+n-1}^{n+m} \binom{j-i-1}{n-2} \\
&= \sum_j x_j \binom{j-1}{n-1} - \sum_i x_i \binom{n+m-i}{n-1}.
\end{aligned}$$

The proof is complete upon making the change of indices $k = n + m - j$ and $k = i - 1$ in the last two summations. □

By dropping any number of terms off the rhs of (3) we obtain an approximation to the lhs of (3) at various degrees of accuracy; for $m \leq n - 1$, the approximation is actually a lower bound since the summand in the rhs of (3) is *nonnegative*. For large m ($> n - 1$), some spacings $x_{n+m-k} - x_{1+k}$ will be negative. They are, however, the negative of quasiranges, allowing (3) to be put in the form

$$\sum_i R_n^{(i)} = \sum_{k=0}^{l} c_k (x_{n+m-k} - x_{1+k}),$$

where $l = [(n+m)/2]$, the greatest integer not exceeding $(n+m)/2$, and c_k is either a binomial coefficient or the difference of two. That the c_k are positive follows easily from the fact that for fixed n and m the binomial coefficient $\binom{n+m-1-k}{n-1}$ is decreasing in k. Thus, regardless of whether $m \leq n - 1$ or not, a series of lower bounds results. The simplest—and quite useful—lower bound is obtained by dropping all but the first term ($k = 0$).

COROLLARY. *For $n \geq 2$,*

$$\sum_i R_n^{(i)} \geq \binom{m+n-1}{m} R_{m+n}. \tag{4}$$

Notice that the lower bound (4) is tighter than (2). It also leads to a stronger RSM property.

THEOREM 1. *$\{n^{-1}R_n, n \geq 2\}$ is RSM if $E|X_1| < \infty$.*

PROOF. Rewriting (4) in the form

$$\binom{m+n}{n}^{-1} \sum_i \frac{R_n^{(i)}}{n} \geq \frac{R_{m+n}}{m+n},$$

and taking the conditional expectation of both sides given $R_{n+m}, \ldots, R_{n+m+k}$, the proof then proceeds in exactly the same way as in Bhattacharyya (1970). □

REMARK 1. We are indebted to the referee for a simpler proof. He points out that the proof requires only the $m = 1$ case of inequality (4). For that case the inequality (and Lemma 1a) is reduced to a triviality.

REMARK 2. Let $H_n \equiv R_n/n$ and let

$$B_n \equiv R_n \Big/ \binom{n}{2} = \frac{2}{n-1} H_n.$$

Clearly, if $\{H_n\}$ is RSM, then $\{B_n\}$ is RSM as well:

$$\begin{aligned} E(B_n | B_{n+1}, B_{n+2}, \ldots) &= \frac{2}{n-1} E(H_n | H_{n+1}, H_{n+2}, \ldots) \\ &\geq \frac{2}{n-1} H_{n+1} = \frac{n}{n-1} B_{n+1} \geq B_{n+1} \quad \text{a.s.} \end{aligned}$$

Thus our Theorem 1 implies Bhattacharyya's. It is also a more "natural" result, putting $\{R_n\}$ on par with the maximal order statistics $\{X_{n,n}\}$ (see Theorem 2 of Bhattacharyya). Finally, as an immediate consequence of our Theorem 1 we have the following improvement over his Theorem 3(i) and the corollary [see also David (1981), page 106, footnote].

THEOREM 2. *If $E|X_1|^k < \infty$, then $E(R_j^k) \leq (j/i)^k E(R_i^k)$, $2 \leq i \leq j$, $k \geq 1$.*

COROLLARY. *If $E|X_1| < \infty$, then $E(R_{n+1}) \leq [(n+1)/n] E(R_n)$, $n \geq 2$.*

The bound is actually tight at $n = 2$ in view of the fact that $E(R_3) = \frac{3}{2}E(R_2)$.

3. Example. Let $\{A(t), t \geq 0\}$ be a mixed renewal process. The sequence of interarrival times $\{\xi_i, i \geq 1\}$ then forms an exchangeable sequence [Huang (1990)], and $\{n^{-1} \max_{1 \leq i,j \leq n} |\xi_i - \xi_j|\}$ is RSM. Mixed renewal processes occur naturally in survey sampling [Sugden (1982)]. They have also been used in describing the lifetimes of some replacement models [Shanthikumar (1985)].

REFERENCES

BHATTACHARYYA, B. B. (1970). Reverse submartingale and some functions of order statistics. *Ann. Math. Statist.* **41** 2155–2157.

DAVID, H. A. (1981). *Order Statistics*, 2nd ed. Wiley, New York.

HUANG, W. J. (1990). On the characterization of point processes with the exchangeable and Markov properties. *Sankhyā Ser. A* **52** 16–27.

SHANTHIKUMAR, J. G. (1985). Lifetime distribution of consecutive-k-out-of-n: F systems with exchangeable lifetimes. *IEEE Transactions on Reliability* **34** 480–483.

SUGDEN, R. A. (1982). Exchangeability and survey sampling inference. In *Exchangeability in Probability and Statistics* (G. Koch and F. Spizzichino, eds.) 321–328. North-Holland, Amsterdam.

DEPARTMENT OF MATHEMATICS
AND STATISTICS
UNIVERSITY OF GUELPH
COLLEGE OF PHYSICAL
AND ENGINEERING SCIENCE
GUELPH, ONTARIO
CANADA N1G 2W1

INSTITUTE OF APPLIED MATHEMATICS
NATIONAL SUN YAT-SEN UNIVERSITY
KAOHSIUNG
TAIWAN, R.O.C.