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LOCAL ASYMPTOTIC MINIMAX RISK BOUNDS FOR ASYMMETRIC LOSS FUNCTIONS

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Hájek established a local asymptotic minimax risk bound for appropriate symmetric loss functions and also gave a necessary condition for the risk of an estimator to attain the lower bound. We extend these results to the case of asymmetric loss functions. The asymmetry brings about the shift of location of the loss functions. Besides, the optimal estimator that attains the bound is shown to have asymptotic normal distribution with asymptotic bias.

1. Introduction. In asymptotic theory of estimation problems, local asymptotic minimax risk is often used as a measure of asymptotic optimality. The concept has been discussed in detail by Hájek (1972), Le Cam (1972) and Ibragimov and Has'minskii (1981), among others. Hájek (1972) has established a local asymptotic minimax risk bound for appropriate symmetric loss functions under the assumptions of local asymptotic normality (LAN). It is also proved that the maximum likelihood estimator and Bayes estimator attain this lower bound under certain regularity assumptions [see, e.g., Ibragimov and Has'minskii (1981)].

There are some cases where asymmetric loss functions may be more appropriate than symmetric ones. Asymmetric linear loss functions have been considered by several authors, including Ferguson (1967), Aitchison and Dunsmore (1975) and Berger (1980). Varian (1975) introduced an asymmetric LINEX loss function that rises approximately exponentially on one side of zero and approximately linearly on the other side [for details, see Zeller (1986)]. Since then, many asymmetric loss functions have been applied to various situations. However, different from the symmetric case, it seems that a unified theory about asymptotic optimality in the case of asymmetric loss functions has not been established. The only interesting approach, which is based on the asymptotic minimax theory in the normal distribution case, has been presented by Lepskii (1987). In this paper, we provide a local asymptotic minimax risk bound for appropriate asymmetric loss functions under the assumptions of LAN. Moreover, we give a necessary condition to attain the lower bound (see Theorem 4.1). The asymmetry of loss functions causes the shift of location of the loss function involved in the minimax risk bound. Another result of our paper is that if there exists an optimal estimator, its asymptotic distribution must be the normal distribution with asymptotic bias.

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In Section 2, we describe notation and assumptions. Section 3 contains some preliminary lemmas. In Section 4, we provide main results on the risk bound. Minimax bounds are calculated for two types of loss functions, namely, linear and LINEX, in Section 5.

2. Notation and assumptions. Let $X_1, X_2, \ldots, X_n$ be a sequence of independently and identically distributed real-valued random variables with family of distributions $\mathcal{P} = \{P(\cdot,\theta)\colon \theta \in \Theta\}$. The parameter space Θ is an open interval of the real line. We consider a sequence of statistical experiments $(\mathcal{X}_n, \mathcal{A}_n, P_n(\cdot,\theta)), n \geq 1$. Take a point $t \in \Theta$ which represents the true value of the parameter θ. We simply write $P_n = P_n(\cdot, t)$ and $P_{n,h} = P_n(\cdot, t + h/\sqrt{n})$.

Let us introduce a family of likelihood ratios

$$r_n(h, x_n) = \frac{dP_{n,h}}{dP_n}(x_n), \qquad h \in \mathbb{R}, n \geq n_h, x_n \in \mathcal{X}_n,$$

where n_h denotes the smallest integer such that $n \geq n_h$ entails $t + h/\sqrt{n} \in \Theta$. In what follows, the argument x_n will usually be omitted.

Hájek (1972) defined local asymptotic normality as follows.

DEFINITION 2.1. The sequence $\{P_n\}$ is said to be locally asymptotic normal (LAN) if

$$r_n(h) = \exp\{h\Delta_{n,t} - \tfrac{1}{2}\gamma_t^2 h^2 + Z_n(h,t)\}, \qquad h \in \mathbb{R}, n \geq n_h,$$

for $\gamma_t > 0$. Here the random variable $\Delta_{n,t}$ satisfies $\mathcal{L}(\Delta_{n,t} | P_n) \to N(0, \gamma_t^2)$, and $Z_n(h,t) \to 0$ in P_n probability for every $h \in \mathbb{R}$.

Let $\ell(z)$ be a loss function that may be asymmetric. In this paper, the following Assumptions A and B on $\ell(z)$ are made:

ASSUMPTION A.

(A1) $\ell(z) \geq 0$ and $\ell(z)$ is nonincreasing for $z < 0$ and is nondecreasing for $z > 0$.

(A2) $\int_{-\infty}^{\infty} \ell(z) \exp\{-\frac{1}{2}cz^2\} dz < \infty$, for any $c > 0$.

(A3) $\int_{-\infty}^{\infty} z^2 \ell(d - z) \exp\{-\frac{1}{2}cz^2\}\, dz < \infty$, for any $c > 0$ and any d.

We shall also introduce a truncated version of $\ell(\cdot)$:

$$\ell_a(y) = \min(\ell(y), a), \quad \text{for } 0 < a \leq \infty.$$

ASSUMPTION B.

(B1) For any constant $k > 0$, the function $g(\beta)$ defined as

$$g(\beta) = \frac{1}{\sqrt{2\pi}k} \int_{-\infty}^{\infty} \ell(\beta - y) \exp\left\{-\frac{y^2}{2k^2}\right\} dy$$

attains its minimum at $\beta = \beta_0$. The minimizer β_0, which depends on k, is finite and unique.

(B2) For any $k > 0$, any large $a, b > 0$ and any small $\lambda > 0$, the function $\tilde{g}(\beta)$,

$$\tilde{g}(\beta) = \frac{\sqrt{1+\lambda}}{\sqrt{2\pi}k} \int_{-\sqrt{b}}^{\sqrt{b}} \ell_a(\beta - y) \exp\left\{-\frac{1}{2}(1+\lambda)\frac{y^2}{k^2}\right\} dy,$$

attains its minimum at $\beta = \tilde{\beta}(a, b, \lambda)$. The minimizer $\tilde{\beta}(a, b, \lambda)$, which also depends on k, is finite.

(B3) It holds that

$$\lim_{a\to\infty, b\to\infty, \lambda\to 0} \tilde{\beta}(a, b, \lambda) = \beta_0.$$

While Assumption A is very general, Assumption B looks atypical, and one may feel anxious about how restrictive Assumption B is. It will be shown in the Appendix that Assumption B is satisfied by very general asymmetric loss functions, such as convex ones. We note that if $\ell(\cdot)$ is symmetric, Assumption B is satisfied with $\beta_0 = \tilde{\beta}(a, b, \lambda) = 0$.

Throughout this paper, we will consider a randomized estimator $\xi(Z, U)$, where U is a randomized variable that is uniformly distributed on (0, 1) and independent of Z and θ.

3. Preliminary lemmas.

LEMMA 3.1. *Let a loss function $\ell(\cdot)$ satisfy Assumptions* A *and* B. *Let Z be a normal random variable with mean $\theta + \beta_0$ and variance k^2, where β_0 is defined in Assumption* (B1). *Then for any $\varepsilon > 0$ there exist positive numbers a, b, α and a one-dimensional probability density $\pi(\cdot)$ with the following property*:

For any randomized estimator $\xi(Z, U)$ of θ such that

$$\Pr\big(|\xi(Z, U) - Z| > \varepsilon | \theta = 0\big) > \varepsilon, \tag{3.1}$$

$$\int_{-b}^{b} \pi(\theta) E\{\ell_a(\xi(Z, U) - \theta) | \theta\}\, d\theta \geq g(\beta_0) + \alpha. \tag{3.2}$$

PROOF. We shall assume θ to be normally distributed as $N(0, \sigma^2)$, where the variance σ^2, which depends on ε, will be appropriately chosen later. Then the conditional density $\psi(\theta|z)$ of θ given $Z = z$ is distributed as

$$N\left(\frac{z - \beta_0}{k^2 + \sigma^2}\sigma^2, \frac{k^2\sigma^2}{k^2 + \sigma^2}\right)$$

and the marginal density $f(z)$ of Z is also distributed as $N(\beta_0, k^2 + \sigma^2)$.

From now on we assume that $|z-\beta_0| \leq b-\sqrt{b}$. Setting

$$\theta - \frac{z-\beta_0}{k^2+\sigma^2}\sigma^2 = y,$$

it follows that

$$\begin{aligned}(3.3)\quad &\int_{-b}^{b} \ell_a(\xi(z,u)-\theta)\psi(\theta|z)\,d\theta \\ &\geq \int_{-\sqrt{b}}^{\sqrt{b}} \ell_a\left(\xi(z,u)-\frac{z-\beta_0}{k^2+\sigma^2}\sigma^2 - y\right)\frac{\sqrt{k^2+\sigma^2}}{\sqrt{2\pi}k\sigma}\exp\left\{-\frac{k^2+\sigma^2}{2k^2\sigma^2}y^2\right\}dy.\end{aligned}$$

On the other hand, if $|\xi(z,u)-z| > \varepsilon$ and $|z-\beta_0| < M$, we have for $(k^2+\sigma^2)/k^2 > 2M/\varepsilon$ that

$$\left|\xi(z,u) - \frac{z-\beta_0}{k^2+\sigma^2}\sigma^2 - \beta_0\right| > \frac{\varepsilon}{2}.$$

Since $|\tilde{\beta}(a,b,k^2/\sigma^2)-\beta_0| < \frac{1}{8}\varepsilon$ for sufficiently large a, b and σ^2 by Assumption (B3), we have that

$$(3.4)\qquad \left|\xi(z,u) - \frac{z-\beta_0}{k^2+\sigma^2}\sigma^2 - \tilde{\beta}\left(a,b,\frac{k^2}{\sigma^2}\right)\right| > \frac{3}{8}\varepsilon.$$

Now note that the distance between $\tilde{\beta}(a,b,k^2/\sigma^2)$ and any other minimizer of $\tilde{g}(\beta)$, if it exists, is less than $\varepsilon/4$. Then we have from Assumption (B2) that if $|\beta-\tilde{\beta}(a,b,k^2/\sigma^2)| > \frac{3}{8}\varepsilon$,

$$\begin{aligned}(3.5)\quad &\frac{\sqrt{k^2+\sigma^2}}{\sqrt{2\pi}k\sigma}\int_{-\sqrt{b}}^{\sqrt{b}} \ell_a(\beta-y)\exp\left\{-\frac{k^2+\sigma^2}{2k^2\sigma^2}y^2\right\}dy \\ &\geq \frac{\sqrt{k^2+\sigma^2}}{\sqrt{2\pi}k\sigma}\int_{-\sqrt{b}}^{\sqrt{b}} \ell_a\left(\tilde{\beta}\left(a,b,\frac{k^2}{\sigma^2}\right)-y\right)\exp\left\{-\frac{k^2+\sigma^2}{2k^2\sigma^2}y^2\right\}dy + \delta \\ &= \tilde{g}(a,b,\lambda)+\delta,\end{aligned}$$

where $\delta > 0$ depends only on ε, but not on a, b and σ^2 when a, b and σ^2 are sufficiently large. Altogether, we have from (3.3), (3.4) and (3.5) that, for $|z-\beta_0| \leq b-\sqrt{b}$, $|\xi(z,u)-z| > \varepsilon$ and $|z-\beta_0| < M$,

$$(3.6)\qquad \int_{-b}^{b} \ell_a(\xi(z,u)-\theta)\psi(\theta|z)\,d\theta \geq \tilde{g}\left(a,b,\frac{k^2}{\sigma^2}\right)+\delta.$$

On the other hand, it follows that

$$\begin{aligned}
&\int_{-b}^{b} \pi(\theta) E\left\{\ell_a(\xi(Z,U)-\theta)|\theta\right\}\, d\theta \\
&= \int_0^1 \int_{-\infty}^{\infty} \int_{-b}^{b} \ell_a(\xi(z,u)-\theta)\psi(\theta|z) f(z)\, d\theta\, dz\, du \\
&\geq \tilde{g}\left(a,b,\frac{k^2}{\sigma^2}\right) \Pr(|Z-\beta_0| < b-\sqrt{b}) \\
&\quad + \delta \Pr(|\xi(Z,U)-Z| > \varepsilon, |Z-\beta_0| < M).
\end{aligned} \tag{3.7}$$

Denote $D = \{(z,u) \in \mathbb{R} \times (0,1) |\ |\xi(z,u)-z| > \varepsilon, |z-\beta_0| < M\}$. Then we have, for any (z,u) in D,

$$\begin{aligned}
\frac{f(z)}{(1/\sqrt{2\pi}k)\exp\{-(z-\beta_0)^2/2k^2\}} &= \frac{k}{\sqrt{k^2+\sigma^2}} \exp\left\{\frac{\sigma^2(z-\beta_0)^2}{2(k^2+\sigma^2)k^2}\right\} \\
&\geq \frac{k}{\sqrt{k^2+\sigma^2}}.
\end{aligned} \tag{3.8}$$

Since $\Pr(D|\theta=0) > \frac{1}{2}\varepsilon$ for sufficiently large M by (3.1), we have from (3.8) that

$$\begin{aligned}
\Pr(D) &= \int\int_D f(z)\, dz\, du \\
&\geq \int\int_D \frac{k}{\sqrt{k^2+\sigma^2}} \frac{1}{\sqrt{2\pi}k} \exp\left\{-\frac{(z-\beta_0)^2}{2k^2}\right\} dz\, du \\
&= \frac{k}{\sqrt{k^2+\sigma^2}} \Pr(D|\theta=0) \geq \frac{1}{2}\varepsilon \frac{k}{\sqrt{k^2+\sigma^2}}.
\end{aligned} \tag{3.9}$$

Now, by utilizing the inequality

$$\exp\left\{-\frac{k^2+\sigma^2}{2k^2\sigma^2}y^2\right\} \geq \left(1-\frac{y^2}{2\sigma^2}\right)\exp\left\{-\frac{y^2}{2k^2}\right\},$$

we have from Assumptions (A3) and (B3) that, for sufficiently large a, b and σ^2,

$$\begin{aligned}
\tilde{g}\left(a,b,\frac{k^2}{\sigma^2}\right) &\geq \int_{-\sqrt{b}}^{\sqrt{b}} \left(1-\frac{y^2}{2\sigma^2}\right) \ell_a\left(\tilde{\beta}\left(a,b,\frac{k^2}{\sigma^2}\right)-y\right) \frac{1}{\sqrt{2\pi}k} \exp\left\{-\frac{y^2}{2k^2}\right\} dy \\
&\geq g(\beta_0) - \frac{\alpha}{2} - \frac{K}{\sigma^2},
\end{aligned}$$

for some $\alpha > 0$ and $K > 0$ which do not depend on a, b and σ^2, and

$$\Pr(|Z-\beta_0| < b-\sqrt{b}) \geq 1 - \frac{\alpha}{2g(\beta_0)}.$$

Therefore we have that

$$\tilde{g}\left(a, b, \frac{k^2}{\sigma^2}\right) \Pr(|Z - \beta_0| < b - \sqrt{b}) \geq g(\beta_0) - \alpha - \frac{K}{\sigma^2}. \tag{3.10}$$

Now, putting

$$2\alpha = \frac{1}{2}\delta\varepsilon \frac{k}{\sqrt{k^2 + \sigma^2}} - \frac{K}{\sigma^2}, \tag{3.11}$$

we establish (3.2) in view of (3.7), (3.9), (3.10) and (3.11). □

LEMMA 3.2. *Let Z be the same random variable as in Lemma 3.1. Then, for every $\delta > 0$, there exist positive numbers a and b and a probability density $\pi(\cdot)$ such that, for any randomized estimator $\xi(Z, U)$,*

$$\int_{-b}^{b} \pi(\theta) E\{\ell_a(\xi(Z, U) - \theta) | \theta\} \, d\theta \geq g(\beta_0) - \delta.$$

LEMMA 3.3. *Assume that the joint probability measure $P_n(\cdot)$ is LAN. For any sequence of statistics $\{S_n\}$, define*

$$s_n(x, u) = \inf\{y \colon P_n[S_n \leq y | \Delta_{n,t} = x] \geq u\},$$

for $x \in \mathbb{R}, 0 < u < 1$. Denote by $F_{n,h}$ the distribution of S_n under $P_{n,h}$ and by $F^*_{n,h}$ the distribution of $s_n(\Delta_{n,t}, U)$ also under $P_{n,h}$, where U is uniformly distributed on $(0, 1)$ and independent of $\Delta_{n,t}$. Then it follows that, for each $h \in \mathbb{R}$,

$$\lim_{n \to \infty} \|F_{n,h} - F^*_{n,h}\| = 0.$$

LEMMA 3.4. *Assume that the joint probability measure $P_n(\cdot)$ is LAN. Denote $G_{n,h}(x) = P_{n,h}(\Delta_{n,t} \leq x)$ and $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} \exp\{-\frac{1}{2}y^2\}\, dy$. Let $G^*_{n,h}$ be the distribution of $G^{-1}_{n,0}\Phi((Z - \beta_0)\gamma_t)$, where Z is $N(h + \beta_0, 1/\gamma_t^2)$. Then it follows that, for each $h \in \mathbb{R}$,*

$$\lim_{n \to \infty} \|G_{n,h} - G^*_{n,h}\| = 0.$$

Lemma 3.2 can be proved along the same lines as in the proof of Lemma 3.1. Refer to Hájek (1972) for the proofs of Lemmas 3.3 and 3.4. We note that our lemmas are slightly different from those in Hájek by the effect β_0 of asymmetry.

4. Main theorem.

THEOREM 4.1. *Assume that the joint probability measure $P_n(\cdot)$ is LAN and that a loss function $\ell(\cdot)$ meets Assumptions* A *and* B. *Let $\{T_n\}$ be any sequence of estimators for θ based on $X_1, \dots, X_n$. Then a lower bound of the risk of $\{T_n\}$ is given as follows*:

$$(4.1)\qquad \begin{aligned} &\lim_{\delta\to 0} \liminf_{n\to\infty} \sup_{|\theta-t|<\delta} E_\theta\{\ell[\sqrt{n}(T_n-\theta)]\} \\ &\quad \ge \frac{\gamma_t}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \ell(\beta_0 - y)\exp\left\{-\frac{1}{2}\gamma_t^2 y^2\right\} dy. \end{aligned}$$

Furthermore, if the lower bound is attained, then

$$(4.2)\qquad \sqrt{n}(T_n - t) - \frac{\Delta_{n,t}}{\gamma_t^2} - \beta_0 \to 0$$

in P_n-probability.

PROOF. Since the upper bound of values of a function over a set is at least its mean value on this set, we may write, for n sufficiently large,

$$(4.3)\qquad \sup_{|\theta-t|<\delta} E_\theta\{\ell[\sqrt{n}(T_n-\theta)]\} \ge \int_{-b}^{b} \pi(h) E\left\{\ell_a[\sqrt{n}(T_n-t)-h] \Big| t + \frac{h}{\sqrt{n}}\right\} dh,$$

whatever the constants a and b and density $\pi(\cdot)$ may be. Now suppose that the random variable Z satisfies that $\mathcal{L}(Z|t+h/\sqrt{n}) = N(h+\beta_0, 1/\gamma_t^2)$. Then, by Lemma 3.2, we fix some $\bar{\delta} > 0$ and choose a, b and π in such a way that

$$(4.4)\qquad \begin{aligned} &\int_{-b}^{b} \pi(h) E\left\{\ell_a[\xi(Z,U)-h] \;\Big|\; t + \frac{h}{\sqrt{n}}\right\} dh \\ &\quad \ge \frac{\gamma_t}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \ell(\beta_0 - y)\exp\left\{-\frac{1}{2}\gamma_t^2 y^2\right\} dy - \bar{\delta}, \end{aligned}$$

for any estimator $\xi(Z,U)$.

Next we identify $S_n = \sqrt{n}(T_n - t)$ in Lemma 3.3. Then we have from Lemma 3.3 and the boundedness of $\ell(\cdot)$ that, for every $h \in \mathbb{R}$,

$$(4.5)\qquad \left| E\left\{\ell_a[\sqrt{n}(T_n-t)-h] \Big| t + \frac{h}{\sqrt{n}}\right\} - E\left\{\ell_a[s_n(\Delta_{n,t},U)-h] \Big| t + \frac{h}{\sqrt{n}}\right\}\right| \to 0.$$

Furthermore, by Lemma 3.4, setting

$$\xi_n(Z,U) = s_n(G_{n,0}^{-1}\Phi((Z-\beta_0)\gamma_t), U),$$

we obtain for every $h \in \mathbb{R}$

$$(4.6)\qquad \left| E\left\{\ell_a[s_n(\Delta_{n,t},U)-h] \Big| t + \frac{h}{\sqrt{n}}\right\} - E\left\{\ell_a[\xi_n(Z,U)-h] \Big| t + \frac{h}{\sqrt{n}}\right\}\right| \to 0.$$

It follows from (4.5) and (4.6) that, for $n > n(a, b, \pi, \bar{\delta})$,

$$\begin{aligned}(4.7)\qquad &\int_{-b}^{b} \pi(h) E\left\{\ell_a[\sqrt{n}(T_n - t) - h]\Big| t + \frac{h}{\sqrt{n}}\right\} dh \\ &\geq \int_{-b}^{b} \pi(h) E\left\{\ell_a[\xi_n(Z, U) - h]\Big| t + \frac{h}{\sqrt{n}}\right\} dh - \bar{\delta}.\end{aligned}$$

Combining (4.3), (4.7) and (4.4), we obtain (4.1).

Now we shall prove the necessity of (4.2) for the attainment of the lower bound. Recalling Lemma 3.3, we can see by putting $S_n = \sqrt{n}(T_n - t)$ and Lemma 3.3 that $\sqrt{n}(T_n - t) - \Delta_{n,t}/\gamma_t^2 - \beta_0$ has the asymptotically same distribution under P_n as

$$(4.8)\qquad s_n(\Delta_{n,t}, U) - \frac{\Delta_{n,t}}{\gamma_t^2} - \beta_0.$$

Assume that (4.8) fails to converge to zero in probability. Then $\xi_n(Z, U) - Z$ also fails to converge to zero in view of Lemma 3.4 and asymptotic distributions of $\Delta_{n,t}$ and Z. In this case, there is an $\varepsilon > 0$ such that for every n there exists an $m > n$ such that

$$\Pr\big(|\xi_m(Z, U) - Z| > \varepsilon\big) > \varepsilon.$$

Therefore, according to Lemma 3.1, we can choose a, b, α and $\pi(\cdot)$ so that

$$\begin{aligned}(4.9)\qquad &\int_{-b}^{b} \pi(h) E\left\{\ell_a[\xi_n(Z, U) - h]\,\Big| t + \frac{h}{\sqrt{n}}\right\} dh \\ &> \frac{\gamma_t}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \ell(\beta_0 - y) \exp\left\{-\frac{1}{2}\gamma_t^2 y^2\right\} dy + \alpha.\end{aligned}$$

This fact in connection with (4.3) and (4.7) contradicts the attainment of the bound, since $\bar{\delta}$ can be made smaller than α. □

REMARK 4.2. Comparing our result (4.1) with Hájek's, we are conscious of the important role of asymmetry. At first sight it seems that asymmetry of loss functions causes a large change of the risk function. However, indeed, the asymmetry does not change the general form of the minimax bound and brings about only the shift of location function in the minimax bound. If it is symmetric, the two results coincide.

REMARK 4.3. If there is an optimal estimator T_n in the sense that its risk attains the lower bound, T_n satisfies (4.2) and has asymptotic normal distribution with asymptotic bias β_0 and asymptotic variance $1/\gamma_t^2$.

5. Examples. Now we shall obtain the local asymptotic minimax risk bound for two familiar types of loss functions, namely, linear and LINEX.

5.1. *Linear loss function.* We consider the linear loss functions as follows:

$$\ell(z) = \begin{cases} c_1 z, & \text{if } z \geq 0, \\ -c_2 z, & \text{if } z < 0, \end{cases}$$

where c_1 and c_2 are positive constants. We note that Assumptions A and B are met since this loss function satisfies the conditions given in Corollary A.3 in the Appendix.

Simple calculation leads to

$$g(\beta) = (c_1 + c_2)\beta\Phi(\gamma_t\beta) + (c_1 + c_2)\frac{1}{\sqrt{2\pi}\gamma_t} \exp\left\{-\frac{1}{2}\gamma_t^2\beta^2\right\} - c_2\beta.$$

Therefore, $g(\beta)$ has a minimum value $g(\beta_0)$ at $\beta_0 = (1/\gamma_t)\Phi^{-1}(c_2/(c_1 + c_2))$, where

$$g(\beta_0) = (c_1 + c_2)\frac{1}{\sqrt{2\pi}\gamma_t} \exp\left[-\frac{1}{2}\left\{\Phi^{-1}\left(\frac{c_2}{c_1 + c_2}\right)\right\}^2\right].$$

5.2. *LINEX loss function.* We consider the LINEX loss function as follows:

$$\ell(z) = c_2\{\exp(c_1 z) - c_1 z - 1\},$$

where $c_1 \neq 0$ and $c_2 > 0$. We note that Assumptions A and B are satisfied in virtue of Corollary A.3.

In this case, simple calculation shows

$$g(\beta) = c_2 \exp\left\{\frac{c_1^2}{2\gamma_t^2} + \beta c_1\right\} - c_1 c_2 \beta - c_2.$$

Therefore, $g(\beta)$ has a minimum value $g(\beta_0) = c_1^2 c_2/2\gamma_t^2$ at $\beta_0 = -c_1/2\gamma_t^2$.

APPENDIX

Here we explore sufficient conditions for $\ell(z)$ to satisfy Assumption B.

LEMMA A.1. *Let $f(x,y)$ and $f(x)$ be continuous, and let $f(x,y)$ converge uniformly to $f(x)$ as $y \to y_0$, where y_0 is a real constant and may be infinity. Furthermore, $f(x)$ attains its minimum at a finite and unique value x_0 and satisfies*

$$\lim_{x \to \pm\infty} f(x) > f(x_0).$$

Then, for a fixed y sufficiently close to y_0, $f(x,y)$ attains its minimum at $x_0(y)$, and it holds that $x_0(y) \to x_0$ as $y \to y_0$.

Lemma A.1 is shown by standard techniques in real analysis and thus the proof is omitted. Now, we establish the next theorem, using Lemma A.1 and the definitions of $g(\beta)$ and $\tilde{g}(\beta)$.

THEOREM A.2. *Suppose that $\ell(\cdot)$ satisfies Assumption A and that $g(\beta)$ attains its minimum at a finite and unique value β_0 and satisfies*

$$\lim_{\beta \to \pm\infty} g(\beta) > g(\beta_0). \tag{A.1}$$

Then, $\ell(\cdot)$ meets Assumption B.

COROLLARY A.3. *Assume that $\ell(\cdot)$ satisfies Assumption A. Assume further that $\ell(\cdot)$ is convex and that $\ell(\cdot)$ is neither constant for $x \geq 0$ nor for $x \leq 0$, for example, excluding $\ell(\cdot)$ that is convex for $x \geq 0$ and is constant for $x \leq 0$. Then $\ell(\cdot)$ meets Assumption B.*

PROOF. It is easily shown that $g(\beta)$ is strictly convex. This fact along with the assumptions of the corollary shows that $\ell(z) \to \infty$ as $z \to \pm\infty$. Therefore we have that $g(\beta) \to \infty$ as $\beta \to \pm\infty$. Altogether, $g(\beta)$ attains its minimum value $g(\beta_0)$ at a finite and unique value β_0 and (A.1) is also satisfied. Thus, the proof is complete by using Theorem A.2. □

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