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AN EXACT DECOMPOSITION THEOREM AND A UNIFIED VIEW OF SOME RELATED DISTRIBUTIONS FOR A CLASS OF EXPONENTIAL TRANSFORMATION MODELS ON SYMMETRIC CONES[1]

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A class of exponential transformation models is defined on symmetric cones Ω with the group of automorphisms on Ω as the acting group. We show that these models are reproductive and the exponent of their joint distribution for a given sample of size q can be split into q independent components, introducing one sample point at a time. The automorphism group can be factorized into the group of positive dilation and another group. Accordingly, the symmetric cone Ω can be seen as the direct product of $\mathbb{R}^+$ and a unit orbit, and every x in Ω can be identified by its orbital decomposition. We derive the distributions of the independent components of the exponent, of the "length" of x, the "direction" of x, the conditional distribution of the direction given the length and other distributions for a given sample. The Wishart distribution and the hyperboloid distribution are two special cases in the class we define.

We also give a unified view of several distributions which are usually treated separately.

1. Introduction. Reproductive exponential models with density of the form $b(x)\exp(\theta u(x) + \varphi x + \log a(\theta, \varphi))$, where x and φ are in $\mathbb{R}^k$ and where θ and $u(x)$ have common dimensions k_1, has been studied in Barndorff-Nielsen and Blaesild (1983a, b). It is shown there that these densities with respect to the Lebesgue measure have in fact the form

$$f(x, \theta, m) = b(x)\exp\Big\{\theta\big(u(x) - \langle u'(m), x - m\rangle - u(m)\big) - N(\theta)\Big\}, \tag{1.1}$$

where $\langle x, y\rangle$ denotes the appropriate inner product of x and y in the given space, m is the mean of x and $\theta(u(x)\cdots - u(m))$ denotes the inner product in the Euclidean k_1-dimensional space.

Recently, in Massam (1989b), it has been proved that for these distributions, the quantity $\sum_{i=1}^{q}\{u(x_i) - \langle u'(m), x_i - m\rangle - u(m)\}$ in the exponent of the joint distribution $f(x_1, \ldots, x_q, \theta, m)$ of the sample $(x_1, \ldots, x_q)$ can be split into q independent components,

$$\text{(S)}\qquad \begin{aligned} R_1 &= q\big(u(\bar{x}_q) - \langle u'(m), \bar{x}_q - m\rangle - u(m)\big)\\ R_i &= u(x_i) + (i-1)u(\bar{x}_{i-1}) - iu(\bar{x}_i), \qquad i = 2, \ldots, q, \end{aligned}$$

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where $\bar{x}_i = i^{-1}(x_1 + \cdots + x_i)$. Each one of the $R_i, i = 2, \ldots, q$ incorporates one more sample point at a time. This result was known before for only the three one-dimensional reproductive distributions, the normal, the gamma and the inverse Gaussian. In Massam (1989b), two more examples are given, the Wishart distribution and an exponential model on the Lorentz cone

$$\Lambda_n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > 0, x_1^2 - x_2^2 - \cdots - x_n^2 > 0\},$$

denoted the L-distribution.

The first purpose of this paper is to define a family of reproductive exponential transformation models on symmetric cones Ω and show that they are of the form (1.1) and therefore have the splitting property (S), that is, the exponent in the joint probability density function for a sample of size q splits into q independent components. The first component involves only the average of the q points while each one of the remaining $q - 1$ components incorporates one more sample point at a time. There are only five irreducible symmetric cones and therefore only five families of distributions as we define them. These families are not new; they are the Wishart distribution on $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ (the space of real numbers, complex numbers, quaternions and Cayley numbers, respectively) and an exponential model on the Lorentz cone, studied in some detail in Letac (1994). These distributions also appear in Jensen (1988) as the distributions of the maximum likelihood estimate of the covariance matrix when the hypothesis is linear in the covariance and the inverse covariance.

The distribution of the quantities R_i will lead us to define the generalized beta and Dirichlet distributions on symmetric cones while the structure of the group of automorphisms on Ω will allow us to define a maximal invariant and corresponding orbits and to derive the corresponding distributions. When Ω is the Lorentz cone, the distribution of the direction conditional on the orbit will give the hyperboloïd distribution [see Jensen (1981) for a study of this distribution]. When Ω is the cone of positive definite matrices, we obtain the distribution of a point in the cone given its determinant.

The second purpose of the paper is to give a unified view of the five distributions on symmetric cones and their related distributions such as the generalized Dirichlet for Ω and the distributions on the unit orbits defined by decomposition of the group of automorphisms on Ω. This is done with the ulterior goal of finding out whether some properties of the distributions on the cone can be transmitted to the conditional distributions on the unit orbit. The splitting property (S) is one of these properties.

In Massam (1989a) a splitting property similar to (S) was given for the non-reproductive three-dimensional hyperboloid distribution, which is a distribution derived from (1.1) by conditioning on the unit orbit, in a very special case. In Casalis and Massam (1994), we prove that we do have, in general, a property similiar to (S) when we condition on the unit orbit, but with conditional independence only.

To define the class of distributions on symmetric cones, we need to give a summary of the mathematical properties of these cones. This is done in Section 2; the material is taken from Faraut (1988). In Section 3, we define the distributions and show they are of the form (1.1) and therefore have the splitting property (S). In Section 4, using the theory of exponential transformation models, we derive the generalized Dirichlet distribution and the distribution obtained by conditioning on a maximal invariant statistic, and we derive other distributions related to a sample from the conditional distribution given the maximal invariant.

2. Symmetric cones and their properties. We give here, without proof, the results and properties needed to define a class of exponential distributions on symmetric cones and to derive other related distributions. All results are taken from Faraut (1988), where proofs and detailed explanations can be found.

2.1. *Symmetric cones and their automorphism group.* Let E be a finite-dimensional real Euclidean space and let $(x \mid y)$ denote the inner product of x and y. Let Ω be an open convex cone, its dual Ω^* is defined by

$$\Omega^* = \left\{y \in E \,\middle|\, (x \mid y) > 0, \forall\, x \in \overline{\Omega} - \{0\}\right\},$$

where $\overline{\Omega}$ denotes the closure of Ω.

An open convex cone Ω is *self-dual* if $\Omega = \Omega^*$. The *automorphism group* is $G(\Omega) = \{g \in \mathrm{GL}(E) \mid g\Omega = \Omega\}$. The open cone Ω is said to be *homogeneous* if $G(\Omega)$ acts transitively, that is, if, for all x and y in Ω, there exists $g \in G(\Omega)$ such that $gx = y$. An open cone Ω is *symmetric* if it is homogeneous and self-dual. Let G be the connected component of the group $G(\Omega)$ containing the identity id_E; G acts transitively on Ω. Two examples of symmetric cones are the cone Π_m of positive definite $m \times m$ matrices and the Lorentz cone $\Lambda_n = \{x = (x_1, \dots, x_n) \mid x_1 > 0, x_1^2 - x_2^2 - \cdots - x_n^2 > 0\}$. In Section 3, we define on Π_m and Λ_n, respectively, the Wishart and the L-distribution. The group G on Π_m is

$$G = \{\rho(g), g \in \mathrm{GL}(m, \mathbb{R}) \mid \rho(g)x = gxg'\}.$$

The group G on Λ_n is $G = \mathbb{R}^+ \times \mathrm{SO}^\uparrow(1, n-1)$, where

$$\mathrm{SO}^\uparrow(1, n-1) = \{A \in \mathrm{GL}(\mathbb{R}^n) \mid \det A = 1, A_{11} > 0, AI_{1,n-1}A' = I_{1,n-1}\}$$

and $I_{1,n-1}$ is the diagonal matrix with first element 1 and other diagonal elements equal to -1. The *characteristic function* φ of Ω is defined on Ω as

$$\varphi(x) = \int_{\Omega^*} e^{-(x \mid y)} dy,$$

where dy is the Lebesgue measure in E. One can prove that $\varphi(x)$ is well defined and that, for all g in G,

$$\varphi(gx) = |\det g|^{-1} \varphi(x).$$

Two important consequences of this are as follows:

1. $\varphi(\lambda x) = \lambda^{-n}\varphi(x)$, that is, $\varphi(x)$ is homogeneous of degree $-n$, where n is the dimension of E.
2. $\varphi(x)dx$ is an *invariant measure* under G; this is the invariant measure with respect to which we will define the distribution on Ω [see (2.12) for the expression of $\varphi(x)$].

2.2. *Jordan algebras and Euclidean Jordan algebras.* Let F be the field $\mathbb{R}$ or $\mathbb{C}$. An algebra V over F is said to be a *Jordan algebra* if, for all x and y in V, the product $x \circ y$ of x and y satisfies $x \circ y = y \circ x$ and $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$. The space of symmetric $m \times m$ matrices with product $x \circ y = \frac{1}{2}(xy + yx)$ is a Jordan algebra and so is $\mathbb{R}^n$ with the product

$$(2.1)\qquad \begin{aligned}&(x_1,x_2,\dots,x_n)\circ(y_1,y_2,\dots,y_n)\\ &\quad= \big(x_1y_1 + x_2y_2 + \cdots + x_ny_n, x_1(y_2,\dots,y_n) + y_1(x_2,\dots,x_n)\big).\end{aligned}$$

We assume that V has an identity element e such that, for all x in V, $x = x \circ e = e \circ x$. Let $\mathbb{C}(X)$ be the algebra over $\mathbb{C}$ of polynomials in one variable with complex coefficients. For an element x in V, define $\mathbb{C}(x) = \{p(x), p \in \mathbb{C}(X)\}$. The subalgebra $\mathbb{C}(x)$ generated by x and the identity element e is equal to $\mathbb{C}(X)/I(x)$, where $I(x) = \{p \in \mathbb{C}(X), p(x) = 0\}$ is an ideal generated by a polynomial called the *minimal polynomial* of x. A complex number λ is an *eigenvalue* of x in V if there exists v in $\mathbb{C}(x)$, nonzero, such that $xv = \lambda v$. Then v is called an *eigenvector* of x. The *rank* of V is $r = \max\{m(x) \mid x \in V\}$, where $m(x) = \inf\{k \mid (e,x,\dots,x^k) \text{ are linearly dependent}\}$. An element is said to be regular if $m(x) = r$. The set of regular elements is open and dense in the algebra V. For regular elements x in V, the minimal polynomial can be written as

$$f(X,x) = X^r - a_1(x)X^{r-1} + a_2(x)X^{r-2} + \cdots + (-1)^r a_r(x).$$

An element λ of $\mathbb{C}$ is an eigenvalue of x if and only if $X = \lambda$ is a root of f. The a_j's are unique and homogeneous of degree j. We will use two of these a_j's:

1. $a_1(x)$ is called the trace of x and is homogeneous of degree 1.
2. $a_r(x)$ is called the determinant of x and is homogeneous of degree r.

An element x is invertible iff $\det x \neq 0$. When it exists, the inverse of x is denoted by x^{-1} and is such that $x \circ x^{-1} = x^{-1} \circ x = e$. For the algebra of symmetric $m \times m$ matrices with the product given above, the trace, determinant and inverse are defined as usual. For the Jordan algebra $\mathbb{R}^n$ with the product $x \circ y$ given above,

$$\operatorname{tr} x = 2x_1, \det x = x_1^2 - x_2^2 - \cdots - x_n^2 \quad\text{and}\quad x^{-1} = \frac{(x_1, -x_2, \dots, -x_n)}{(x_1^2 - x_2^2 - \cdots - x_n^2)}.$$

We can define two maps on V: for x and y in V.

$$L(x)y = xy \quad\text{and}\quad P(x) = 2L(x)^2 - L(x^2).$$

We will not use these maps explicitly except in formal calculation. It will help to remember that, for the algebra of symmetric $m \times m$ matrices,

$$P(x)y = xyx \quad \text{and, in general,} \quad P(x)e = x^2.$$

We will use the following properties:

(2.2) The differential of the map $x \to \log\det x$ is x^{-1}.

(2.3) The differential of the map $x \to x^{-1}$ is $-P(x)^{-1} = -P(x^{-1})$.

(2.4) $$\det(P(y)x) = (\det y)^2 \det x.$$

The proof of (2.2) and (2.3) due to Faraut can be found in Casalis (1990). A Jordan algebra V is Euclidean if there exists an inner product $(u \mid v)$, defined by a positive definite associative bilinear form on V, such that for all x, u, v in V, $(L(x)u \mid v) = (u \mid L(x)v)$. From now on, we will take $(u \mid v) = \mathrm{tr}(uv)$.

The isotropy group is the subgroup $K = \{g \in G | ge = e\}$ of G. The subgroup K is equal to $G \cap \mathrm{O}(E)$, where $\mathrm{O}(E)$ denotes the orthogonal group of E and plays a crucial role in the polar decomposition of G given in subsection 2.4.

2.3. *The Pierce decomposition of a Jordan algebra V.* An element c in V is said to be idempotent if $c^2 = c$. The only possible eigenvalues of an indempotent c are 0, $\frac{1}{2}$ and 1, and the corresponding eigenspaces are denoted $V(c,0)$, $V(c,\frac{1}{2})$, and $V(c,1)$. The decomposition

$$V = V(c,0) \oplus V\left(c,\frac{1}{2}\right) \oplus V(c,1)$$

is called the *Pierce decomposition* of V with respect to c; V is the direct sum of the three eigenspaces of c. In the Jordan algebra of $m \times m$ real symmetric matrices, an idempotent is an orthogonal projection. In $\mathbb{R}^n$ with product (2.1) the only idempotents are $e = (1,0,\dots,0)$ and $c = \left(\frac{1}{2}, x_2, \dots, x_n\right)$, where $x_2^2 + x_3^2 + \cdots + x_n^2 = \frac{1}{4}$. One can show that $\mathrm{tr}(c) = \mathrm{rank}\, V(c,1)$. If $\mathrm{rank}\, V(c,1) = 1$, c is said to be primitive idempotent. Two idempotents a and b are said to be orthogonal if $ab = 0$. A system $(e_1, \dots, e_m)$ of orthogonal idempotents is complete if $e_1 + \cdots + e_m = e$. A Jordan algebra V is said to be simple if it does not contain any nontrivial ideal and this is so if and only if $V(c,\frac{1}{2}) \neq \{0\}$, for any nontrivial idempotent c. In a simple algebra, we have the following fundamental properties:

1. Two complete systems of orthogonal primitive idempotents have the same number r of elements. This number r is equal to the rank of V.
2. If $\{e_1, \dots, e_r\}$ and $\{f_1, \dots, f_r\}$ are two systems of orthogonal primitive idempotents, there exists an automorphism A such that $Ae_i = f_i, i = 1, \dots, r$. [Recall that an automorphism in a Jordan algebra satisfies $A(x \circ y) = A(x) \circ A(y)$.]
3. If (a,b) and (c,d) are two couples of orthogonal primitive idempotents,

$$\dim V\left(a,\frac{1}{2}\right) \cap V\left(b,\frac{1}{2}\right) = \dim V\left(c,\frac{1}{2}\right) \cap V\left(d,\frac{1}{2}\right).$$

This fixed dimension is denoted d.

4. $V = \oplus_{j\le k} V_{jk}$, where $V_{jk} = V\big(e_j, \frac{1}{2}\big) \cap V\big(e_k, \frac{1}{2}\big)$ when $j \neq k$, $V_{jj} = V(e_j, 1)$ and $\dim V_{jj} = 1$. [A proof of this can be derived from Braun and Koecher (1966), XI, Lemma 3.3, page 320.]

It is then clear that we have the following relationship between the dimension n of V, its rank r and the fixed dimension d:

$$n = r + d\frac{r(r-1)}{2}. \tag{2.5}$$

These numbers n, r and d are characteristics of the symmetric cone Ω and therefore of the class of distributions we are going to define on it.

2.4. *Decomposition of G and polar decomposition of x in Ω.* For the orbital decomposition of Ω in Section 4, we will need to use the polar decomposition of any g in G. Any element in G can be written as $g = P(y)k$, for some y in Ω and k in the isotropy group K. This is easy to see if we observe that, for any g in G, $ge = y^2 = P(y)e$, for some y in Ω, [the first equality is due to the fact that any symmetric cone is the interior of the cone of squares in a Euclidean Jordan algebra—see Faraut (1988), III.3 and III.4]. Put $k = P(y)^{-1}g$, then $ke = e$; therefore $k \in K = G(\Omega)_e$ and $g = P(y)k$. This decomposition is unique.

Let us turn to the polar decomposition of x in Ω. Let $c_1, \dots, c_r$ be a given complete system of primitive orthogonal idempotents in V. The spectral theorem states that for any $x \in V$ there exists a complete system of orthogonal idempotents $c_1, \dots, c_r$ such that $x = \sum_{j=1}^r a'_j c_j$. The isotropy group K acts transitively on the set of complete systems of orthogonal primitive idempotents. Therefore, any x in V can be written

$$x = ka, \qquad k \in K, \qquad a = \sum_{j=1}^r a_j c_j, \quad a_j \in \mathbb{R},$$

where $\{a_j, j = 1, \dots, r\} = \{a'_j, j = 1, \dots r\}$.

If we set

$$R = \left\{ a = \sum_{j=1}^r a_j c_j \;\middle|\; a_j \in \mathbb{R} \right\},$$

$$R_+ = \left\{ a = \sum_{j=1}^r a_j c_j \;\middle|\; a_1 < \cdots < a_r \right\},$$

$$M = \{k \in K \mid \forall a \in R, ka = a\},$$

one can prove that the Jacobian of the map $(kM, a) \to ka = x$ from $K/M \times R_+$ to V is equal to

$$2^{2r-2n} \prod_{j<k} (a_k - a_j)^d. \tag{2.6}$$

Since the symmetric cone Ω is the interior of the cone of squares in V, for $x \in \Omega$, each a_j can be written $a_j = e^{t_j}$, $t_j \in \mathbb{R}$. So any point x in Ω can be written

$$x = k\sum_{j=1}^{r} e^{t_j}c_j, \qquad k \in K,\ t_j \in \mathbb{R}. \tag{2.7}$$

Moreover the measure $\varphi(x)\,dx$, invariant with respect to the group G, is as we shall see later proportional to $(\det x)^{-n/r}dx$. Using (2.6) and (2.7) one then obtains the following integral formula with respect to the G-invariant measure $(\det x)^{-n/r}dx$:

$$\begin{aligned}&\int_\Omega f(x)(\det x)^{-n/r}dx\\ &\quad = 2^{n-r}c\int_{K\times\mathbb{R}_+} f(ke^t)\prod_{j<k}\sinh\left[\frac{t_k-t_j}{2}\right]dk\,dt_1\cdots dt_r,\end{aligned} \tag{2.8}$$

where c is a constant to be determined, dk is the normalized Haar measure of K and e^t is a short notation for $\Sigma_{j=1}^{r}e^{t_j}c_j$. We will use this formula when computing a conditional distribution given a maximal invariant statistic in Section 4.

We also need to introduce the triangular subgroup of G. Let x be in V. According to Property 4 in Section 2.3, x can be written as $x = \Sigma_{j\le k}x_{jk}$, where $x_{jk} \in V_{jk}$. The triangular subgroup T of G is the set of elements t in G such that, for all x in V, $(tx_{k\ell})_{ij} = 0$, if $(i,j) < (k,\ell)$ for the lexicographic order, and $(tx_{ij})_{ij} = \lambda_{ij}x_{ij}\ \forall i,j$ with positive numbers λ_{ij} which do not depend on x. In the algebra of $m\times m$ symmetric matrices, an element t in T is such that $tx = \alpha x\alpha'$, where α is a lower triangular matrix with positive diagonal elements. One can prove that, for any x in Ω, there exists a unique t in T such that

$$x = te. \tag{2.9}$$

We will use this decomposition in the derivation of the generalized beta and Dirichlet distribution.

2.5. *The gamma function of a symmetric cone and some formulas for differentials.* The usual gamma function for $\Omega = \mathbb{R}^+$ is $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$. For a general symmetric cone Ω and $s = (s_1,\dots,s_r)$ in $\mathbb{C}^r$, one can define the gamma function for Ω by

$$\Gamma_\Omega(s) = \int_\Omega e^{-\operatorname{tr}(x)}\Delta_s(x)(\det x)^{-n/r}dx,$$

where $\Delta_s(x) = \lambda_1^{s_1}\cdots\lambda_r^{s_r}$ if $x = \lambda_1c_1 + \cdots + \lambda_rc_r$.

For $s = (0,\dots,0)$ in $\mathbb{C}^r$ and p in $\mathbb{C}$, $\Delta_{s+p}(x) = \Delta_p(x) = (\det x)^p$ and the preceding integral becomes

$$\Gamma_\Omega(p) = \int_\Omega e^{-\operatorname{tr}(x)}(\det x)^{p-n/r}dx,$$

where dx is the Lebesgue measure on Ω. This integral converges absolutely for $\mathrm{Re}(p) > (r-1)d/2$ and is equal to

$$\Gamma_\Omega(p) = (2\pi)^{(n-r)/2}\Gamma(p)\Gamma\left(p-\frac{d}{2}\right)\cdots\Gamma\left(p-(r-1)\frac{d}{2}\right). \tag{2.10}$$

We also have, for any y in Ω and $\mathrm{Re}(p) > (r-1)d/2$,

$$\int_\Omega e^{-\mathrm{tr}(xy)}(\det x)^{p-n/r}dx = \Gamma_\Omega(p)(\det y)^{-p}. \tag{2.11}$$

In particular, for $p = n/r$, we obtain the expression of the characteristic function

$$\varphi(y) = \int_\Omega e^{-\mathrm{tr}(xy)}\,dx = \Gamma_\Omega\left(\frac{n}{r}\right)(\det y)^{-n/r}, \tag{2.12}$$

from which we derive immediately the G-invariant measure $(\det x)^{-n/r}\,dx$.

Similarly, we can define the beta function of the symmetric cone Ω by

$$B_\Omega(p,q) = \int_{\Omega\cap(e-\Omega)}(\det x)^{p-n/r}\left(\det(e-x)\right)^{q-n/r}dx. \tag{2.13}$$

This integral converges for $\mathrm{Re}(p) > (r-1)d/2$ and $\mathrm{Re}(q) > (r-1)d/2$ and is equal to

$$B_\Omega(p,q) = \frac{\Gamma_\Omega(p)\Gamma_\Omega(q)}{\Gamma_\Omega(p+q)}. \tag{2.14}$$

In the course of our calculations, we will also need the following formulae:

$$\text{for any } x \text{ in } \Omega, \quad \det P(x) = (\det x)^{2n/r}; \tag{2.15}$$

$$\text{for any } g \text{ in } G, \quad \det gx = (\det g)^{r/n}\det x; \tag{2.16}$$

$$\text{for any } g \text{ in } G, \text{ the differential} \quad d(gx) = (\det ge)^{n/r}dx. \tag{2.17}$$

For more properties on differentials, the reader is referred to Farrell (1985) and Wijsman (1990).

3. A class of reproductive exponential transformation models on a symmetric cone Ω. Given a symmetric cone Ω, let G be the connected component of $G(\Omega)$ containing the identity. The measure $d\mu(x) = (\det x)^{-n/r}dx$ is G-invariant. Let us define the following natural exponential family, for some $p > (r-1)d/2$:

$$F_p = \left\{\frac{1}{\Gamma_\Omega(p)}e^{-(x|y)}(\det x)^p(\det y)^p\,d\mu(x), y \in \Omega\right\}.$$

Let us now consider both p and y as parameters. The density of a distribution in this family with respect to the Lebesgue measure is

$$(3.1)\quad f(x,y,p) = \frac{1}{\Gamma_\Omega(p)}(\det x)^{-n/r}(\det x)^p \exp[-(x \mid y)](\det y)^p$$

$$(3.2)\quad = \frac{1}{\Gamma_\Omega(p)}(\det x)^{-n/r} \exp p\left\{ \log \det x - \frac{1}{p}(x \mid y) + \log \det y \right\}.$$

We will show now that this density is of the form (1.1) and therefore, by Massam [(1989b), Theorem 2.1] has the splitting property (S) mentioned in Section 1.

THEOREM 3.1. *Given Ω and G as before, $p > (r - 1)d/2$, the distribution defined on Ω with density $[1/\Gamma_\Omega(p)]e^{-(x|y)}(\det x)^p(\det y)^p$ with respect to the G-invariant measure $d\mu(x) = (\det x)^{-n/r}dx$ has density, with respect to the Lebesgue measure on Ω, equal to*

$$f(x,m,p)$$
$$= \frac{1}{\Gamma_\Omega(p)}\left| \det D^2u(x)\right|^{1/2} \exp\left\{p\big(u(x) - (u'(m) \mid x - m) - u(m)\big) - N(p)\right\},$$

where $u(x) = \log\det x$, m is the mean of the distribution and $N(p)$ is a quantity independent of m. Given a sample $(x_1,\dots,x_q)$ from this distribution, the exponent

$$E = \sum_{i=1}^{q} \big(u(x_i) - (u'(m) \mid x_i - m) - u(m)\big)$$

of the joint distribution of $(x_1,\dots,x_q)$ can be decomposed into q independent components $E = R_1 + \Sigma_{i=2}^q R_i$ with

$$R_1 = q\big(\log \det \bar{x}_q + \log \det y\big) - \frac{1}{p}\operatorname{tr} y\bar{x}_q + C,$$
$$R_i = \log\left[\frac{(\det x_i)\big(\det(\bar{x}_{i-1})\big)^{i-1}}{(\det \bar{x}_i)^i}\right],$$

where C is a constant. The cumulant generating function (i.e., the logarithm of the moment generating function) of the quantities

$$\log\frac{\det(x_i)\det(x_1 + \cdots + x_{i-1})^{i-1}}{\det(x_1 + \cdots + x_i)^i} \quad \textit{is} \quad \log\frac{B_\Omega(p+s,(i-1)(p+s))}{B_\Omega(p,(i-1)p)}.$$

PROOF.

Step 1. We prove that $|\det D^2u(x)|^{1/2} = (\det x)^{-n/r}$. Let $u(x) = \log\det x$. From (2.2) and (2.3), it follows that $D^2u(x) = -P(x)^{-1}$. Since $P(x)^{-1} = P(x^{-1})$

[see Faraut (1988), II, 3.1] and $\det P(x^{-1}) = (\det x^{-1})^{2n/r}$ [see (2.15)], we obtain $|\det D^2u(x)| = |\det x^{-1}|^{2n/r}$, for any x in Ω, and the result follows.

Step 2. $(u'(x) \mid x) = (D\log \det(x) \mid x) = r$, for all x in Ω. This follows immediately from (2.12) and the fact that $(x \mid -D \log \varphi(x)) = n$ for all x in Ω [see Faraut (1988), I.3.4, page 11].

Step 3. The mean m is equal to py^{-1}. Indeed, the cumulant generating function of a distribution in F_p, with respect to the measure $\mu_p(dx) = (\det x)^{p-n/r}\,dx$ is $k(\cdot)$ such that

$$\begin{aligned}\exp\left(k(-y)\right) &= \int_\Omega \exp\left(-(x \mid y)\right)(\det x)^{p-n/r}\,dx\\ &= \Gamma_\Omega(p)(\det y)^{-p}, \quad \text{from (2.11)}.\end{aligned}$$

So $k(-y) = -p\log\det(y) + \log\Gamma_\Omega(p)$ and $m = dk(-y)/d(-y) = py^{-1} = pu'(y)$, or equivalently, $y = pm^{-1} = pu'(m)$. Then $\log\det y = \log\det(pm^{-1}) = \log(p^r \det m^{-1}) = r\log p - \log\det m$, since $\det x$ is homogeneous of degree r. Moreover,

$$\begin{aligned}\frac{1}{p}\Big(x \mid pD(\log\det(m))\Big) &= \Big(x \mid D(\log\det(m))\Big)\\ &= \Big(x-m \mid D(\log\det(m))\Big) + \Big(m \mid D(\log\det(m))\Big)\\ &= \left(x-m \mid u'(m)\right) + r, \quad \text{from Step 2.}\end{aligned}$$

It is easy to see that

$$f(x,y,p) = \frac{1}{\Gamma_\Omega(p)}(\det x)^{-n/r}(\det x)^p(\det y)^p e^{-(x|y)}$$

becomes

$$\begin{aligned}f(x,y,p) &= \frac{(\det x)^{-n/r}}{\Gamma_\Omega(p)}\ \exp p\left\{\log\det x - \frac{1}{p}\Big(x \mid pD(\log\det(m))\Big)\right.\\ &\qquad\left. + \log\det y\right\}\\ &= \frac{(\det x)^{-n/r}}{\Gamma_\Omega(p)}\exp\ p\Big\{u(x) - (x-m \mid u'(m)) - r + \log\det(p)\\ &\qquad - \log\det(m)\Big\}\\ &= \frac{|\det D^2u(x)|^{1/2}}{\Gamma_\Omega(p)}\exp\Big\{p\Big(u(x) - (x-m \mid u'(m)) - u(m)\Big) - pr\\ &\qquad + pr\ \log p\Big\}\\ &= |\det D^2u(x)|^{1/2}\exp\Big\{p\Big(u(x) - (u'(m) \mid x-m) - u(m)\Big) - N(p)\Big\},\end{aligned}$$

where $N(p) = pr - pr \log p + \log \Gamma_\Omega(p)$.

The first part of the theorem is proved. It now follows directly from Theorem 2.1 in Massam (1989b) that R_1 and $R_i, i = 2, \dots, q$, are independent [recall that $(x \mid y) = \operatorname{tr} xy$] and the R_i's, $i = 2, \dots, q$, have cumulant generating function

$$N(p+s) - N(p) + N((i-1)(p+s)) - N((i-1)p) - (N(i(p+s)) - N(ip))$$
$$= rs \log \frac{i^i}{(i-1)^{(i-1)}} + \log \frac{B_\Omega(p+s,(i-1)(p+s))}{B_\Omega(p,(i-1)p)}.$$

Rewriting $\bar{x}_i$ as $(x_1 + \cdots + x_i)/i$, we obtain

$$R_i = \log \left[\frac{\det x_i \ \det(x_1 + \cdots + x_{i-1})^{(i-1)}}{\det(x_1 + \cdots + x_i)^i} \frac{i^{ri}}{(i-1)^{r(i-1)}} \right],$$

and the second part of the theorem follows. We can rewrite the last result as

$$E\left[\left[\frac{\det x_i \ \det(x_1 + \cdots + x_{i-1})^{(i-1)}}{\det(x_1 + \cdots + x_i)^i}\right]^s\right] = \frac{B_\Omega(p+s,(i-1)(p+s))}{B_\Omega(p,(i-1)p)}. \tag{3.3}$$ □

There are five irreducible symmetric cones. We partially reproduce, as Table 1, a table from Faraut (1988) giving these cones and their characteristic numbers n, r and d. The different cones are as follows:

1. $\Pi_m(\mathbb{R})$, the cone of $m \times m$ positive definite symmetric matrices with entries in $\mathbb{R}$;
2. $\Pi_m(\mathbb{C})$, the cone of $m \times m$ positive definite Hermitian matrices with entries in $\mathbb{C}$;
3. $\Pi_m(\mathbb{H})$, the cone of $m \times m$ positive definite Hermitian matrices with entries in the space of quaternions;
4. $\Lambda_n(\mathbb{R})$, the Lorentz cone of $\mathbb{R}^n$;
5. $\Pi_3(\mathbb{O})$, the cone of positive definite Hermitian 3×3 matrices with entries in $\mathbb{O}$, the algebra of Cayley numbers.

The form of the density in F_p gives immediately the density on each one of these cones Ω once we know what is meant by $(x \mid y) = \operatorname{tr} xy$ and $\det x$.

TABLE 1

Ω	n	r	d
$\Pi_m(\mathbb{R})$	$\frac{1}{2}m(m+1)$	m	1
$\Pi_m(\mathbb{C})$	m^2	m	2
$\Pi_m(\mathbb{H})$	$m(2m-1)$	m	4
Λ_n	n	2	$n-2$
$\Pi_3(\mathbb{O})$	27	3	8

For $\Omega = \Pi_m(\mathbb{R})$, the trace and determinant are the usual ones for matrices. For the cones of Hermitian matrices on $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, in order to find $\det x$ and $\operatorname{tr} x$ we define some real representation of the matrices x in $\Pi_m(\mathbb{R}), \Pi_m(\mathbb{C})$ or $\Pi_m(\mathbb{H})$ as follows.

Any element $a = a_0 + ia_i$ of $\mathbb{C}$ can be represented by the 2×2 matrix $r(a) = \begin{bmatrix} a_0 & -a_i \\ a_i & a_0 \end{bmatrix}$, which is the matrix of the endomorphism $b \to ab$ from $\mathbb{C}$ to $\mathbb{C}$. Similarly, any element $a = a_0 + ia_i + ja_j + ka_k$ in $\mathbb{H}$ can be represented by the 4×4 matrix

$$r(a) = \begin{pmatrix} a_0 & -a_i & -a_j & -a_k \\ a_i & a_0 & -a_k & a_j \\ a_j & a_k & a_0 & -a_i \\ a_k & -a_j & a_i & a_0 \end{pmatrix},$$

which is the matrix of the endomorphism $b \to ab$ from $\mathbb{H}$ to $\mathbb{H}$. Any $m \times m$ matrix $x = (x_{ij}), i, j = 1, \dots, m$, with coefficients in $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ can be represented by a $\delta m \times \delta m$ matrix x_δ defined by blocks as $x_\delta = [r(x_{ij})]$, where $\delta = 1, 2$ or 4 depending on whether the elements x_{ij} are in $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, respectively.

It is easy to see that the following property is true: $(xy)_\delta = x_\delta y_\delta$ and therefore if p is a polynomial with real coefficients,

$$P(x) = 0 \iff P(x_\delta) = 0.$$

Thus the minimal polynomials of x and x_δ are identical. One can also show [see Casalis (1990)] that the dimensions of the eigenspaces of x_δ are multiples of δ. Since x_δ is real symmetric, it can be diagonalized as

$$D_\lambda = \begin{pmatrix} \lambda_1 I_\delta & & 0 \\ & \ddots & \\ 0 & & \lambda_m I_\delta \end{pmatrix},$$

where I_δ is the identity matrix of dimension δ. The determinant and trace of x_δ are

$$\det x_\delta = \prod_{i=1}^m \lambda_i^\delta \quad \text{and} \quad \operatorname{tr} x_\delta = \delta \sum_{i=1}^m \lambda_i,$$

respectively.

From the construction of the matrix x_δ and the fact that its eigenspaces have dimensions which are multiples of δ, it follows that

$$\det x = \prod_{i=1}^m \lambda_i = (\det x_\delta)^{1/\delta},$$

$$\operatorname{tr} x = \sum_{i=1}^m \lambda_i = \frac{1}{\delta} \operatorname{tr} x_\delta$$

and, for each cone $\Pi_m(\mathbb{R}), \Pi_m(\mathbb{C})$ or $\Pi_m(\mathbb{H})$, the rank r is equal to m.

Taking $y = (2\delta\Sigma)^{-1}$, the density

$$f(x,y,p) = \frac{1}{\Gamma_\Omega(p)}(\det x)^{-n/r}(\det x)^p(\det y)^p e^{-\operatorname{tr} xy}$$

immediately gives, within a multiplicative factor, the Wishart distribution on $\mathbb{R}$ for $\delta = 1$. The complex Wishart distribution for $\delta = 2$ [see, e.g., Krishnaiah (1976)] or the Wishart distribution on the space of quaternions for $\delta = 4$ [see Møller (1984) or Anderson (1975)].

Let us now consider $\Pi_3(\mathbb{O})$ the space of 3×3 Hermitian matrices on the algebra of Cayley numbers $\mathbb{O}$. The space $\mathbb{O}$ is a vector space $\mathbb{R} \times V$, where V is a seven-dimensional Euclidean space with an orthonormal basis $(\mathbf{e}_i, i = 1, \dots, 7)$, on which we can define a noncommutative, nonassociative multiplication such that if $a = (a_0, a_1, a_2, \dots, a_7) = (a_0, \mathbf{a})$ and $b = (b_0, b_1, \dots, b_7) = (b_0, \mathbf{b})$ are two elements of $\mathbb{O}$,

$$ab = \big(a_0 b_0 - \mathbf{ab}, a_0\mathbf{b} + b_0\mathbf{a} + \mathbf{a} \wedge \mathbf{b}\big),$$

where $\mathbf{a} \wedge \mathbf{b} = \sum_{i=1}^{7} \Pi_i(\mathbf{a}) \wedge \Pi_i(\mathbf{b}), \Pi_i(\mathbf{a})$ is the orthogonal projection of $\mathbf{a}$ on V_i, the subspace generated by $(\mathbf{e}_i, \mathbf{e}_{i+1}, \mathbf{e}_{i+3})$ and "$\wedge$" in V_i denotes the usual vector product. The endomorphism $b \to ab$ from $\mathbb{O}$ to $\mathbb{O}$ can then be represented by

$$r(a) = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_4 & -a_7 & a_2 & -a_6 & a_5 & a_3 \\ a_2 & a_4 & a_0 & -a_5 & -a_1 & a_3 & -a_7 & a_6 \\ a_3 & a_7 & a_5 & a_0 & -a_6 & -a_2 & a_4 & a_1 \\ a_4 & -a_2 & a_1 & a_6 & a_0 & -a_7 & -a_3 & a_5 \\ a_5 & a_6 & -a_3 & a_2 & a_7 & a_0 & -a_1 & -a_4 \\ a_6 & -a_5 & -a_7 & -a_4 & a_3 & a_1 & a_0 & -a_2 \\ a_7 & -a_3 & -a_6 & a_1 & -a_5 & a_4 & a_2 & a_0 \end{pmatrix}.$$

The conjugate of a is $\bar{a} = (a_0, -\mathbf{a})$, the square norm of a is $a\bar{a} = \bar{a}a = N(a) = a_0^2 + \mathbf{a}^2$, $\operatorname{Re}(a) = a_0$.

The cone $\Pi_3(\mathbb{O})$ is therefore the set of matrices

$$x = \begin{pmatrix} \alpha_1 & c & \bar{b} \\ \bar{c} & \alpha_2 & a \\ b & \bar{a} & \alpha_3 \end{pmatrix},$$

where α_1, α_2 and α_3 are real and a, b, and c are elements of $\mathbb{O}$. The square of x is well defined but since the multiplication in $\mathbb{O}$ is not associative, we define x^3 with Jordan product

$$x^3 = \frac{1}{2}\big(x^2 x + x x^2\big).$$

It can then be verified [see Casalis (1990)] that

$$P_3(x) = x^3 - \alpha x^2 + \beta x - \gamma = 0,$$

where

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3,$$
$$\beta = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 - N(a) - N(b) - N(c),$$
$$\gamma = \alpha_1\alpha_2\alpha_3 - \alpha_1 N(a) - \alpha_2 N(b) - \alpha_3 N(c) + 2\,\mathrm{Re}(abc).$$

It follows that the rank of $\Pi_3(\mathbb{O})$ is equal to 3, tr $x = \alpha_1+\alpha_2+\alpha_3$ and det $x = \gamma$. One can observe that γ is obtained formally by calculating the determinant in the usual way. The family F_p of densities is now well defined on $\Pi_3(\mathbb{O})$.

For $\Omega = \Lambda_n$, let $\bar{y} = (y_1, -y_2, \dots, -y_n)$ be the parameter. Then tr $x\bar{y} = 2B(x,y)$ and det $x = B(x,x)$, where $B(x,y) = x_1 y_1 - x_2 y_2 - \cdots - x_n y_n$. Expression (3.1) becomes

$$\frac{B(x,x)^{-n/2}}{(2\pi)^{(n-2)/2}\Gamma(p)\Gamma(p-(n-2)/2)} B(x,x)^p B(y,y)^p \exp(-2B(x,y)).$$

If we replace y by $y/2$ we obtain the distribution on Λ_n as defined in Letac (1994) and used in Massam (1989b).

The models in the family F_p are obviously exponential transformation models on Ω with acting group G and with density

$$\frac{1}{\Gamma_\Omega(p)} \exp(-(x \mid y))(\det x)^p(\det y)^p,$$

with respect to the G-invariant measure $(\det x)^{-n/r}\,dx$. Under this group G, there is only one orbit in Ω. However, we will show that the group G on Ω can always be decomposed into the direct product of $\mathbb{R}^+$ and another group, so that we can define a set of orbits in Ω and give an orbital decomposition of x into $x = us$, where u is a maximal invariant statistic and $s = x/u$. In the next section, we will investigate the distribution of u, s and s given u. We will also look at the distributions of quantities related to the R_i, $i = 2, \dots, q$, in Theorem 3.1. These will lead to the generalized beta and Dirichlet distributions.

4. Some distributions derived from the family F_p.

4.1. *The generalized beta and Dirichlet distributions.* We observed in (3.2) that the sth moment of

$$D_i = \frac{\det x_i \det(x_1 + \cdots + x_{i-1})^{i-1}}{(\det(x_1 + \cdots + x_i))^i} \quad \text{is} \quad \frac{B_\Omega(p+s, (i-1)(p+s))}{B_\Omega(p, (i-1)p)}.$$

In the case where the x_i are positive $m \times m$ matrices with real entries, D_i can be rewritten as

$$D_i = \det U_i(I - U_i)^{i-1}, \quad \text{where } U_i = (x_1 + \cdots + x_i)^{-1/2} x_i (x_1 + \cdots + x_i)'^{-1/2},$$

and $x^{1/2}$ is a lower triangular matrix with positive diagonal elements such that $x^{1/2}(x^{1/2})' = x$. This could also be expressed by setting $x_1 + \cdots + x_i = t_i I$, where t_i

is an element of the triangular subgroup of G, and defining U_i by $x_i = t_i U_i$. It is a classical result that U_i has the multivariate beta$(p,(i-1)p)$ distribution [see Muirhead (1982), Theorem 3.3.1]. Similarly, we can define a generalized beta distribution for any symmetric cone Ω and even more generally a Dirichlet distribution.

If an element x in Ω has the distribution with density (3.1) on Ω, with respect to the Lebesgue measure, we will say, for convenience, that it has the $W(p,y)$ distribution.

Consider q independent variables $x_1,\dots,x_q$, each x_j with the $W(p_j,y)$ distribution. We show later how we can generate $q-1$ variables $s_1,\dots,s_{q-1}$ with a generalized Dirichlet distribution.

THEOREM 4.1. *Let $x_1,\dots,x_q$ be q independent variables, where each x_j has the $W(p_j,y)$ distribution on the symmetric cone Ω. Let z_q be the sum $z_q = x_1+\cdots+x_q$, and let t be the unique element in the triangular subgroup of G such that $z_q = te$. Define the variables $s_1,\dots,s_q$ as $x_i = ts_i$, $i=1,\dots,q$. Then z_q is independent of $(s_1,\dots,s_{q-1})$ and the joint distribution of $(s_1,\dots,s_{q-1})$ with respect to the Lebesgue measure on*

$$\Omega_{q-1,e} = \left\{(s_1,\dots,s_{q-1})\colon s_i \in \Omega, i=1,\dots,q-1 \text{ and } \sum_{i=1}^{q-1} s_i \in e-\Omega\right\}$$

is

$$\begin{aligned}\frac{\Gamma_\Omega\left[\sum_{i=1}^q p_i\right]}{\prod_{i=1}^q \Gamma_\Omega(p_i)}\left(\det s_1\right)^{p_1-n/r}\cdots\left(\det s_{q-1}\right)^{p_{q-1}-n/r}&\\ \times\det\left(e-s_1-\cdots-s_{q-1}\right)^{p_q-n/r}.&\end{aligned}\tag{4.1}$$

PROOF. The joint density of $x_1,\dots,x_q$ with respect to the Lebesgue measure is

$$\begin{aligned}\frac{1}{\prod_{i=1}^q \Gamma_\Omega(p_i)}\left(\det x_1\right)^{p_q-n/r}\cdots\left(\det x_q\right)^{p_q-n/r}&\\ \times\exp\left[-\left(x_1+\cdots+x_q\mid y\right)\right]\left(\det y\right)^{p_1+\cdots+p_q}.&\end{aligned}$$

Using (2.16) and (2.17) we find that the Jacobian for the change of variables, $(x_1,\dots,x_q)\to(te,s_1,\dots,s_{q-1})$ is equal to $(\det te)^{(q-1)n/r}$. Observing that $s_q = e-s_1-\cdots-s_{q-1}$, the density with respect to $dte\, ds_1\cdots ds_{q-1}$ is

$$\begin{aligned}\frac{1}{\prod_{i=1}^q \Gamma_\Omega(p_i)}\left(\det s_1\right)^{p_1-n/r}\cdots\left(\det s_{q-1}\right)^{p_{q-1}-n/r}\det\left(e-s_1-\cdots-s_{q-1}\right)^{p_q-n/r}&\\ \times e^{-(te|y)}\left(\det y\right)^{p_1+\cdots p_q}\left(\det te\right)^{p_1+\cdots+p_q-q(n/r)+(q-1)n/r},&\end{aligned}$$

which can be rewritten as

$$\frac{\Gamma_\Omega\left[\sum_{i=1}^q p_i\right]}{\prod_{i=1}^q \Gamma_\Omega(p_i)}\left(\det s_1\right)^{p_1-n/r}\cdots\left(\det s_{q-1}\right)^{p_{q-1}-n/r}\det\left(e-s_1-\cdots-s_{q-1}\right)^{p_q-n/r}$$
$$\times\frac{1}{\Gamma_\Omega\left[\sum_{i=1}^q p_i\right]}e^{(te|y)}\left(\det te\right)^{p_1+p_2+\cdots+p_q-n/r}\left(\det y\right)^{p_1+\cdots+p_q}.$$

It is now clear that $te = x_1 + \cdots + x_q$ and $(s_1, \cdots, s_{q-1})$ are independent, that $x_1 + \cdots + x_q$ has the $W\left[\sum_{i=1}^q p_i, y\right]$ distribution and that $(s_1, \ldots, s_{q-1})$ have the generalized Dirichlet density on $\Omega_{q-1,e}$ with density (4.1). Since $x_1 + \cdots + x_q$ has the $W(\Sigma p_i, y)$ distribution, the $W(p,y)$ model is reproductive. From this and the fact that $E(x) = py^{-1}$, it follows that the first part of Theorem 3.1 is a direct consequence of the results in Barndorff-Nielsen and Blaesild (1983a). (Our proof of Theorem 3.1 is more constructive.)

Applying Theorem 4.1 to two variables only, one generates immediately the generalized beta distribution on $\Omega_{1,e}$. □

THEOREM 4.2. *Let x_1 and x_2 be independent variables with distribution $W(p,y)$ and $W(q,y)$, respectively. Let $x_1 + x_2 = te$, where t is the unique element in the triangular subgroup of G satisfying this identity. The variable s_1 defined by $x_1 = ts_1$ has the generalized beta(p,q) distribution on $\Omega_{1,e}$ with density*

$$\frac{1}{B_\Omega(p,q)}\left(\det s_1\right)^{p-n/r}\det\left(e-s_1\right)^{q-n/r}.$$

Theorem 4.2 is an immediate consequence of Theorem 4.1.

If $(x_1, \ldots, x_q)$ is a sample from the distribution $W(p,y)$, then by Theorem 4.2 we have that, for each $i = 2, \ldots, q, s_i$ defined by $x_i = t_i s_i$, where $t_i e = x_1 + \cdots + x_i$, has the generalized beta$(p,(i-1)p)$ distribution. The variable s_i is the variable appearing in the quantity R_i, $i = 2, \ldots, q$, in Theorem 3.1, which could be rewritten as follows.

COROLLARY. *Let $(x_1, \ldots, x_q)$ be a sample from the distribution $W(p,y)$ on Ω. Then the exponent $E = \sum_{i=1}^q \left(u(x_i) - \left(u'(m) \mid x_i - m\right) - u(m)\right)$ of the joint distribution can be split into q independent components*:

$$q\left(\log\det\bar{x}_q + \log\det y\right) - \frac{1}{p}\operatorname{tr}\bar{x}_q y = R_1 + D,$$
$$\log\left(\det(s_i)\det\left(e-s_i\right)^{i-1}\right) = R_i + D_i, \qquad i = 2, \ldots, q,$$

where $s_i = t_i^{-1}x_i, i = 2, \ldots, q$, $t_i e = x_1 + \cdots + x_i$, D is a constant and $D_i, i = 2, \ldots, q$ are constants depending on i, and where R_1 and $R_i, i = 2, \ldots, q$, are as defined in Theorem 3.1. Moreover $s_i, i = 2, \ldots, q$, has the generalized beta$(p,(i-1)p)$ distribution.

4.2. *Distributions of the maximal invariant and of the direction in the orbital decomposition of x in Ω.* Consider the cone Ω and the connected subgroup G of $G(\Omega)$, containing the identity, and let x have the $W(p,y)$ distribution with density

$$\frac{1}{\Gamma_\Omega(p)}(\det x)^p e^{-(x|y)}(\det y)^p$$

with respect to the invariant measure $d\mu(x) = (\det x)^{-n/r}\,dx$. Similarly to what is done in Barndorff-Nielsen (1988) or Barndorff-Nielsen, Blaesild and Eriksen (1989) in general for exponential transformation models, we will derive the distribution of the statistic $u(x) = (\det x)^{1/r}$, which is maximal invariant under the action of a subgroup $G^\dagger$ of G that we will define later and the distribution of the direction $s(x) = x/u(x)$. We will also show how the hyperboloid distribution as defined in Jensen (1981) is a special case of the conditional distribution of s given u. We will also derive the distributions of the sum $s_{.q}$ of $s_1, \dots, s_q$, the "length" $R_q = (\det s_{.q})^{1/r}$ and $s_{.q}/R_q$ given R_q. From now on, R_q is the quantity just defined. This notation has been used to follow the notation in Jensen (1981) and should not be confused with the qth component R_i mentioned so far in this paper.

We will now give an orbital decomposition of x. According to the polar decomposition of the automorphism group (see Section 2), any g in G can be written as $g = P(y)k$ for some y in Ω and $k \in K = G \cap O(E)$. Let us decompose g further into

$$g = (\det y)^{2/r}\frac{P(y)k}{(\det y)^{2/r}} \in \mathbb{R}^+ \times G^\dagger,$$

where $G^\dagger = \left\{P(y)k/(\det y)^{2/r}, y \in \Omega, k \in K\right\}$ is a subgroup of G, as can be readily verified. Indeed, from (2.15) and (2.16) and the fact that $\det k = 1$ if $k \in K$ it follows that $\det h = 1$ if $h \in G^\dagger$. The identity element belongs to $G^\dagger$ and if $P(y)k/(\det y)^{2/r}$ belongs to $G^\dagger$, its inverse $P(y^{-1})k^{-1}/(\det y^{-1})^{2/r}$ also belongs to $G^\dagger$.

Consider the function $u^r(x) = \det x$. It is invariant under $G^\dagger$. Indeed

$$\det\left[\frac{P(y)}{(\det y)^{2/r}}k(x)\right] = \frac{1}{(\det y)^{2r/r}}\det\left(P(y)k(x)\right),$$

by homogeneity of degree r of the determinant function. Let $h = P(y)k/(\det y)^{2/r}$ in $G^\dagger$ be given, for some $y \in \Omega$ and $k \in K$. By (2.15) and (2.16),

$$u^r(h(x)) = \frac{(\det y)^2}{(\det y)^2}\det k(x)$$

and since $k \in K$, $\det k(e) = 1$ and $u^r(hx) = \det x = u^r(x)$. The statistic $u(x)$ is therefore invariant under $G^\dagger$ and we can define orbits in Ω by $\det x$ = constant.

The statistic $u(x)$ is a maximal invariant statistic under $G^{\uparrow}$ and $(u, s = x/u)$ is an orbital decomposition of x. The unit orbit is the set $H_1 = \{x \in \Omega \mid \det x = 1\}$. Clearly Ω is the direct product of $\mathbb{R}^+$ and H_1 and therefore the G-invariant measure $d\mu(x) = (\det x)^{-n/r}dx$ factorizes into the product $d\rho(u) \times d\sigma(s)$, where $d\rho(u)$ is an $\mathbb{R}^+$-invariant measure on $\mathbb{R}^+$ and $d\sigma(s)$ is an invariant measure on H_1. To find this factorization, we use the polar decomposition of $x = k\sum_{j=1}^{r} e^{t_j} c_j$ given in (2.7) and the integral formula (2.8), giving

$$(4.2) \qquad (\det x)^{-n/r}\, dx = c2^{n-r} \prod_{j<k} \left[\sinh \left[\frac{t_k - t_j}{2} \right] \right]^d dk\, dt_1 \cdots dt_r.$$

To split this into the product of a measure for u and a measure for s, we change variables. Since the c_j are orthogonal idempotents one can easily prove that $\det x = \prod_{j=1}^{r} e^{t_j} = \exp\left(\sum_{j=1}^{r} t_j\right)$. Therefore $\log u^r = \sum_{j=1}^{r} t_j$. Let us introduce the new variables

$$s_j = t_r - t_j, \qquad j = 1, \ldots, r-1,$$
$$u = \exp\left[\frac{t_1 + \cdots + t_{r-1} + t_r}{r}\right].$$

The Jacobian is $J = \left|\partial(s_1, \ldots, s_{r-1}, u)/\partial(t_1, \ldots, t_{r-1}, t_r)\right| = (r/r)u$.

So $dt_1 \cdots dt_r = (du/u)\, ds_1 \cdots ds_{r-1}$. Moreover, for $j < k < r$, $t_k - t_j = s_j - s_k$ and therefore

$$(4.3) \qquad \begin{aligned} &(\det x)^{-n/r}\, dx \\ &= c2^{n-r} \prod_{j<k<r} \left[\sinh \left[\frac{s_j - s_k}{2} \right] \right]^d \prod_{i=1}^{r-1} \left[\sinh \frac{s_i}{2} \right]^d \frac{du}{u}\, dk\, ds_1 \cdots ds_{r-1}, \end{aligned}$$

which gives us immediately the required splitting $d\mu(x) = (\det x)^{-n/r}\, dx = d\rho(u) \times d\sigma(s)$, where $d\rho(u) = du/u$ and $d\sigma(s)$ is the cofactor of du/u in the right-hand side of (4.3). Letting $y = \kappa\xi$, where $\kappa = (\det y)^{1/r}$ and $\xi \in H_1$, the density of x w.r.t. $d\mu$ is $[1/\Gamma_\Omega(p)]\kappa^{rp}u^{rp} \exp[-\kappa u(s|\xi)]$ and therefore the density of u w.r.t. $d\rho$ is

$$(4.4) \qquad \frac{1}{\Gamma_\Omega(p)} \kappa^{rp} u^{rp} \int_{H_1} \exp\left[-\kappa u(s|\xi) \right] d\sigma(s).$$

Let us show that the integral in (4.4) is independent of ξ and therefore that the density of u is independent of ξ. Indeed, for any $h \in G^{\uparrow}$, there exists an $h_1 \in G^{\uparrow}$ such that $(s|h\xi) = (h_1 s|\xi)$, where $h_1 = h^{-1}$ if $h = k \in K$ and $h_1 = h$ if $h = P(y)$, with $y \in \Omega$, $\det y = 1$; since σ is invariant under $G^{\uparrow}$, we have

$$\begin{aligned} \int_{H_1} \exp\left[-\kappa u(s|h\xi) \right] d\sigma(s) &= \int_{H_1} \exp\left[-\kappa u(h_1 s|\xi) \right] d\sigma(s) \\ &= \int_{H_1} \exp\left[-\kappa u(s|\xi) \right] d\sigma(s). \end{aligned}$$

Moreover, $G^\dagger$ acts transitively on H_1; for any ξ, ξ' in H_1, there exists an $h \in G^\dagger$ such that $\xi' = h\xi$ and therefore $\int_{H_1} \exp[-\kappa u(s|\xi)]\, d\sigma(s) = \int_{H_1} \exp[-\kappa u(s|\xi')]\, d\sigma(s)$, that is, the density of u is independent of ξ.

We can also regard the family of densities of x as a transformation model with acting group $G^\dagger$: This is clear if we write the density of x as

$$\frac{1}{\Gamma_\Omega(p)} \kappa^{rp} u^{rp} \exp\left[-\kappa u(s|\xi)\right] d\rho(u)\, d\sigma(s)$$

and recall the following: first, that any ξ in H_1 can be written as $\xi = he$ for some h in $G^\dagger$; second, that for any $h \in G^\dagger$ there exists $h_1 \in G^\dagger$ such that $(s|h\xi) = (h_1 s|\xi)$; finally, that σ is invariant under $G^\dagger$. Now since κ is invariant under $G^\dagger$ it follows that the marginal density of u is independent of $G^\dagger$. This is a special case of a general result concerning the density of the maximal invariant in an exponential transformation model, given in Barndorff-Nielsen, Blaesild and Eriksen [(1989), Theorem 2.1; henceforth abbreviated as BNBE].

To compute $\int_{H_1} \exp[-\kappa u(s|\xi)]\, d\sigma(s)$, we need to express $(s|\xi)$ in terms of $s_1, \dots, s_{r-1}$. We have

$$\begin{aligned}
t_1 &= \log u - \frac{s_1}{r} + \frac{s_2 - s_1}{r} + \cdots + \frac{s_{r-2} - s_1}{r} + \frac{s_{r-1} - s_1}{r},\\
t_2 &= \log u + \frac{s_1 - s_2}{r} - \frac{s_2}{r} + \frac{s_3 - s_2}{r} + \cdots + \frac{s_{r-1} - s_2}{r},\\
&\cdots\\
t_{r-1} &= \log u + \frac{s_1 - s_{r-1}}{r} + \frac{s_2 - s_{r-1}}{r} + \cdots + \frac{s_{r-2} - s_{r-1}}{r} - \frac{s_{r-1}}{r},\\
t_r &= \log u + \frac{s_1}{r} + \frac{s_2}{r} + \cdots + \frac{s_{r-2}}{r} + \frac{s_{r-1}}{r};
\end{aligned}$$

therefore,

$$x = k\sum_{j=1}^{r} e^{t_j} c_j = ku\left(\sum_{j=1}^{r-1} \exp\left[-\frac{s_j}{r} + \frac{1}{r}\sum_{i \neq j}(s_i - s_j)\right] c_j + \exp\left(\sum_{i=1}^{r-1} \frac{s_i}{r}\right) c_r\right).$$

A special case—the Lorentz cone and the hyperboloid distribution. The integral $I = \int_{H_1} \exp\left[-\kappa u(s|\xi)\right] d\sigma(s)$ is not very pleasant except in the case of the Lorentz cone. Then indeed $r = 2$. The only variables are u and s.

Since I is independent of ξ,

$$\int_{H_1} \exp[-\kappa u(s|\xi)]\, d\sigma(s) = \int_{H_1} \exp[-\kappa u(s|e)]\, d\sigma(s).$$

Moreover, $k \in K$ is an automorphism and preserves the trace [see Faraut (1988), II.4.3] therefore $(s|e) = \operatorname{tr} s = \operatorname{tr} k(e^{s_1/2}c_1 + e^{-s_1/2}c_2) = \operatorname{tr}(e^{s_1/2}c_1 + e^{-s_1/2}c_2) = 2\cosh(s_1/2)$. Following the pattern of integration in (2.8), we then have

$$\int_{H_1} \exp[-\kappa u(s|\xi)]\,d\sigma(s) = \int_{H_1} \exp[-\kappa u(s|e)]\,d\sigma(s)$$

$$(4.5)\qquad = \int_K \int_0^\infty c2^{n-2} \exp\left(-2\kappa u \cosh \frac{s_1}{2}\right)\left(\sinh \frac{s_1}{2}\right)^{n-2} dk\,ds_1$$

$$= \int_K c2^{n-2}\,dk \int_0^\infty \exp\left(-2\kappa u \cosh \frac{s_1}{2}\right)\left(\sinh \frac{s_1}{2}\right)^{n-2} ds_1.$$

Since $\int_K dk = 1$ for the normalised Haar measure, we have

$$\int_{H_1} \exp[-\kappa u(s|\xi)]\,d\sigma(s)$$

$$= c2^{n-1} \int_0^\infty \exp\left(-2\kappa u \cosh \frac{s_1}{2}\right)\left(\sinh \frac{s_1}{2}\right)^{2(n-2)/2} d\left(\frac{s_1}{2}\right)$$

$$= c2^{n-1} \frac{\Gamma\left(\frac{n-2}{2}+\frac{1}{2}\right)}{(\kappa u)^{(n-2)/2}\Gamma\left(\frac{1}{2}\right)} K_{(n-2)/2}(2\kappa u)$$

$$= c\frac{2^{n-1}\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}(\kappa u)^{(n-2)/2}} K_{(n/2)-1}(2\kappa u),$$

where K_ν is the modified Bessel function of the third kind of index ν [see Abramovitz and Stegun (1970), formula 9.6.23, for this integral representation of the modified Bessel function of the third kind]. We still have to compute the constant c, which depends only on the Jordan algebra containing Ω. The simplest way to compute this constant is to look upon it as a normalizing constant for the marginal density of u. The condition $\int_0^\infty f(u)\,du/u = 1$, where

$$f(u) = \frac{1}{\Gamma_\Omega(p)} (\kappa u)^{2p} \int_{H_1} \exp[-\kappa u(s|\xi)]\,d\sigma(s),$$

yields

$$\frac{c2^{n-1}\Gamma((n-1)/2)}{\sqrt{\pi}\Gamma_\Omega(p)} \int_0^\infty (\kappa u)^{2p-(n/2)+1} K_{(n/2)-1}(2\kappa u)\frac{du}{u}$$

$$= \frac{c2^{n-1}\Gamma((n-1)/2)}{\Gamma(\frac{1}{2})\Gamma_\Omega(p)} 2^{-(2p-(n/2)+1)} \int_0^\infty (2\kappa u)^{2p-n/2} K_{(n/2)-1}(2\kappa u)\,d(2\kappa u)$$

$$= 1.$$

Since

$$\int_0^\infty t^\mu K_\nu(t)\,dt = 2^{\mu-1}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)$$

[formula 11.4.22 in Abramovitz and Stegun (1970)], the normalization condition gives

$$\frac{c2^{n/2}\Gamma\left(\frac{n-1}{2}\right)}{\pi^{(n/2)-1}2^2\Gamma\left(\frac{1}{2}\right)} = 1.$$

Thus

$$c = \frac{2^2\Gamma\left(\frac{1}{2}\right)\pi^{(n/2)-1}}{2^{n/2}\Gamma\left((n-1)/2\right)},$$

and the density of u with respect to $d\rho(u) = du/u$ is

$$f(u) = \frac{2^{(n/2)+1}\pi^{(n/2)-1}}{\Gamma_\Omega(p)}(\kappa u)^{2p}\frac{K_{(n/2)-1}(2\kappa u)}{(\kappa u)^{(n/2)-1}}. \tag{4.6}$$

The conditional density of s given u, with respect to $d\sigma$, is easily obtained by dividing the density of x by the marginal density of u,

$$f(s|u) = \frac{(\kappa u)^{(n-2)/2}\exp\left[-\kappa u(s|\xi)\right]}{2^2(2\pi)^{(n-2)/2}K_{(n/2)-1}(2\kappa u)}.$$

For $u = 1$, when we condition on the unit orbit, we obtain exactly the density for the hyperboloid distribution given in Jensen [(1981), formulae (6) and (8)], where $s * \xi = \frac{1}{2}(s|\xi)$. The hyperboloid distribution has been obtained as a conditional distribution in a similar way in BNBE (1989).

The general case. Since we cannot compute the integral in (4.4) directly, we consider the moments of $\det x$ or, more conveniently, of $\det x\ \det y$. The moments are

$$\begin{aligned} E\left((\det x\ \det y)^s\right) &= \frac{1}{\Gamma_\Omega(p)}\int_\Omega \left((\det x)(\det y)\right)^{p+s} e^{-(x|y)}(\det x)^{-(n/r)}\,dx \\ &= \frac{\Gamma_\Omega(p+s)}{\Gamma_\Omega(p)} = \prod_{i=0}^{r-1}\frac{\Gamma(p+s-id/2)}{\Gamma(p-id/2)}, \end{aligned}$$

from which it follows immediately that $\det x\ \det y$ is distributed like the product of r random variables U_i, where $2U_i$ has a chi-square $(2p-id)$ distribution. This result is well known in the special case of the Wishart distribution, where $y = (2\Sigma)^{-1}$, $d = 1$ and $r = m$: then $\det x\Sigma^{-1}$ is distributed like the product of m random variables with the chi-square $(2p - i)$ density, $i = 0, \dots, m-1$.

We can also obtain the explicit expression of the density of $\det x\ \det y$ by using the inverse Mellin transforms. Using again the notation $\kappa^r = \det y$, from the expression of the moments above, we obtain immediately the density of $\kappa^r u^r$ with respect to $d(\kappa^r u^r)/\kappa^r u^r = r\,du/u$ in terms of the G-function

$$h(\kappa^r u^r) = \frac{1}{\prod_{i=0}^{r-1}\Gamma(p-id/2)} G^{r0}_{0r}\left(\kappa^r u^r \middle| p, p-\frac{d}{2}, \dots, p-(r-1)\frac{d}{2}\right),$$

where G^{r0}_{0r} is a generalized hypergeometric function [see Mathai and Saxena (1973)] for a thorough treatment of hypergeometric functions and their application in statistics). The density of u with respect to the measure du/u is therefore

$$\frac{(2\pi)^{(n-r)/2}r}{\Gamma_\Omega(p)}G^{r0}_{0r}\left(\kappa^r u^r \middle| p, p-\frac{d}{2},\ldots,p-(r-1)\frac{d}{2}\right), \tag{4.7}$$

and the conditional density of $s|u$ with respect to $d\sigma$ is

$$\frac{(\kappa u)^{rp}\ \exp\left[-\kappa u(s|\xi)\right]}{(2\pi)^{(n-r)/2}rG^{r0}_{0r}\left(\kappa^r u^r|p,p-\frac{d}{2},\ldots,p-(r-1)\frac{d}{2}\right)}. \tag{4.8}$$

There are several interesting special cases when G^{r0}_{0r} is particularly simple. Of course, we can derive the hyperboloid distribution in the case of the Lorentz cone, in yet another way when $r = 2$ and $d = n-2$. Letting $u = 1$, (4.8) becomes

$$\frac{\kappa^{2p}\exp\left[-\kappa(s|\xi)\right]}{(2\pi)^{d/2}2G^{20}_{02}(\kappa^2|p,p-d/2)},$$

with $G^{20}_{02}(\kappa^2 \mid p,p-d/2) = 2\kappa^{2p-d/2}K_{d/2}(2\kappa)$, and we obtain the density of s as

$$\frac{(\kappa)^{d/2}\exp\left[-\kappa(s|\xi)\right]}{(2\pi)^{d/2}2^2K_{d/2}(2\kappa)} = \frac{(\kappa)^{(n/2)-1}\exp\left[-\kappa(s|\xi)\right]}{(2\pi)^{(n/2)-1}2^2K_{(n/2)-1}(2\kappa)},$$

which is of course the same as the density derived previously in the special case of the Lorentz cone and the hyperboloid distribution.

When $r = 4$,

$$G^{40}_{04}\left(x \mid a,a+\frac{1}{2},b,b+\frac{1}{2}\right) = 4\pi x^{(a+b)/2}K_{2(a-b)}(4x^{1/4}).$$

Then if x is a 4×4 symmetric matrix with the Wishart $W(p,\xi)$ distribution, the conditional density of $x = s$ given that $\det x = 1$, with respect to $d\sigma$, is

$$\frac{\kappa^{4p}\exp\left[-\kappa(s|\xi)\right]}{(2\pi)^{(n-r)/2}4^2\pi\kappa^{4p-2}K_2(4\kappa)}.$$

Since $n = (4\times 3)/2$, $\kappa\xi = 2\Sigma^{-1}$ and $\kappa^4 = \det(\Sigma^{-1}/2) = (1/2^4)\det\Sigma^{-1}$, the preceding density can be written as

$$\frac{\det\Sigma^{-1/2}\exp\left(-\operatorname{tr} s\Sigma^{-1}\right)}{2^7\pi^2K_2\left(2\det\Sigma^{-1/4}\right)}.$$

These results can be summarized in the following theorem.

THEOREM 4.3. *Let x have a $W(p,y)$ distribution on a symmetric cone Ω. Then (u,s), where $u = (\det x)^{1/r}$ and $s = x/u$, forms an orbital decomposition of x. The group G factorizes into $\mathbb{R}^+\times G^\dagger$, where*

$$G^\dagger = \left\{\frac{P(y)}{(\det y)^{2/r}}k \text{ for some } y\in\Omega,\ k\in K = G\cap O(E)\right\}.$$

The G-invariant measure $(\det x)^{-n/r}\,dx$ *factorizes into* $(du/u)\,d\sigma(s)$, *where* du/u *is invariant on* $\mathbb{R}^+$ *and* $d\sigma(s)$ *is invariant on the unit orbit defined by* $\det x = 1$. *The density of the distribution of* $u = (\det x)^{1/r}$, *with respect to* du/u, *is*

$$\frac{(2\pi)^{(n-r)/2}r}{\Gamma_\Omega(p)}G^{r0}_{0r}\left(\kappa^r u^r \middle| p, p-\frac{d}{2},\dots,p-(r-1)\frac{d}{2}\right).$$

The conditional density of $s = x/(\det x)^{1/r}$ *given* $u = (\det x)^{1/r}$, *with respect to* $d\sigma(s)$, *is*

$$\frac{(\kappa u)^{rp}\exp\left[-\kappa u(s|\xi)\right]}{(2\pi)^{(n-r)/2}rG^{r0}_{0r}(\kappa^r u^r | p, p-d/2,\dots,p-(r-1)d/2)}.$$

Moreover, $\det x\ \det y$ *is distributed like the product of r random variables* U_i, *each* U_i, *being such that* $2U_i$ *has the chi-square* $(2p - id)$ *distribution.*

To carry on with the unified treatment of different distributions, we now consider a sample $(s_1,\dots,s_q)$ from the conditional distribution of s given $u = 1$, henceforth denoted for convenience as the H-distribution. We will show that in a manner exactly parallel to what is happening for the hyperboloid distribution, the density of $s_{.q}/R_q$ given R_q has the same density but with precision parameter κ replaced by $R_q\kappa$.

4.3. *Distribution of the resultant length and direction for a sample from the H-distribution.* We denote the density (4.8) for $u = 1$ as

$$h(s) = \frac{\kappa^{rp}\exp\left[-\kappa(s|\xi)\right]}{(2\pi)^{(n-r)/2}rG^{r0}_{0r}(\kappa^r)},$$

where we use the notation $G^{r0}_{0r}(\kappa^r \mid p, p-d/2,\dots,p-(r-1)d/2) = G^{r0}_{0r}(\kappa^r)$ since the parameters are always the same from now on. Consider a sample $(s_1,\dots,s_q)$ from this H-distribution. Let $s_{.q} = s_1+\cdots+s_q$ and $R_q = (\det s_{.q})^{1/r}$. The joint density of $(s_1,\dots,s_q)$ with respect to the $G^\dagger$ invariant measure $\otimes^q d\sigma(s)$ is

$$h(s_1,\dots,s_q) = \frac{\kappa^{rpq}\exp\left[-R_q\kappa(s_{.q}/R_q|\xi)\right]}{\left[(2\pi)^{(n-r)/2}r\right]^q\left[G^{r0}_{0r}(\kappa^r)\right]^q}.$$

Obviously $s_{.q}\in\Omega$, $s_{.q}/R_q$ belongs to the unit orbit H_1, R_q is a maximal invariant under $G^\dagger$ and $(R_q, s_{.q}/R_q)$ forms an orbital decomposition of $s_{.q}$ in Ω.

Let φ be the mapping from H_1^q to $H_1\times\mathbb{R}^+$ defined by $\varphi(s_1,\dots,s_q) = (s_{.q}/R_q, R_q)$, and let ν be the image by φ of $\otimes^q\sigma$. Then ν is invariant on $H_1\times\mathbb{R}^+$. To decompose ν into the product of the invariant measure σ on H_1 and a measure Ψ_q on $\mathbb{R}^+$ we first need to prove that the action of $G^\dagger$ on H_1 is proper (so that an invariant measure on H_1 is unique up to a positive multiplicative constant) and that the mapping φ is proper.

Let us first prove that φ is proper. Since $\varphi(s_1,\dots,s_q) = (s_{.q}/R_q, R_q)$ is an orbital decomposition of $t(s_1,\dots,s_q) = \Sigma_{i=1}^q s_i$, it is equivalent to prove that $t: H_1^q \to \Omega$ is proper, that is, the inverse image of a compact set in Ω is compact. Since t is continuous, the inverse image is closed; let us prove it is also bounded. Suppose that $\|\Sigma s_i\|$ for $(s_1,\dots,s_q) \in H_1^q$ is bounded. Let $x = \Sigma\lambda_i c_i$, with $\lambda_i > 0$, be the spectral decomposition of $x \in \Omega$. On Ω, $\|x\| = (\operatorname{tr} x^2)^{1/2} = (\Sigma\lambda_i^2)^{1/2}$ and $\operatorname{tr} x = \Sigma\lambda_i$ are equivalent norms. Indeed, since $\lambda_i > 0$, we clearly have $\Sigma\lambda_i \geq (\Sigma\lambda_i^2)^{1/2}$ and by Schwarz inequality, we have $\operatorname{tr} x = \operatorname{tr} xe = \Sigma\lambda_i \leq (\Sigma\lambda_i^2)^{1/2}(\operatorname{tr} e)^{1/2}$. So $\|\Sigma s_i\|$ bounded is equivalent to $\operatorname{tr}\Sigma s_i = \Sigma \operatorname{tr} s_i$ bounded and, therefore, by the equivalence of norms each $\|s_i\|$ is bounded, which proves that t is proper and therefore φ is proper.

Let us now prove that the action of $G^\dagger$ on H_1 is proper. The isotropy group K of e, the identity element in H_1, is a compact subgroup of $G^\dagger$. By Proposition 2.3.11 of Wijsman (1990), it follows that the action of $G^\dagger$ on $G^\dagger/K$ is proper ($G^\dagger$ is a locally compact group). Moreover, from Lemma 2.3.17 of Wijsman (1990), since K is the isotropy group of e in H_1 we have that the 1–1 correspondence Ψ_e: $G^\dagger/K \to H_1$ defined by $\Psi_e(gK) = ge$ is a homeomorphism; therefore the action of $G^\dagger$ on H_1 is proper.

We can now apply Theorem 7.5.1 of Wijsman (1990). Indeed, φ is a proper mapping so that the image ν of $\otimes^q\sigma$ is a measure on Ω; $G^\dagger$ acts transitively and properly on H_1 so that an invariant measure on H_1 is unique up to a multiplicative constant and $G^\dagger$ acts trivially on $\mathbb{R}^+$; it follows from Theorem 7.5.1 that there exists a measure Ψ_q on $\mathbb{R}^+$ such that $\nu = \sigma \otimes \Psi_q$. Therefore the image of

$$\prod_{i=1}^q \frac{\kappa^{rp}\exp\left[-\kappa(s_i|\xi)\right]}{(2\pi)^{(n-r)/2} r G_{0r}^{r0}(\kappa^r)}\, d\sigma(s_i)$$

by the mapping φ is

$$\left(\frac{\kappa^{rp}}{(2\pi)^{(n-r)/2} r G_{0r}^{0r}(\kappa^r)}\right)^q \exp\left[-\kappa R_q\left(\frac{s_{.q}}{R_q}\Big|\xi\right)\right] d\sigma\left(\frac{s_{.q}}{R_q}\right) d\psi_q(R_q).$$

The density of R_q with respect to $d\psi_q$ is obtained by integration w.r.t. $d\sigma$,

$$f(R_q) = \left(\frac{\kappa^{rp}}{(2\pi)^{(n-r)/2} r G_{0r}^{r0}(\kappa^r)}\right)^q \int_{H_1} \exp\left[-\kappa R_q(s|\xi)\right] d\sigma(s).$$

From (4.4) and (4.7) we deduce that

$$f(R_q) = \frac{\kappa^{rpq} G_{0r}^{r0}(R_q^r\kappa^r)}{\left((2\pi)^{(n-r)/2} r\right)^{q-1} (R_q\kappa)^{rp} \left(G_{0r}^{r0}(\kappa^r)\right)^q}.$$

It follows that the conditional density of $s_{.q}/R_q$ given R_q, with respect to $d\sigma$, is equal to

$$f\left[\frac{s_{.q}}{R_q}\Big|R_q\right] = \frac{(R_q\kappa)^{rp}}{(2\pi)^{(n-r)/2} r G_{0r}^{r0}(R_q^r\kappa^r)} \exp\left(-R_q\kappa\left[\frac{s_{.q}}{R_q}\Big|\xi\right]\right).$$

The density of $s._q/R_q$ given R_q is obviously an H-distribution like the distribution of s but with precision parameter κ replaced by $R_q\kappa$. In the case of the Lorentz cone, this is obviously the same result as in Jensen [(1981), Theorem 2]. In the general case, we formulate the result as the following theorem.

THEOREM 4.4. *Let* $(s_1, \dots, s_q)$ *be a sample from the H-distribution on the unit orbit in* Ω. *Let* $s._q/R_q$ *and* R_q *be the resultant direction and "length," respectively, of the sum* $s._q = s_1 + \cdots + s_q$. *Then the density of* R_q *with respect to* $d\psi_q$ *is*

$$f(R_q) = (2\pi)^{(n-r)/2} r \left[\frac{\kappa^{rp}}{(2\pi)^{(n-r)/2} r G_{0r}^{r0}(\kappa^r)}\right]^q \frac{G_{0r}^{r0}(R_q^r \kappa^r)}{(R_q\kappa)^{rp}},$$

and the conditional distribution of $s._q/R_q$ *given* R_q *has density, with respect to* $d\sigma$, *equal to*

$$f\left[\frac{s._q}{R_q}\middle| R_q\right] = \frac{(R_q\kappa)^{rp}}{(2\pi)^{(n-r)/2} r G_{0r}^{r0}(R_q^r\kappa^r)} \exp\left(-R_q\kappa\left[\frac{s._q}{R_q}\middle|\xi\right]\right).$$

It should be noted that the fact that $s._q/R_q$ given R_q has a distribution of the same form as that of s on H_1 but with precision parameter κR_q instead of κ is due to the following: if s on H_1 has density

$$P_\theta(ds) = \alpha(\kappa)\exp\left[-\kappa(s|\xi)\right]\sigma(ds),$$

where $\theta = (\kappa, \xi), (\alpha(\kappa))^{-1} = \int \exp\left[-\kappa(s|\xi)\right]\sigma(ds)$ and $(s_1, \dots, s_q)$ is a sample from this distribution, then

$$P_\theta(ds_1, \dots, ds_q) = \alpha(\kappa)^q \exp\left[-\kappa R_q\left(\frac{s._q}{R_q}\middle|\xi\right)\right]\sigma(ds_1)\cdots\sigma(ds_q).$$

As was done before, the image of $\otimes^q\sigma$ can be decomposed into $\sigma \times \psi_q$. The conditional density of $s._q/R_q$ given R_q is immediately obtained as

$$\alpha(\kappa R_q)\exp\left[-\kappa R_q(s|\xi)\right]\sigma(ds).$$

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