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A UNIFIED APPROACH TO IMPROVING EQUIVARIANT ESTIMATORS

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In the point and interval estimation of the variance of a normal distribution with an unknown mean, the best affine equivariant estimators are dominated by Stein's truncated and Brewster and Zidek's smooth procedures, which are separately derived. This paper gives a unified approach to this problem by using a simple definite integral and provides a class of improved procedures in both point and interval estimation of powers of the scale parameter of normal, lognormal, exponential and Pareto distributions. Finally, the same method is applied to the improvement on the James–Stein rule in the simultaneous estimation of a multinormal mean.

1. Introduction. Let S and X be independent random variables, where S/σ^2 has a chi-square distribution χ_n^2 with n degrees of freedom and X has a normal distribution $N(\mu, \sigma^2)$ with an unknown mean μ. Assume that we want to estimate the unknown variance σ^2 by an estimator δ under the quadratic loss $(\delta/\sigma^2 - 1)^2$. Stein (1964) showed that the best equivariant estimator relative to the full affine transformation group is $\delta_0 = (n+2)^{-1}S$ and that it can be dominated by considering a class of scale equivariant estimators

$$\delta = \phi(W)S, \qquad W = X^2/S, \tag{1.1}$$

for a positive function ϕ. In the equivariant class, Stein (1964) found an improved estimator $\delta_S = \phi_S(W)S$ for

$$\phi_S(W) = \min\{(n+2)^{-1}, (n+3)^{-1}(1+W)\}. \tag{1.2}$$

Brewster and Zidek (1974) used an idea of Brown (1968) to develop the generalized Bayes and better estimator $\delta_{BZ} = \phi_0(W)S$ for

$$\phi_0(W) = \frac{\int_0^\infty v f_n(v) F_1(Wv)\, dv}{\int_0^\infty v^2 f_n(v) F_1(Wv)\, dv}, \tag{1.3}$$

where $f_k(v)$ and $F_k(v)$ designate the density and the distribution functions of χ_k^2. These procedures have been separately applied to the interval estimation of the normal variance and the estimation of the scale parameter of exponential and Pareto distributions. For the references, see Shorrock (1990), Goutis and Casella (1991) and Maatta and Casella (1990).

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In this paper, we employ a definite integral to derive a broader class of improved estimators $\delta(\phi) = \phi(W)S$, $\phi(\cdot)$ satisfying the following conditions:

CONDITION A. $\phi(w)$ is nondecreasing and $\lim_{w\to\infty} \phi(w) = 1/(n+2)$;
CONDITION B. $\phi(w) \geq \phi_0(w)$.

Taking $\phi(w) = \phi_0(w)$, the bound in Condition B, gives δ_{BZ}. Also since $\phi_0(w) \leq (1+w)/(n+3)$, the function $\phi_S(w)$ satisfies Conditions A and B, so that Stein's result follows. In this way, these two procedures of Stein and of Brewster and Zidek can be unified. Brewster and Zidek (1974) pointed out this fact in the normal case. More directly we derive it based on a simple definite integral. Furthermore, this approach allows us to generalize the underlying distributions and loss functions as well as to treat the problem of interval estimation.

In Section 2, we describe the distributional assumptions—for instance, a monotone likelihood ratio property, which is essential in this problem. The assumptions are satisfied by normal, lognormal, exponential and Pareto distributions. For loss functions with bowl shape, a definite integral is utilized to get unified conditions on $\phi(w)$ for the domination, and Brewster–Zidek–type estimators and Stein-type truncated ones are automatically presented. Similar results are given for interval estimation, which is discussed in Section 3. In Section 4, we deal with the problem of improving the James–Stein rule in the simultaneous estimation of a mean vector of a multivariate normal distribution and present analogous conclusions.

2. Point estimation of the scale parameter. Let S and T be independent random variables, where S/σ and T/σ have densities

$$g(v)I_{[v>0]} \quad \text{and} \quad h(u;\lambda)I_{[u>k(\lambda)]}, \tag{2.1}$$

for an unknown real parameter λ, a real function $k(\lambda)$ and the indicator function $I_{[\cdot]}$. It is noted that $T = X^2$, $\lambda = \mu^2/(2\sigma^2)$ and $k(\lambda) = 0$ in the normal case stated in Section 1. For $\alpha \neq 0$, we want to estimate σ^α by an estimator $\delta = \delta(S, T)$ relative to the loss function $L(\delta/\sigma^\alpha)$, where $L(t)$ is absolutely continuous and strictly bowl-shaped, that is, strictly decreasing for $t < 1$ and strictly increasing for $t > 1$. As a consequence L is differentiable almost everywhere. To guarantee interchange of limit and integration in the proofs, we assume that

$$\int_0^\infty L(cv^\alpha)g(v)\,dv < \infty \quad \text{and} \quad \int_0^\infty |L'(cv^\alpha)|v^\alpha g(v)\,dv < \infty \quad \text{for } c > 0.$$

It is also assumed that $\int_0^\infty |L'(cv^\alpha)|v^{\alpha+1}g(v)h(v)\,dv < \infty$ for $d > 0$ and $h(u) = h(u;0)$.

Let $H(x;\lambda) = \int_0^x h(u;\lambda)I_{[u>k(\lambda)]}\,du$ and $H(x) = \int_0^x h(u)\,du$. Assume the following:

(A.1) $H(x;\lambda)/H(x)$ is nondecreasing in $x > 0$.

If $h(x;\lambda)/h(x)$ is nondecreasing, (A.1) is guaranteed. Among estimators cS^{α}, for $c > 0$, the best c_0 is given as a solution to the equation

$$\int_0^{\infty} L'(c_0 v^{\alpha}) v^{\alpha} g(v)\, dv = 0. \tag{2.2}$$

From Brewster and Zidek (1974), it is seen that such a c_0 uniquely exists under the following assumption:

(A.2) $g(c_1 x)/g(c_2 x)$ is strictly increasing in x for $0 < c_1 < c_2$.

For improving on the estimator $\delta_0 = c_0 S^{\alpha}$, consider a class of estimators

$$\delta(\phi) = \begin{cases} \phi(W) S^{\alpha}, & \text{if } W > 0, \\ c_0 S^{\alpha}, & \text{otherwise,} \end{cases} \tag{2.3}$$

for $W = T/S$ and a positive function ϕ. It is assumed that $\phi(\cdot)$ and $L(\phi(\cdot))$ are absolutely continuous and that the risk of $\delta(\phi)$ is finite. Based on the following lemma, we can get the theorem.

LEMMA 2.1. *For positive functions $p(x)$ and $q(x)$, assume that $p(x)/q(x)$ is increasing. If $K(x) < 0$ for $x < x_0$ and $K(x) > 0$ for $x > x_0$, then*

$$\int_0^{\infty} K(x) \frac{p(x)}{q(x)}\, dx \geq \frac{p(x_0)}{q(x_0)} \int_0^{\infty} K(x)\, dx,$$

where the equality holds if and only if $p(x)/q(x)$ is a constant almost everywhere.

THEOREM 2.2. *For $\alpha > 0$ (resp., < 0), assume* (A.1) *and the following conditions*:

(a) *$\phi(w)$ is nondecreasing (resp., nonincreasing) and* $\lim_{w\to\infty} \phi(w) = c_0$;
(b) $\int_0^{\infty} L'(\phi(w) v^{\alpha}) v^{\alpha} g(v) H(wv)\, dv \geq$ (*resp.* $\leq$) 0.

Then $\delta(\phi)$ dominates δ_0.

PROOF. The case $\alpha > 0$ only is treated. Observe that

$$\begin{aligned} &R(\lambda, \delta(\phi)) - E\left[L(c_0 S^{\alpha}/\sigma^{\alpha}) I_{[W \leq 0]}\right] \\ &\quad = \int_0^{\infty}\int_0^{\infty} L(\phi(w) v^{\alpha}) v g(v) h(wv;\lambda) I_{[wv > k(\lambda)]}\, dw\, dv. \end{aligned} \tag{2.4}$$

Since $\phi(w)$ is differentiable almost everywhere, by a definite integral, we get

$$
\begin{aligned}
&E\left[L\left(\frac{c_0 S^\alpha}{\sigma^\alpha}\right)I_{[W>0]}\right]\\
&= \left[\int_0^\infty L(\phi(w)v^\alpha)g(v)H(wv;\lambda)\,dv\right]_{w=0}^\infty\\
&= \int_0^\infty \frac{d}{dw}\left\{\int_0^\infty L(\phi(w)v^\alpha)g(v)H(wv;\lambda)\,dv\right\}dw \qquad (2.5)\\
&= \int_0^\infty\int_0^\infty L'(\phi(w)v^\alpha)\phi'(w)v^\alpha g(v)H(wv;\lambda)\,dv\,dw\\
&\quad + \int_0^\infty\int_0^\infty L(\phi(w)v^\alpha)vg(v)h(wv;\lambda)I_{[wv>k(\lambda)]}\,dw\,dv,
\end{aligned}
$$

which gives that

$$
\begin{aligned}
&R(\lambda,\delta(\phi)) - \mathrm{R}(\lambda,\delta_0)\\
&= -\int_0^\infty\int_0^\infty L'(\phi(w)v^\alpha)\phi'(w)v^\alpha g(v)H(wv;\lambda)\,dv\,dw\\
&= -\int_0^\infty \phi'(w)\int_0^\infty L'(\phi(w)v^\alpha)\frac{H(wv;\lambda)}{H(wv)}v^\alpha g(v)H(wv)\,dv\,dw \qquad (2.6)\\
&\le -\int_0^\infty \phi'(w)\frac{H(wv_0;\lambda)}{H(wv_0)}\int_0^\infty L'(\phi(w)v^\alpha)v^\alpha g(v)H(wv)\,dv\,dw,
\end{aligned}
$$

where $v_0 = \{\phi(w)\}^{-1/\alpha}$. The inequality in (2.6) follows from Lemma 2.1, (A.1) and condition (a). Hence Theorem 2.2 is established. □

Define $\phi_0(w)$ and $\phi_1(w)$, respectively, by solutions of the equations

$$\int_0^\infty L'(\phi_0(w)v^\alpha)v^\alpha g(v)H(wv)dv = 0, \qquad (2.7)$$

$$\int_0^\infty L'(\phi_1(w)v^\alpha)v^{\alpha+1}g(v)h(wv)dv = 0, \qquad (2.8)$$

and let $\phi_S(w) = \min\{c_0, \phi_1(w)\}$ for $\alpha > 0$, and $\phi_S(w) = \max\{c_0, \phi_1(w)\}$ for $\alpha < 0$. The uniqueness of ϕ_0 can be shown based on (A.2) and the following assumption:

(A.3) $H(c_1x)/H(c_2x)$ is strictly increasing in x for $0 < c_1 < c_2$.

Also, the uniqueness of ϕ_1 is guaranteed by (A.2) and the assumption that:

(A.4) $h(c_1x)/h(c_2x)$ is strictly increasing in x for $0 < c_1 < c_2$.

Note that (A.4) implies (A.3). The resulting estimators $\delta(\phi_0)$ and $\delta(\phi_S)$ correspond to the Brewster–Zidek–type and the Stein-type truncated rules, respectively. From (2.6), it is seen that $R(0,\delta(\phi_0)) = R(0,\delta_0)$ at $\lambda = 0$.

PROPOSITION 2.3. *Under* (A.1), (A.2) *and* (A.3), *the estimator* $\delta(\phi_0)$ *dominates* δ_0.

PROPOSITION 2.4. *Under* (A.1), (A.2) *and* (A.4), *the estimator* $\delta(\phi_S)$ *dominates* δ_0.

The proofs are straightforward, where in the proof of Proposition 2.4, it will be noted that (A.4) implies that $H(x)/\{xh(x)\}$ is strictly increasing. For the details, see Kubokawa (1991b).

The estimators $\delta(\phi_0)$ and $\delta(\phi_S)$, respectively, were given by Brewster and Zidek (1974) and Stein (1964) in the normal case as stated in Section 1, by Brewster (1974) and Arnold (1970) in the exponential case and by Kubokawa, Honda, Morita and Saleh (1993) and Kubokawa, Robert and Saleh (1992) in a multivariate extension. Assumptions (A.1)–(A.4) are also satisfied by lognormal and Pareto distributions and the corresponding procedures are given.

It is remarked that if we impose convexity on loss functions, all the results obtained here hold without assuming (A.2). Also the results can be extended to the case where S and T are not independent by replacing (A.1)–(A.4) with suitable assumptions.

3. Interval estimation of the scale parameter. In this section, we shall deal with the interval estimation of $\sigma^\alpha, \alpha \neq 0$, in the model described in Section 2. A confidence interval we look at is of the form

$$I_0 = \left\{\sigma^\alpha \,\Big|\, \left(\frac{S}{a}\right)^\alpha \le (\ge)\sigma^\alpha \le (\ge)\left(\frac{S}{b}\right)^\alpha \text{ for } \alpha > 0 \text{ (resp., } \alpha < 0)\right\},$$

where the constants a and b $(a > b)$ satisfy

$$ag(a) = bg(b) \quad \text{and} \quad P[\sigma^\alpha \in I_0] = 1 - \gamma, \tag{3.1}$$

for $0 < \gamma < 1$. This is a confidence interval such that the ratio of endpoints is minimized and is also the shortest unbiased [Tate and Klett (1959)]. The uniqueness of a and b is due to (A.2). Here, by using the statistic T, we want to construct a confidence interval with the same ratio of endpoints as I_0 and uniformly higher probability of coverage. For this purpose, consider the interval

$$I(\phi) = \begin{cases} \left\{\sigma^\alpha \,\Big|\, \left(\dfrac{\phi(W)S}{a}\right)^\alpha \le (\ge)\sigma^\alpha \le (\ge)\left(\dfrac{\phi(W)S}{b}\right)^\alpha \right. \\ \qquad\qquad \left. \text{for } \alpha > 0 \text{ (resp., } \alpha < 0)\right\}, & \text{if } W > 0, \\ I_0, & \text{otherwise,} \end{cases} \tag{3.2}$$

for an absolutely continuous and positive function ϕ. Based on the following lemma due to Cohen (1972), we can get the theorem.

LEMMA 3.1. *For positive functions $p(x)$ and $q(x)$, assume that $p(x)/q(x)$ is increasing. Then, for $c_1, c_2 > 0$ and $0 < x_1 < x_2$,*

$$c_2 p(x_2) - c_1 p(x_1) \geq \{p(x_1)/q(x_1)\}\{c_2 q(x_2) - c_1 q(x_1)\},$$

where the equality holds if and only if $p(x_1)/q(x_1) = p(x_2)/q(x_2)$.

THEOREM 3.2. *Assume* (A.1) *and that, for $\alpha > 0$ ($\alpha < 0$), the following hold*:

(a) *$\phi(w)$ is nondecreasing (nonincreasing) and* $\lim_{w\to\infty} \phi(w) = 1$.
(b) $ag(a/\phi)H((a/\phi)w) - bg(b/\phi)H((b/\phi)w) \geq (\leq) 0$.

Then $I(\phi)$ improves upon I_0 in terms of coverage probability, that is,

$$P[\sigma^\alpha \in I(\phi)] \geq P[\sigma^\alpha \in I_0]$$

uniformly with respect to $\lambda > 0$.

PROOF. For $\alpha > 0$,

$$\begin{aligned} &P[\sigma^\alpha \in I(\phi)] - P[\{\sigma^\alpha \in I(\phi)\} \cap \{W \leq 0\}] \\ &\quad = \int_0^\infty \int_{b/\phi}^{a/\phi} v g(v) h(wv; \lambda) I_{[wv > k(\lambda)]}\, dv\, dw. \end{aligned}$$

Similar to (2.5),

$$\begin{aligned} &\left[\int_{b/\phi}^{a/\phi} g(v) H(wv; \lambda)\, dv\right]_{w=0}^{\infty} \\ &\quad = -\int_0^\infty \frac{\phi'}{\phi^2}\left\{ag\left(\frac{a}{\phi}\right)H\left(\frac{a}{\phi}w; \lambda\right) - bg\left(\frac{b}{\phi}\right)H\left(\frac{b}{\phi}w; \lambda\right)\right\} dw \\ &\qquad + \int_0^\infty \int_{b/\phi}^{a/\phi} v g(v) h(wv; \lambda) I_{[wv > k(\lambda)]}\, dv\, dw, \end{aligned} \tag{3.3}$$

so that

$$\begin{aligned} &P[\sigma^\alpha \in I(\phi)] - P[\sigma^\alpha \in I_0] \\ &\quad = \int_0^\infty \frac{\phi'}{\phi^2}\left\{ag\left(\frac{a}{\phi}\right)H\left(\frac{a}{\phi}w; \lambda\right) - bg\left(\frac{b}{\phi}\right)H\left(\frac{b}{\phi}w; \lambda\right)\right\} dw, \end{aligned} \tag{3.4}$$

which can be verified to be nonnegative by (A.1) and Lemma 3.1, and we get Theorem 3.2. □

Define $\phi_0(w)$ and $\phi_1(w)$, respectively, by solutions of the equations

$$ag\left(\frac{a}{\phi_0}\right)H\left(\frac{a}{\phi_0}w\right) = bg\left(\frac{b}{\phi_0}\right)H\left(\frac{b}{\phi_0}w\right), \tag{3.5}$$

$$a^2g\left(\frac{a}{\phi_1}\right)h\left(\frac{a}{\phi_1}w\right) = b^2g\left(\frac{b}{\phi_1}\right)h\left(\frac{b}{\phi_1}w\right), \tag{3.6}$$

and let $\phi_S(w) = \min\{1, \phi_1(w)\}$ for $\alpha > 0$, and $\phi_S(w) = \max\{1, \phi_1(w)\}$ for $\alpha < 0$. Note that (A.2), (A.3) and (A.4) guarantee the uniqueness of ϕ_0 and ϕ_1. From (3.4), it is seen that $P[\sigma^\alpha \in I(\phi_0)] = P[\sigma^\alpha \in I_0]$ at $\lambda = 0$.

PROPOSITION 3.3. *Under* (A.1), (A.2) *and* (A.3), $I(\phi_0(W))$ *improves on* I_0 *in terms of coverage probability.*

PROPOSITION 3.4. *Under* (A.1), (A.2) *and* (A.4), $I(\phi_S(W))$ *improves on* I_0 *in terms of coverage probability.*

The proofs are straightforward. The improved intervals of the type $I(\phi_S)$ were given by Nagata (1989, 1991) for the normal and the exponential cases. The intervals of the type $I(\phi_0)$ can be also derived. It may be noted that a similar result can be obtained in improvement on the minimum-length confidence interval in terms of coverage probability.

4. James–Stein estimator and its improvements. In this section, the simultaneous estimation of a mean vector of a multivariate normal distribution is treated and it is shown that the approach used in Sections 2 and 3 is effectively employed in the improvements on the well-known James–Stein estimator.

Let $\mathbf{X}$ be a p-variate random vector having $N_p(\theta, \mathbf{I})$ and suppose that we want to estimate θ by $\delta = \delta(\mathbf{X})$ under the loss $\|\delta-\theta\|^2 = (\delta-\theta)'(\delta-\theta)$. Stein (1956) showed that equivariant estimators relative to the orthogonal transformation group are of the forms $\delta(\phi) = (1 - \phi(\|\mathbf{X}\|^2)/\|\mathbf{X}\|^2)\mathbf{X}$ and that there exists an estimator dominating $\mathbf{X}$ among these when $p \geq 3$. James and Stein (1961) constructed the improved procedure $\delta_{\mathrm{JS}} = (1 - (p-2)/\|\mathbf{X}\|^2)\mathbf{X}$, and since then, many shrinkage rules dominating $\mathbf{X}$ have been presented and their properties have been studied. Baranchick (1964) proposed that the positive-part Stein estimator $\delta_{\mathrm{JS}}^+ = (1 - (p-2)/\|\mathbf{X}\|^2)^+\mathbf{X}$ is better than δ_{JS}, where $a^+ = \max(0, a)$. Recently, based on the Brown–Brewster–Zidek method, Kubokawa (1991a) derived a smooth estimator superior to δ_{JS}, which is identical to the admissible procedure given by Strawderman (1971) and Berger (1976). Here we obtain a class of improved estimators along the approach used in Sections 2 and 3. Let $f_p(t)$ be a density of χ_p^2 and assume the absolute continuity of ϕ.

THEOREM 4.1. *Assume that the following hold*:

(a) $\phi(t)$ *is nondecreasing and* $\lim_{t\to\infty}\phi(t) = p - 2$.

(b) $\phi(t) \geq \phi_0(t)$, *where*

$$\phi_0(t) = p - 2 - \frac{2f_p(t)}{\int_0^t s^{-1} f_p(s)\, ds}.$$

Then $\delta(\phi(\|\mathbf{X}\|^2))$ *dominates* δ_{JS}.

PROOF. By the Stein identity,

$$R(\lambda, \delta(\phi)) = p + \int_0^\infty \left\{ \frac{\phi^2 - 2(p-2)\phi}{t} - 4\phi' \right\} f_p(t; \lambda)\, dt, \tag{4.1}$$

where $f_p(t; \lambda)$ designates a density of $\chi_p^2(\lambda)$ with noncentrality $\lambda = \|\theta\|^2/2$. By a definite integral,

$$\begin{aligned} &\left[\varphi(t) \int_0^t \frac{1}{s} f_p(s; \lambda)\, ds \right]_0^\infty \\ &\quad = \int_0^\infty \varphi'(t) \int_0^t \frac{1}{s} f_p(s; \lambda)\, ds\, dt + \int_0^\infty \frac{\varphi(t)}{t} f_p(t; \lambda)\, dt, \end{aligned} \tag{4.2}$$

for a differentiable function φ. Letting $\varphi(t) = \phi^2(t) - 2(p-2)\phi(t)$, we can see from (4.1) that

$$\begin{aligned} &R(\lambda, \delta_{\mathrm{JS}}) - R(\lambda, \delta(\phi)) \\ &\quad = 2 \int_0^\infty \phi'(t) \left[\left\{ \phi(t) - (p-2) \right\} \int_0^t \frac{1}{s} f_p(s; \lambda)\, ds + 2 f_p(t; \lambda) \right] dt, \end{aligned} \tag{4.3}$$

which is nonnegative since

$$\frac{f_p(t; \lambda)}{\int_0^t (1/s) f_p(s; \lambda)\, ds} \geq \frac{f_p(t)}{\int_0^t (1/s) f_p(s)\, ds}. \qquad \square$$

Theorem 4.1 presents the improved procedure $\delta(\phi_0(\|\mathbf{X}\|^2))$ [the admissible estimator given by Strawderman (1971) and Berger (1976)] and the positive-part rule $\delta_{\mathrm{JS}}^{+} = \delta(\min(p-2, \|\mathbf{X}\|))$.

These results will be extended to the case where the covariance matrix Σ of $\mathbf{X}$ is of the form $\Sigma = \sigma^2\mathbf{I}$, σ^2 being unknown. The same notation as before is used for simplicity. Let $\mathbf{X}$ and S be independent random variables where $\mathbf{X}$ has $\mathrm{N}_p(\theta, \sigma^2\mathbf{I})$ and S has $\sigma^2\chi_n^2$. For the loss $\|\delta(\mathbf{X}, S) - \theta\|^2/\sigma^2$, $\mathbf{X}$ is dominated by the James–Stein estimator $\delta_{\mathrm{JS}} = (1 - (p-2)/\{(n+2)W\})\mathbf{X}$, for $W = \|\mathbf{X}\|^2/S$. For improving on δ_{JS}, consider the estimator $\delta(\phi(W)) = (1 - \phi(W)/W)\mathbf{X}$. Let $h(w) = \int_0^\infty v f_n(v) f_p(vw)\, dv$.

THEOREM 4.2. *Assume that the following hold*:

(a) $\phi(w)$ *is nondecreasing and* $\lim_{w\to\infty} \phi(w) = (p-2)/(n+2)$;

(b) $\phi(w) \geq \phi_0(w)$, *where*

$$\phi_0(w) = \frac{(p-2)\int_0^w s^{-1}h(s)\,ds - 2h(w)}{(n+2)\int_0^w s^{-1}h(s)\,ds + 2h(w)}.$$

Then $\delta(\phi(W))$ *dominates* δ_{JS}.

The proof is omitted. The class given by Theorem 4.2 includes $\delta(\phi_0(W))$, which is the generalized Bayes estimator given by Lin and Tsai (1973), and a positive-part version of δ_{JS}. Note that Theorem 4.2 can be applied to the case where Σ is fully unknown.

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REFERENCES

ARNOLD, B. C. (1970). Inadmissibility of the usual scale estimate for a shifted exponential distribution. *J. Amer. Statist. Assoc.* **65** 1260–1264.

BARANCHICK, A. (1964). Multiple regression and estimation of the mean of a multivariate normal distribution. Technical Report 51, Dept. Statistics, Stanford Univ.

BERGER, J. O. (1976). Admissible minimax estimation of a multivariate normal mean with arbitrary quadratic loss. *Ann. Statist.* **4** 223–226.

BREWSTER, J. F. (1974). Alternative estimators for the scale parameter of the exponential distribution with unknown location. *Ann. Statist.* **2** 553–557.

BREWSTER, J. F. and ZIDEK, J. V. (1974). Improving on equivariant estimators. *Ann. Statist.* **2** 21–38.

BROWN, L. D. (1968). Inadmissibility of the usual estimators of scale parameters in problems with unknown location and scale parameters. *Ann. Math. Statist.* **39** 29–48.

COHEN, A. (1972). Improved confidence intervals for the variance of a normal distribution. *J. Amer. Statist. Assoc.* **67** 382–387.

GOUTIS, C. and CASELLA, G. (1991). Improved invariant confidence intervals for a normal variance. *Ann. Statist.* **19** 2015–2031.

JAMES, W. and STEIN, C. (1961). Estimation with quadratic loss. In *Proc. Fourth Berkeley Symp. Math. Statist. Probab.* **1** 361–379. Univ. California Press, Berkeley.

KUBOKAWA, T. (1991a). An approach to improving the James–Stein estimator. *J. Multivariate Anal.* **36** 121–126.

KUBOKAWA, T. (1991b). A unified approach to improving equivariant estimators. Technical Report METR 91-1, Dept. Mathematical Engineering, Univ. Tokyo.

KUBOKAWA, T., HONDA, T., MORITA, K. and SALEH, A. K. MD. E. (1993). Estimating a covariance matrix of a normal distribution with unknown mean. *J. Japan Statist. Soc.* **23**. To appear.

KUBOKAWA, T., ROBERT, C. and SALEH, A. K. MD. E. (1992). Empirical Bayes estimation of the covariance matrix of a normal distribution with unknown mean under an entropy loss. *Sankhyā Ser. A* **54** 402–410.

LIN, P.-E. and TSAI, H.-L. (1973). Generalized Bayes minimax estimators of the multivariate normal mean with unknown covariance matrix. *Ann. Statist.* **1** 142–145.

MAATTA, J. M. and CASELLA, G. (1990). Developments in decision-theoretic variance estimation. *Statist. Sci.* **5** 90–120.

NAGATA, Y. (1989). Improvements of interval estimations for the variance and the ratio of two variances. *J. Japan Statist. Soc.* **19** 151–161.

NAGATA, Y. (1991). Improved interval estimation for the scale parameter of exponential distribution with an unknown location-scale parameter. *Journal of the Japanese Society for Quality Control* **21** 5–10. (In Japanese.)

SHORROCK, G. (1990). Improved confidence intervals for a normal variance. *Ann. Statist.* **18** 972–980.

STEIN, C. (1956). Inadmissibility of the usual estimator for the mean of a multivariate normal distribution. In *Proc. Third Berkeley Symp. Math. Statist. Probab.* **1** 197–206. Univ. California Press, Berkeley.

STEIN, C. (1964). Inadmissibility of the usual estimator for the variance of a normal distribution with unknown mean. *Ann. Inst. Statist. Math.* **16** 155–160.

STRAWDERMAN, W. E. (1971). Proper Bayes minimax estimators of the multivariate normal mean. *Ann. Math. Statist.* **42** 385–388.

TATE, R. F. and KLETT, G. W. (1959). Optimal confidence intervals for the variance of a normal distribution. *J. Amer. Statist. Assoc.* **54** 674–682.

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