The Annals of Statistics
1995, Vol. 23, No. 2, 670–689

AUTOREGRESSION QUANTILES AND RELATED RANK-SCORES PROCESSES[1]

BY HIRA L. KOUL AND A. K. MD. E. SALEH

Michigan State University and Carleton University

This paper develops extensions of the regression quantiles of Koenker and Bassett (1978) to autoregression. It generalizes several results of Jurečková (1992a) and Gutenbrunner and Jurečková (1992) in linear regression to autoregression models. In particular, it gives the asymptotic uniform linearity of linear rank-scores statistics based on residuals suitable in autoregression. It also discusses the two types of L-statistics appropriate in autoregression.

1. Introduction and summary. The regression quantiles (RQ's) of Koenker and Bassett (1978) (KB) have been accepted as an appropriate extension of the one-sample quantiles in location models to multiple linear regression models. They characterized RQ's as solutions of a parameterized family of linear programs. Gutenbrunner and Jurečková (1992) (GJ) call the solutions of the dual of this problem *regression rank scores* (RRS's). They point out that RQ's and RRS's are related to each other in the same fashion as the one-sample order statistics and the ordinary rank scores [Hájek and Šidák (1967), page 186] are in the one-sample location model. Jurečková (1992b) further points out that unlike the ordinary rank-scores statistics, one of the major advantages of using RRS statistics based on the residuals is that the corresponding estimators of some of the components of the regression parameter vector when others are treated as nuisance parameters do not require the estimation of the latter. The focus of this paper is to *develop* and *investigate* analogs of these procedures in the pth order autoregression [AR(p)] models.

Using RQ's and RRS's, different types of L-estimators of the slope parameters in linear regression were proposed by Ruppert and Carroll (1980), Koenker and Portnoy (1987), Portnoy and Koenker (1989) and GJ. Recently, Gutenbrunner, Jurečková, Koenker and Portnoy (1993) showed that tests of subhypotheses in linear regression based on RRS statistics are asymptotically distribution-free and do not require estimation of the parameters that are not under test. The present paper also contains analogs of some of these estimators and tests in AR(p) models.

Received December 1992; revised June 1994.

[1]Research partially supported by the NSF Grants DMS-91-02041 and DMS-94-02904 and by NSERC Grant A3088.

AMS 1991 *subject classifications*. Primary 62M10; secondary 62G30.

Key words and phrases. Regression quantiles, asymptotic uniform linearity of autoregression rank-score processes and statistics, L-statistics.

More precisely, let $p \geq 1$ be a fixed integer, $\mathbb{R}^p$ denote the p-dimensional Euclidean space, $\mathbb{R} = \mathbb{R}^1$ and $\mathbf{t}'$ denote the transpose of a $p \times 1$ vector $\mathbf{t} \in \mathbb{R}^p$. Let F be a continuous distribution function (d.f.) on $\mathbb{R}$ and $\varepsilon, \varepsilon_i$, $i = 0, \pm 1, \pm 2, \ldots$, be independent and identically distributed (i.i.d.) F random variables (r.v.'s). In an AR(p) model with a location parameter one observes $\{X_i\}$ satisfying the relation

$$(1.1) \qquad X_i = \rho_0 + \rho_1 X_{i-1} + \cdots + \rho_p X_{i-p} + \varepsilon_i, \qquad i = 0, \pm 1, \pm 2, \ldots,$$

where $\boldsymbol{\rho} := (\rho_0, \rho_1, \ldots, \rho_p)' \in \mathbb{R}^{p+1}$ is the unknown parameter vector of interest. Throughout we shall also assume the following:

$$(\text{a1}) \qquad E(\varepsilon) = 0, \qquad E(\varepsilon^2) < \infty;$$

(a2) all roots of the equation $x^p - \rho_1 x^{p-1} - \cdots - \rho_p = 0$ are inside the unit circle.

Furthermore, let $\mathbf{Y}'_{i-1} := (1, X_{i-1}, \ldots, X_{i-p})$, $i = 0, \pm 1, \pm 2, \ldots$, and $\mathscr{X}_n$ be the $n \times (p+1)$ matrix whose ith row is $\mathbf{Y}'_{i-1}$, $1 \leq i \leq n$. Recall, say from Brockwell and Davis (1987), that under (1.1), (a1) and (a2), the process $\{X_i\}$ is causal and invertible. This and the continuity of F imply that the rows of $\mathscr{X}_n$ are linearly independent and the columns of $\mathscr{X}_n$ are linearly independent, with probability 1 (w.p.1), so that the various inverses in the sequel exist w.p.1. It also implies that the process $\{X_i\}$ is stationary and ergodic [Hannan (1970), Theorem IV.2.3, page 204].

Now, akin to the definition of KB in linear regression, define an αth *autoregression quantile* (ARQ) as any member $\hat{\boldsymbol{\rho}}_n(\alpha)$ of the set

$$(1.2) \qquad \hat{\mathscr{R}}_n(\alpha) = \left\{\mathbf{b} \in \mathbb{R}^{p+1}: \sum_i h_\alpha(X_i - \mathbf{Y}'_{i-1}\mathbf{b}) = \text{minimum}\right\},$$

where $h_\alpha(u) := \alpha u I(u > 0) - (1 - \alpha) u I(u \leq 0)$, $u \in \mathbb{R}$, $\alpha \in (0, 1)$. Note that $\hat{\boldsymbol{\rho}}_n(1/2)$ is the well known least absolute deviation estimator of $\boldsymbol{\rho}$. Here, and in the sequel, $I(A)$ denotes the indicator of an event A and the index i in the summation and the maximum varies from 1 to n, unless specified otherwise.

Similar to Theorem 3.1 of KB, one obtains the following linear programming version of the above minimization problem:

$$(1.3) \qquad \begin{aligned} &\text{minimize } \alpha \mathbf{1}'_n \mathbf{r}^+ + (1 - \alpha)\mathbf{1}'_n \mathbf{r}^-, \text{ with respect to } (\mathbf{b}, \mathbf{r}^+, \mathbf{r}^-), \\ &\text{subject to } \mathbf{X}_n - \mathscr{X}_n \mathbf{b} = \mathbf{r}^+ - \mathbf{r}^-, \ (\mathbf{b}, \mathbf{r}^+, \mathbf{r}^-) \in \mathbb{R}^{p+1} \times \mathbb{R}^n_+ \times \mathbb{R}^n_+, \end{aligned}$$

where $\mathbf{1}'_n = [1, \ldots, 1]_{1 \times n}$ and $\mathbf{X}'_n := (X_1, \ldots, X_n)$ is the response vector. As in KB, $\hat{\mathscr{R}}_n(\alpha)$ is the convex hull of one or more basic solutions of the form

$$(1.4) \qquad \mathbf{b}_{\mathbf{h}} = \mathscr{X}_{\mathbf{h}}^{-1} \mathbf{X}_{\mathbf{h}},$$

where $\mathbf{h}$ is a subset of $\{1, 2, \ldots, n\}$ of size $p + 1$ and $\mathscr{X}_{\mathbf{h}}$ ($\mathbf{X}_{\mathbf{h}}$) denotes the subdesign matrix (subresponse vector) with rows $\mathbf{Y}'_{i-1}$, $i \in \mathbf{h}$ (coordinates X_i, $i \in \mathbf{h}$).

In general, there will be "break-points" $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{J_n} = 1$, for some $J_n \leq \binom{n}{p+1}$ such that $\hat{\boldsymbol{\rho}}_n(\cdot)$ is a stochastic process, herein called *ARQ process*, that is constant over each interval (α_i, α_{i+1}), $0 \leq i \leq J_n - 1$, and takes values in $[D(0,1)]^{p+1}$.

An analog of the corresponding dual program, mentioned in the Appendix of KB, is the following:

$$\text{(1.5)} \qquad \begin{aligned} &\text{maximize } \mathbf{X}_n'\mathbf{a}, \text{ with respect to } \mathbf{a}, \\ &\text{subject to } \mathscr{X}_n'\mathbf{a} = (1-\alpha)\mathscr{X}_n'\mathbf{1}_n, \ \mathbf{a} \in [0,1]^n. \end{aligned}$$

By the linear programming theory the optimal solution $\hat{\mathbf{a}}_n(\alpha)$ of (1.5) can be computed in terms of $\hat{\boldsymbol{\rho}}_n(\alpha)$ as follows: If $\hat{\boldsymbol{\rho}}_n(\alpha) = \mathscr{X}_{\mathbf{h}_n(\alpha)}^{-1}\mathbf{X}_{\mathbf{h}_n(\alpha)}$ for some $(p+1)$-dimensional subset $\mathbf{h}_n(\alpha)$ of $\{1, 2, \ldots, n\}$ then, for $i \in \mathbf{h}_n^c(\alpha)$, the complement of $\mathbf{h}_n(\alpha)$,

$$\text{(1.6)} \qquad \begin{aligned} \hat{a}_{ni}(\alpha) &= 1, \qquad X_i > \mathbf{Y}_{i-1}'\hat{\boldsymbol{\rho}}_n(\alpha), \\ &= 0, \qquad X_i < \mathbf{Y}_{i-1}'\hat{\boldsymbol{\rho}}_n(\alpha), \end{aligned}$$

and, for $i \in \mathbf{h}_n(\alpha)$, $\hat{a}_{ni}(\alpha)$ is the solution of the $p+1$ linear equations

$$\text{(1.7)} \qquad \sum_{j \in h_n(\alpha)} \mathbf{Y}_{j-1}\hat{a}_{nj}(\alpha) = (1-\alpha)\sum_{j=1}^{n} \mathbf{Y}_{j-1} - \sum_{j=1}^{n} \mathbf{Y}_{j-1} I\big(X_j > \mathbf{Y}_{j-1}'\hat{\boldsymbol{\rho}}_n(\alpha)\big).$$

The scores $\{\hat{\mathbf{a}}_n(\alpha);\ \alpha \in (0,1)\}$ are the analogs of GJ's RRS's, herein called the *autoregression rank scores* (ARRS's). The continuity of the error d.f. F and the causality of the underlying process imply that $\hat{\mathbf{a}}_n(\alpha)$'s are unique for all $\alpha \in (0,1)$, w.p.1. The process $\hat{\mathbf{a}}_n \in [0,1]^n$ has piecewise linear paths in $[C(0,1)]^n$ and $\hat{\mathbf{a}}_n(0) = \mathbf{1}_n = \mathbf{1}_n - \hat{\mathbf{a}}_n(1)$. It is invariant in the sense that $\hat{\mathbf{a}}_n(\alpha)$ based on the observation vector $\mathbf{X}_n + \mathscr{X}_n\mathbf{t}$ is the sme as the $\hat{\mathbf{a}}_n(\alpha)$ based on $\mathbf{X}_n$, for all $\mathbf{t} \in \mathbb{R}^{p+1}$, $0 < \alpha < 1$. An obvious modification of the computation algorithms of Koenker and d'Orey (1987, 1994) for computing RQ's and RRS's can be adapted to compute the ARQ's and ARRS's. The comments of GJ on the duality of order statistics and ranks scores from the one-sample location model to the linear regression model by the RQ and RRS processes apply equally here to ARQ's and ARRS's.

Section 2 obtains the asymptotic joint distribution of a finite number of suitably normalized ARQ's and *asymptotic representations* of ARQ and ARRS processes. Section 3 applies these results to yield the asymptotic behavior of analogs of some of the above-mentioned L-estimators and ARRS statistics. It is observed that under appropriate conditions, the asymptotic equivalence between various L-estimators that exists in the linear regression setup continues to hold in the AR(p) setup.

A sequence of stochastic processes $\{\mathscr{Y}_n(\mathbf{t}),\ \mathbf{t} \in \mathbb{R}^{p+1}\}$, $n \geq 1$, is said to be *asymptotically uniformly linear* (a.u.l.) if it is approximated by a sequence of stochastic processes that is linear in $\mathbf{t}$, uniformly in $\mathbf{t}$ over compacts, in probability. Jurečková (1992a) proved that the regression rank-scores statistics based on residuals in the linear regression setup is a.u.l. in the standardized slope parameter vector, the basic result needed to obtain the asymptotic distributions of the estimators in Jurečková (1992b). The corresponding result for ARRS processes and statistics in the AR(p) model appears in Section 4 below. All of our proofs heavily depend on a *uniform closeness* result of a randomly weighted residual empirical process of Koul and Ossiander (1994).

For a $\mathbf{t} \in \mathbb{R}^{p+1}$, let $\|\mathbf{t}\|$ stand for its usual Euclidean norm. All the limits are taken as $n \to \infty$, unless mentioned otherwise. For a sequence of numbers (r.v.'s), $O(1)$ $[O_p(1)]$ denotes boundedness (in probability) and $o(1)$ $[o_p(1)]$ denotes the convergence to zero (in probability). For an $\mathbb{R}^{p+1}$-valued stochastic process $\{\mathscr{Z}_n(\alpha),\ \alpha \in [0,1]\}$, we say $\mathscr{Z}_n(\alpha) = O_p^*(1)[o_p^*(1)]$, if $\sup\{\|\mathscr{Z}_n(\alpha)\|,\ a \le \alpha \le 1-a\} = O_p(1)\,[o_p(1)]$, $\forall\ a \in (0, 1/2]$.

2. Asymptotic representations of ARQ's and ARRS's. This section obtains the asymptotic joint distribution of ARQ's and an asymptotic representation of ARRS processes. Their proofs are facilitated by Lemma 2.1 below. To state this lemma, let $(\Omega, \mathscr{A}, P)$ be a probability space and $(\varepsilon_i, \gamma_{ni}, \xi_{ni})$, $1 \le i \le n$, be an array of trivariate r.v.'s defined on $(\Omega, \mathscr{A})$ such that $\{\varepsilon_i\}$ are i.i.d. F; ε_i is independent of (γ_{ni}, ξ_{ni}), $1 \le i \le n$. Furthermore, let $\{\mathscr{A}_{ni}\}$ be an array of sub-σ-fields such that $\mathscr{A}_{ni} \subset \mathscr{A}_{ni+1}$, $\mathscr{A}_{ni} \subset \mathscr{A}_{n+1i}$, $1 \le i \le n$, $n \ge 1$, (γ_{n1}, ξ_{n1}) is $\mathscr{A}_{n1}$-measurable, the r.v.'s $\{\varepsilon_1, \dots, \varepsilon_{j-1};\ (\gamma_{ni}, \xi_{ni}),\ 1 \le i \le j\}$ are $\mathscr{A}_{nj}$-measurable, $2 \le j \le n$, and ε_j is independent of $\mathscr{A}_{nj}$, $1 \le j \le n$. Define, for an $x \in \mathbb{R}$,

$$
\begin{aligned}
V_n(x) &:= n^{-1} \sum_i \gamma_{ni} I(\varepsilon_i \le x + \xi_{ni}),\\
V_n^*(x) &:= n^{-1} \sum_i \gamma_{ni} I(\varepsilon_i \le x),\\
J_n(x) &:= n^{-1} \sum_i E\{\gamma_{ni} I(\varepsilon_i \le x + \xi_{ni}) | \mathscr{A}_{ni}\}\\
&= n^{-1} \sum_i \gamma_{ni} F(x + \xi_{ni}), \qquad (2.1)\\
J_n^*(x) &:= n^{-1} \sum_i \gamma_{ni} F(x),\\
U_n(x) &:= n^{1/2}(V_n(x) - J_n(x)),\\
U_n^*(x) &:= n^{1/2}(V_n^*(x) - J_n^*(x)).
\end{aligned}
$$

We are now ready to state our basic preliminary lemma.

LEMMA 2.1. *In addition to the above, assume that F is continuous and* *that the following hold*:

(A1) $$n^{-1/2}\max_i|\gamma_{ni}| = o_p(1),$$

(A2) $$\max_i|\xi_{ni}| = o_p(1),$$

(A3) $$\left(n^{-1}\sum_i \gamma_{ni}^2\right)^{1/2} = \gamma + o_p(1), \qquad \gamma \text{ a positive r.v.}$$

Then, $\forall\ x \in \mathscr{I} := \{x;\ 0 < F(x) < 1\}$,

(2.2) $$|U_n(x) - U_n^*(x)| = o_p(1).$$

If the continuity of F is strengthened to

(F1) *F has a continuous and positive density f on* $\mathscr{I}$,

then the processes $\{U_n\}$ *and* $\{U_n^*\}$ *are ventually tight on every compact subset of* $\mathbb{R}$ *in the uniform metric and*, $\forall\ 0 < K < \infty$,

(2.3) $$\sup_{|x|\le K}|U_n(x) - U_n^*(x)| = o_p(1).$$

If (F1) *is strengthened to*

(F2) *F has a uniformly continuous and a.e. positive density f on* $\mathscr{I}$,

then the processes $\{U_n\}$ *and* $\{U_n^*\}$ *are eventually tight in the uniform metric on* $\mathbb{R}$ *and*

(2.4) $$\sup_{x\in\mathscr{I}}|U_n(x) - U_n^*(x)| = o_p(1).$$

Under the additional assumption (A4),

(2.5) $$U_n \Rightarrow \gamma B(F), \qquad U_n^* \Rightarrow \gamma B(F),$$

where B is a Brownian bridge in $C[0,1]$, *independent of* γ, *and where*:

(A4) *For each* $n \ge 1$, $\{\gamma_{ni};\ 1 \le i \le n\}$ *is square integrable.*

Statements (2.2) and (2.4) are proved in Corollary 2.1 and Theorem 1.1 of Koul and Ossiander (1994), while (2.3) can be deduced from the details of the proof of Theorem 1.1 of the same paper. Now, consider the minimum-distance-type estimator of $\boldsymbol{\rho}$ defined by

$$\hat{\boldsymbol{\rho}}_{\mathrm{md}}(\alpha) = \arg\min_{\mathbf{t}}\|n^{1/2}\mathbf{T}_n(\mathbf{t},\alpha)\|^2,$$

(2.6) $$\mathbf{T}_n(\mathbf{t},\alpha) := n^{-1}\sum_i \mathbf{Y}_{i-1}\{I(X_i - \mathbf{t}'\mathbf{Y}_{i-1} \le 0) - \alpha\},$$

$$0 \le \alpha \le 1, \mathbf{t} \in \mathbb{R}^{p+1}.$$

By the continuity of F, $\mathbf{T}_n(\mathbf{t},\alpha)$ is an almost everywhere differential of the function $\sum_i h_\alpha(X_i - \mathbf{t}'\mathbf{Y}_{i-1})$ with respect to $\mathbf{t}$. The asymptotic equivalence of $\hat{\boldsymbol{\rho}}_{\mathrm{md}}(\alpha)$ to $\hat{\boldsymbol{\rho}}_n(\alpha)$ is given in Lemma 2.2 below. To state and prove this lemma, we need to define

$$\begin{aligned} &\boldsymbol{\rho}(\alpha) := \boldsymbol{\rho} + F^{-1}(\alpha)\mathbf{e}_1, \qquad \mathbf{e}_1 = (1, 0, \dots, 0)', \\ (2.7) \qquad &q(\alpha) := f(F^{-1}(\alpha)), \qquad 0 < \alpha < 1, \\ &\boldsymbol{\Sigma}_n := n^{-1}\mathscr{X}_n'\mathscr{X}_n = n^{-1}\sum_i \mathbf{Y}_{i-1}\mathbf{Y}_{i-1}', \qquad \boldsymbol{\Sigma} := \operatorname*{plim}_n \boldsymbol{\Sigma}_n. \end{aligned}$$

Recall, say from Hannan (1970), that under (a1) and (a2), $\boldsymbol{\Sigma}$ exists, and is positive definite.

LEMMA 2.2. *In addition to* (1.1), (a1) *and* (a2), *assume that F is continuous. Then,* $\forall\ x \in \mathscr{I}$, $\forall\ \mathbf{s} \in \mathbb{R}^{p+1}$,

$$\begin{aligned} &n^{1/2}\Big\| n^{-1}\sum_i \mathbf{Y}_{i-1}\big\{I\big(\varepsilon_i \le x + n^{-1/2}\mathbf{s}'\mathbf{Y}_{i-1}\big) \\ (2.8) \qquad &\qquad -F\big(x + n^{-1/2}\mathbf{s}'\mathbf{Y}_{i-1}\big) - I(\varepsilon_i \le x) + F(x)\big\}\Big\| = o_p(1), \end{aligned}$$

$$(2.9) \qquad \sup\{\|n^{1/2}\mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha)\|; \alpha \in [0,1]\} = O_p(1).$$

If the continuity of F is strengthened to (F1), *then* $\forall\ a \in (0, 1/2]$ *and* $\forall\ 0 < b < \infty$,

$$\begin{aligned} (2.10) \qquad \sup_{\substack{\|\mathbf{s}\| \le b \\ a \le \alpha \le 1-a}} &\Big\| n^{1/2}\big[\mathbf{T}_n\big(\boldsymbol{\rho}(\alpha) + n^{-1/2}\mathbf{s}, \alpha\big) - \mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha)\big] \\ &\qquad -\boldsymbol{\Sigma}_n\mathbf{s}q(\alpha)\Big\| = o_p(1) \end{aligned}$$

and

$$(2.11)(\mathrm{i}) \qquad n^{1/2}(\hat{\boldsymbol{\rho}}_{\mathrm{md}}(\alpha) - \boldsymbol{\rho}(\alpha)) = -\{q(\alpha)\boldsymbol{\Sigma}_n\}^{-1} n^{1/2}\mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha) + o_p^*(1),$$

$$(2.11)(\mathrm{ii}) \qquad n^{1/2}(\hat{\boldsymbol{\rho}}_n(\alpha) - \hat{\boldsymbol{\rho}}_{\mathrm{md}}(\alpha)) = o_p^*(1).$$

If (F1) *is replaced by* (F2), *then for every* $0 < b < \infty$,

$$\begin{aligned} (2.12) \qquad &\sup\big\| n^{1/2}\big[\mathbf{T}_n\big(\boldsymbol{\rho}(\alpha) + n^{-1/2}\mathbf{s}, \alpha\big) - \mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha)\big] - \boldsymbol{\Sigma}_n\mathbf{s}q(\alpha)\big\| \\ &\qquad = o_p(1), \end{aligned}$$

where the supremum is now over $(\alpha, \mathbf{s}) \in [0,1] \times \{\mathbf{s} \in \mathbb{R}^{p+1}; \|\mathbf{s}\| \le b\}$.

PROOF. To simplify the exposition, let $\tilde{\mathbf{T}}_n(\mathbf{s}, \alpha) := n^{1/2}\mathbf{T}_n(\boldsymbol{\rho}(\alpha) + n^{-1/2}\mathbf{s}, \alpha)$, $0 \le \alpha \le 1$, $\mathbf{s} \in \mathbb{R}^{p+1}$, and write $\tilde{\mathbf{T}}_n(\alpha)$ for $\tilde{\mathbf{T}}_n(\mathbf{0}, \alpha)$. Also, let $\mathscr{F}_i :=$ σ-field$\{\varepsilon_j;\ j \le i\}$. If in (2.1) we take

$$\begin{aligned} (2.13) \qquad &\gamma_{ni} \equiv j\text{th coordinate of } \mathbf{Y}_{i-1}, \qquad \xi_{ni} \equiv n^{-1/2}\mathbf{s}'\mathbf{Y}_{i-1}, \\ &\mathscr{A}_{ni} = \mathscr{F}_{i-1}, \qquad i \ge 1, \end{aligned}$$

then $U_n(F^{-1}(\alpha))$ and $U_n^*(F^{-1}(\alpha))$ are equal to the jth coordinate of $\tilde{\mathbf{T}}_n(\mathbf{s}, \alpha)$ and $\tilde{\mathbf{T}}_n(\alpha)$, respectively, $0 \le \alpha \le 1$, $\mathbf{s} \in \mathbb{R}^{p+1}$, $1 \le j \le p$. Using the ergodicity and stationarity of the underlying process it follows that (A1)–(A4) are implied by (a1) and (a2) for the entities given in (2.13). Thus (2.2) and (2.5) applied p times, jth time to the entities given at (2.13), imply (2.8) and (2.9), respectively.

To prove (2.10), apply (2.3) to the entities given at (2.13) to conclude that for every $\mathbf{s} \in \mathbb{R}^{p+1}$, $a \in (0, 1/2]$,

$$\sup_{a \le \alpha \le 1-a} \left\| \tilde{\mathbf{T}}_n(\mathbf{s}, \alpha) - \tilde{\mathbf{T}}_n(\alpha) - \boldsymbol{\Sigma}_n \mathbf{s} q(\alpha) \right\| = o_p(1).$$

The uniformity over $\|\mathbf{s}\| \le b$ is achieved by exploiting the monotonicity of the indicator function together with the compactness of the set $\{\mathbf{s}\colon \|\mathbf{s}\| \le b\}$. The details are similar to those in the proof of Theorem 1.2 of Koul (1991) and are left out for the sake of brevity. The proof of (2.12) follows from (2.4) in the same way as does that of (2.10) from (2.3).

To prove (2.11)(i), it suffices to prove the following: $\forall\, \varepsilon > 0$, $0 < z < \infty$, $\exists\, b$ $(= b_z^{\varepsilon})$ and an N^{ε}, such that $\forall\, a \in (0, 1/2]$,

$$P\left(\inf_{\|\mathbf{s}\| > b} \left\| \tilde{\mathbf{T}}_n(\mathbf{s}, \alpha) \right\| > z, \forall\, \alpha \in [a, 1-a] \right) \ge 1 - \varepsilon, \qquad n \ge N^{\varepsilon}. \tag{2.14}$$

Note that (2.14) implies that $\sup\{n^{1/2}\|\hat{\boldsymbol{\rho}}_{\mathrm{md}}(\alpha) - \boldsymbol{\rho}(\alpha)\|;\ a \le \alpha \le 1-a\} = O_p(1)$. This and (2.10) imply (2.11)(i) in a routine fashion. We proceed to prove (2.14).

Fix an $\varepsilon > 0$ and $a \in (0, 1/2]$. Write an $\mathbf{s} \in \mathbb{R}^{p+1}$ with $\|\mathbf{s}\| > b$ as $\mathbf{s} = r\boldsymbol{\theta}$, $|r| > b$ and $\|\boldsymbol{\theta}\| = 1$. Also note that $\boldsymbol{\theta}'\tilde{\mathbf{T}}_n(r\boldsymbol{\theta}, \alpha)$ is a nondecreasing function of r, $\forall\, \alpha \in (0,1)$, $\boldsymbol{\theta} \in \mathbb{R}^{p+1}$. Therefore, by the Cauchy–Schwarz (C-S) inequality, $\forall\, \alpha \in (0,1)$,

$$\inf_{\|\mathbf{s}\| > b} \left\| \tilde{\mathbf{T}}_n(\mathbf{s}, \alpha) \right\|^2 \ge \inf_{\substack{|r| > b \\ \|\boldsymbol{\theta}\| = 1}} \{\boldsymbol{\theta}'\tilde{\mathbf{T}}_n(r\boldsymbol{\theta}, \alpha)\}^2 \ge \inf_{\substack{|r| = b \\ \|\boldsymbol{\theta}\| = 1}} \{\boldsymbol{\theta}'\tilde{\mathbf{T}}_n(r\boldsymbol{\theta}, \alpha)\}^2. \tag{2.15}$$

Now, let $T_n^*(r, \boldsymbol{\theta}, \alpha) := \boldsymbol{\theta}'\tilde{T}_n(\alpha) - r\boldsymbol{\theta}'\boldsymbol{\Sigma}_n\boldsymbol{\theta} q(\alpha)$, $k_n := \inf\{\boldsymbol{\theta}'\boldsymbol{\Sigma}_n\boldsymbol{\theta};\ \|\boldsymbol{\theta}\| = 1\}$, $k := \operatorname{plim}_n k_n = \inf\{\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta};\ \|\boldsymbol{\theta}\| = 1\}$ and $A_n^{\varepsilon} := [k(1-\varepsilon) \le k_n \le (1+\varepsilon)k]$. By the positive definiteness of $\boldsymbol{\Sigma}$, $k > 0$. This and (2.9) imply that $\exists\, K_{\varepsilon}$, N_1^{ε} and N_2^{ε}, such that

$$\begin{aligned} P(A_n^{\varepsilon}) &\ge 1 - (\varepsilon/3), \qquad n \ge N_1^{\varepsilon}, \\ P\left(\sup_{\substack{\|\boldsymbol{\theta}\| = 1 \\ 0 \le \alpha \le 1}} \left| \boldsymbol{\theta}'\tilde{\mathbf{T}}_n(\alpha) \right| \le K\varepsilon \right) &\ge 1 - (\varepsilon/3), \qquad n \ge N_2^{\varepsilon}. \end{aligned} \tag{2.16}$$

Also, by (2.10), $\exists\, N_3^{\varepsilon}$, such that $\forall\, n > N_3^{\varepsilon}$,

$$\begin{aligned} &P\left(\inf_{\substack{|r| = b \\ \|\boldsymbol{\theta}\| = 1}} \left\{ \boldsymbol{\theta}'\tilde{\mathbf{T}}_n(r\boldsymbol{\theta}, \alpha) \right\}^2 > z, \forall\, a \le \alpha \le 1-a \right) \\ &\quad \ge P\left(\inf_{\substack{|r| = b \\ \|\boldsymbol{\theta}\| = 1}} \left\{ \mathbf{T}_n^*(r, \boldsymbol{\theta}, \alpha) \right\}^2 > z, \forall\, a \le \alpha \le 1-a \right) - \varepsilon/3. \end{aligned} \tag{2.17}$$

But, using the fact that $||d| - |c|| \le |d + c|$, d, c real, we have

$$P\left(\inf_{\substack{|r|=b\\ \|\theta\|=1}} \{\mathbf{T}_n^*(r,\theta,\alpha)\}^2 > z, \forall\, a \le \alpha \le 1-a\right)$$

$$\ge P\left(\left|\theta'\tilde{\mathbf{T}}_n(\alpha)\right| - b\theta'\Sigma_n\theta q(\alpha)\right| > z^{1/2}, \forall\, \|\theta\| = 1, a \le \alpha \le 1-a\right)$$

$$\ge P\left(\left|\theta'\tilde{\mathbf{T}}_n(\alpha)\right| \le -z^{1/2} + bk_n q_a, \forall\, \|\theta\| = 1, a \le \alpha \le 1-a\right)$$

$$\ge P\left(\sup_{\substack{\|\theta\|=1\\ 0\le\alpha\le 1}} \left|\theta'\tilde{\mathbf{T}}_n(\alpha)\right| \le -z^{1/2} + bk(1-\varepsilon)q_a;\, A_n^{\varepsilon}\right)$$

$$\ge 1 - (2\varepsilon/3), \qquad n \ge N^{\varepsilon} := N_1^{\varepsilon} \vee N_2^{\varepsilon} \vee N_3^{\varepsilon},$$

by (2.16), as long as $b \ge (K\varepsilon + z^{1/2})/\{k(1-\varepsilon)q_a\}$, where $q_a := \inf\{q(\alpha); a \le \alpha \le 1-a\}$. This together with (2.15) and (2.17) proves (2.14).

The details of our proof of (2.11)(ii) are very similar to those of Lemmas 4.1 and 4.2 and Corollary 4.1 of Jurečková (1971). Thus we shall first show that $\forall\, a \in (0, 1/2]$,

$$\sup_{a\le\alpha\le 1-a} \left\|n^{1/2}\mathbf{T}_n(\hat{\boldsymbol{\rho}}_n(\alpha),\alpha)\right\| = o_p(1). \tag{2.18}$$

The inequalities in (3.1) of Theorem 3.3 of KB, when adapted to the current setup and after writing $\operatorname{sgn}(x) \equiv 1 - 2I(x \le 0) + I(x = 0)$, state that w.p.1., $\forall\, 0 < \alpha < 1$,

$$(\alpha - 1)\mathbf{1}_p < \sum_{i\in\mathbf{h}_n^c(\alpha)} \mathbf{Y}_{i-1}'\{I(X_i - \mathbf{Y}_{i-1}'\hat{\boldsymbol{\rho}}_n(\alpha) \le 0) - \alpha\}\mathscr{X}_{h_n(\alpha)}^{-1}$$
$$+ \sum_{i\in\mathbf{h}_n^c(\alpha)} \mathbf{Y}_{i-1}'I(X_i - \mathbf{Y}_{i-1}'\hat{\boldsymbol{\rho}}_n(\alpha) = 0)\mathscr{X}_{\mathbf{h}_n(\alpha)}^{-1} < \alpha\mathbf{1}_p,$$

where $\mathbf{h}_n(\alpha)$ is as in (1.4) and $\mathbf{h}_n^c(\alpha)$ denotes its complement. Let $\mathbf{w}_n'(\alpha)$ denote the vector inside the inequalities. Now, from (1.4), it follows that $I(X_i - \mathbf{Y}_{i-1}'\hat{\boldsymbol{\rho}}_n(\alpha) = 0) = 0$, for all $i \in \mathbf{h}_n^c(\alpha)$. Thus we obtain the following: w.p.1, $\forall\, 0 < \alpha < 1$,

$$\left[\sum_i \mathbf{Y}_{i-1}'\{I(X_i' - \mathbf{Y}_{i-1}'\hat{\boldsymbol{\rho}}_n(\alpha) \le 0) - \alpha\}\right.$$
$$\left. - \sum_{i\in\mathbf{h}_n(\alpha)} \mathbf{Y}_{i-1}'\{I(X_i - \mathbf{Y}_{i-1}'\hat{\boldsymbol{\rho}}_n(\alpha) \le 0) - \alpha\}\right]\mathscr{X}_{\mathbf{h}_n(\alpha)}^{-1} = \mathbf{w}_n'(\alpha).$$

Again, by (1.4), $I(X_i - \mathbf{Y}_{i-1}'\hat{\boldsymbol{\rho}}_n(\alpha) \le 0) = 1$, $i \in \mathbf{h}_n(\alpha)$, $0 < \alpha < 1$, w.p.1. Hence, w.p.1, $\forall\, 0 < \alpha < 1$,

$$n^{1/2}\mathbf{T}_n(\hat{\boldsymbol{\rho}}_n(\alpha),\alpha) = n^{-1/2}\sum_{i\in\mathbf{h}_n(\alpha)} \mathbf{Y}_{i-1}(1-\alpha) + n^{-1/2}\mathscr{X}_{\mathbf{h}_n(\alpha)}'\mathbf{w}_n(\alpha),$$

so that

$$\sup_{a \le \alpha \le 1-a} \left\| n^{1/2} \mathbf{T}_n(\hat{\boldsymbol{\rho}}_n(\alpha), \alpha) \right\| \le 2(p+1) \max_i n^{-1/2} \|\mathbf{Y}_{i-1}\|. \tag{2.19}$$

Now (2.18) follows from (2.19) and the fact that $\max_i n^{-1/2} \|\mathbf{Y}_{i-1}\| = o_p(1)$, which is implied by (a1) and (a2). Consequently,

$$\sup_{a \le \alpha \le 1-a} \inf_{\mathbf{t}} \left\| n^{1/2} \mathbf{T}_n(\mathbf{t}, \alpha) \right\| = o_p(1). \tag{2.20}$$

Note that (2.18) and (2.20) are analogs of Lemma 4.1 and Corollary 4.1 of Jurečková. Use them and an argument similar to the proof of Lemma 4.2 of Jurečková (1971) together with (2.8) and (2.10) to conclude (2.11)(ii). The details are left out for the sake of brevity. □

The following theorem gives the asymptotic joint distribution of a finite number of ARQ's and the asymptotic representation of the ARQ process on $D[a, 1-a]$, $\forall\, a \in (0, 1/2]$, as a generalization of Theorem 4.1 of KB to the AR(p) model (1.1). In it, $\mathbf{A} \oplus \mathbf{B}$ denotes the Kronecker product, for any two compatible matrices $\mathbf{A}$ and $\mathbf{B}$.

THEOREM 2.1. *In addition to* (1.1), (a1) *and* (a2), *assume that* (F1) *holds. Then*

$$\begin{aligned} n^{1/2}&(\hat{\boldsymbol{\rho}}_n(\alpha) - \boldsymbol{\rho}(\alpha)) \\ &= -\boldsymbol{\Sigma}_n^{-1} n^{1/2} \mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha)/q(\alpha) + o_p^*(1), \qquad 0 < \alpha < 1. \end{aligned} \tag{2.21}$$

Consequently, for any $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k < 1$, *the asymptotic joint distribution of* $[n^{1/2}(\hat{\boldsymbol{\rho}}_n(\alpha_1) - \boldsymbol{\rho}(\alpha_1)), \ldots, n^{1/2}(\hat{\boldsymbol{\rho}}_n(\alpha_k) - \boldsymbol{\rho}(\alpha_k))]$ *is* $(p+1) \times k$-*variate normal with mean matrix* $\mathbf{0}$ *and the covaraince matrix* $\mathscr{C} := \mathbf{A} \oplus \boldsymbol{\Sigma}^{-1}$, *where* $\mathbf{A}$ *is a* $k \times k$ *matrix whose* (i,j)*th entry* $a_{ij} := (\alpha_i - \alpha_i \alpha_j)/q(\alpha_i) q(\alpha_j)$, $1 \le i \le j \le k$.

PROOF. The proof of (2.21) follows from (2.11) in an obvious fashion. This in turn implies that

$$\begin{aligned} \big[n^{1/2}&(\hat{\boldsymbol{\rho}}_n(\alpha_1) - \boldsymbol{\rho}(\alpha_1)), \ldots, n^{1/2}(\hat{\boldsymbol{\rho}}_n(\alpha_k) - \boldsymbol{\rho}(\alpha_k))\big] \\ &= -\{1/q(\alpha_1), \ldots, 1/q(\alpha_k)\} \oplus \boldsymbol{\Sigma}_n^{-1} n^{1/2} \\ &\quad \times \big[\mathbf{T}_n(\boldsymbol{\rho}(\alpha_1), \alpha_1), \ldots, \mathbf{T}_n(\boldsymbol{\rho}(\alpha_k), \alpha_k)\big] + o_p(1). \end{aligned}$$

The asymptotic normality now follows from this and the fact that $n\mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha)$ is a mean zero square integrable martingale array with respect to the σ-fields $\{\mathscr{F}_{i-1}\}$ of (2.13), Corollary 3.1 of Hall and Heyde (1980) and the Cramér–Wold device. □

We now turn to the ARRS processes. To define these, let $q \ge 1$ be an integer, $\{g_{nij};\ 1 \le i \le n;\ 1 \le j \le q\}$ be r.v.'s such that $\{g_{nij};\ 1 \le j \le q\}$ are $\mathscr{F}_{i-1}$-measurable and independent of ε_i, $1 \le i \le n$. Let $\mathbf{g}'_{ni} := (g_{ni1}, g_{ni2}, \ldots, g_{niq})$, $1 \le i \le n$, and $\mathscr{G}$ be $n \times q$ matrix whose ith row equals $\mathbf{g}'_{ni}$, $1 \le i \le n$. Define sequences of ARRS processes

$$\hat{\mathbf{U}}_{ng}(\alpha) = n^{-1} \sum_i \mathbf{g}_{ni}\{\hat{a}_{ni}(\alpha) - (1-\alpha)\}, \qquad 0 \le \alpha \le 1, \tag{2.22}$$

and an approximating weighted empirical process $\mathbf{U}_{n\mathbf{g}}$ by

$$\mathbf{U}_{n\mathbf{g}}(\alpha) = n^{-1} \sum_i \mathbf{g}_{ni}\{I(\varepsilon_i > F^{-1}(\alpha)) - (1-\alpha)\}, \qquad 0 \le \alpha \le 1. \tag{2.23}$$

Let $\mathbf{U}_{n\mathbf{x}}$ stand for $\mathbf{U}_{n\mathbf{g}}$ when $q = p+1$ and $\mathbf{g}_{ni} \equiv \mathbf{Y}_{i-1}$. Observe that $\mathbf{U}_{n\mathbf{x}}(\alpha) \equiv -\mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha)$.

Before stating the next result, we need to introduce the following two conditions:

(G1) $n^{-1}(\mathscr{G}'\mathscr{G}) = \boldsymbol{\Gamma} + o_p(1)$ where $\boldsymbol{\Gamma}$ is a positive definite $q \times q$ matrix,

(G2) $$n^{-1/2} \max_i \|\mathbf{g}_{ni}\| = o_p(1).$$

Let $\mathscr{W}_n := n^{-1}\mathscr{G}'\mathscr{X}_n$, $\hat{\boldsymbol{\Delta}}_n(\alpha) := n^{1/2}(\hat{\boldsymbol{\rho}}_n(\alpha) - \rho(\alpha))$, $0 < \alpha < 1$. From (2.9) and (2.21) we obtain that $\forall\ a \in (0, 1/2]$,

$$\sup\{\|\hat{\boldsymbol{\Delta}}_n(\alpha)\|;\ a \le \alpha \le 1-a\} = O_p(1). \tag{2.24}$$

We are now ready to state and prove the following theorem.

THEOREM 2.2. *Assume that* (1.1), (a1), (a2), (F1), (G1) *and* (G2) *hold. Then, for every* $a \in (0, 1/2]$,

$$n^{1/2}\left[\hat{\mathbf{U}}_{n\mathbf{g}}(\alpha) - \mathbf{U}_{n\mathbf{g}}(\alpha)\right] = -\mathscr{W}_n \cdot \hat{\boldsymbol{\Delta}}_n(\alpha) q(\alpha) + o_p^*(1). \tag{2.25}$$

Consequently,

$$n^{1/2}\hat{\mathbf{U}}_{n\mathbf{g}}(\alpha) = n^{1/2}\left[\mathbf{U}_{n\mathbf{g}}(\alpha) - \mathscr{W}_n \boldsymbol{\Sigma}_n^{-1}\mathbf{U}_{n\mathbf{x}}(\alpha)\right] + o_p^*(1). \tag{2.26}$$

PROOF. From (1.6) we obtain that w.p.1, $\forall\ 1 \le i \le n$, $0 < \alpha < 1$,

$$\hat{a}_{ni}(\alpha) = I(\varepsilon_i > F^{-1}(\alpha) + \mathbf{Y}_{i-1}'\{\hat{\boldsymbol{\rho}}_n(\alpha) - \boldsymbol{\rho}(\alpha)\}) + \hat{a}_{ni}(\alpha) I(X_i = \mathbf{Y}_{i-1}'\hat{\boldsymbol{\rho}}_n(\alpha)).$$

This in turn gives the following identity:

$$\begin{aligned} &\hat{a}_{ni}(\alpha) - (1-\alpha) \\ &\quad = I(\varepsilon_i > F^{-1}(\alpha)) - (1-\alpha) \\ &\qquad - \{I(\varepsilon_i \le F^{-1}(\alpha) + \mathbf{Y}_{i-1}'(\hat{\boldsymbol{\rho}}_n(\alpha) - \boldsymbol{\rho}(\alpha))) - I(\varepsilon_i \le F^{-1}(\alpha))\} \\ &\qquad + \hat{a}_{ni}(\alpha) I(X_i = \mathbf{Y}_{i-1}'\hat{\boldsymbol{\rho}}_n(\alpha)), \qquad 1 \le i \le n, 0 < \alpha < 1, \text{w.p.1.} \end{aligned} \tag{2.27}$$

From this and (1.4) we obtain that

$$
\begin{aligned}
&n^{1/2}\hat{\mathbf{U}}_{n\mathbf{g}}(\alpha)\\
&\quad= n^{1/2}\mathbf{U}_{n\mathbf{g}}(\alpha) - \mathscr{W}_n\hat{\boldsymbol{\Delta}}_n(\alpha)q(\alpha)\\
&\qquad -\Big[n^{-1/2}\sum_i \mathbf{g}_{ni}\Big\{I\big(\varepsilon_i \le F^{-1}(\alpha) + n^{-1/2}\mathbf{Y}_{i-1}'\hat{\boldsymbol{\Delta}}_n(\alpha)\big)\\
&\qquad\qquad -I\big(\varepsilon_i \le F^{-1}(\alpha)\big)\Big\} - \mathscr{W}_n\hat{\boldsymbol{\Delta}}_n(\alpha)q(\alpha)\Big]\\
&\qquad + n^{-1/2}\sum_{i\in\mathbf{h}_n(\alpha)} \mathbf{g}_{ni}\hat{a}_{ni}(\alpha)I(X_i = \mathbf{Y}_{i-1}'\hat{\boldsymbol{\rho}}_n(\alpha))\\
&\quad= n^{1/2}\mathbf{U}_{n\mathbf{g}}(\alpha) - \mathscr{W}_n\hat{\boldsymbol{\Delta}}_n(\alpha)q(\alpha) - \mathbf{R}_1(\alpha) + \mathbf{R}_2(\alpha) \quad \text{(say)}.
\end{aligned}
$$

From (a1), (a2), (G1) and the C-S inequality it readily follows that $\|\mathscr{W}_n\| = O_p(1)$. Now, apply (2.3) to $\gamma_{ni} \equiv g_{nij}$ and other entities as in (2.13) and an argument like the one used to prove (2.10), to conclude, in view of (2.24), that $\sup\{\|\mathbf{R}_1(\alpha)\|: a \le \alpha \le 1 - a\} = o_p(1)$. By (G2), $\sup\{\|\mathbf{R}_2(\alpha)\|: 0 \le \alpha \le 1\} = o_p(1)$. Hence (2.25) follows. □

COROLLARY 2.1. *Under the assumptions of Theorem* 2.2, *the ARQ and ARRS processes are asymptotically independent. Moreover, for every* $k \ge 1$ *and for every* $0 < \alpha_1 < \cdots < \alpha_k$, *the asymptotic distribution of* $n^{1/2}(\hat{\mathbf{U}}_{n\mathbf{g}}(\alpha_1), \dots, \hat{\mathbf{U}}_{n\mathbf{g}}(\alpha_k))$ *is normal with the mean matrix* $\mathbf{0}$ *and the covariance matrix* $\mathbf{B} \oplus \operatorname{plim} n^{-1}[\mathscr{G}' - \mathscr{W}_n\boldsymbol{\Sigma}_n^{-1}\mathscr{X}_n'][\mathscr{G}' - \mathscr{W}_n\boldsymbol{\Sigma}_n^{-1}\mathscr{X}_n']'$, *where* $\mathbf{B}$ *is a* $k \times k$ *symmetric matrix with* (i,j)*th entry* $b_{ij} = (\alpha_i - \alpha_i\alpha_j)$, $1 \le i \le j \le k$.

PROOF. Let $s_i(\alpha) := I(\varepsilon_i > F^{-1}(\alpha)) - (1 - \alpha)$, $1 \le i \le n$; $\mathbf{s}(\alpha) := (s_1(\alpha), \dots, s_n(\alpha))'$, $0 < \alpha < 1$. Observe that the leading r.v.'s in the right-hand sides of (2.21) and (2.26) are equal to $-\boldsymbol{\Sigma}_n^{-1}n^{-1/2}\mathscr{X}_n'\mathbf{s}(\alpha)/q(\alpha)$ and $n^{-1/2}[\mathscr{G}' - \mathscr{W}_n\boldsymbol{\Sigma}_n^{-1}\mathscr{X}_n']\mathbf{s}(\alpha)$, respectively. By Corollary 3.1 of Hall and Heyde (1980), the stationarity and ergodicity of the underlying processes, and by the Cramér–Wold device, it follows that for every $\alpha \in (0,1)$, the asymptotic joint distribution of $\hat{\boldsymbol{\Delta}}_n(\alpha)$ and $n^{1/2}\hat{\mathbf{U}}_{n\mathbf{g}}(\alpha)$ is $(p + 1 + q)$-dimensional normal with the mean vector $\mathbf{0}$ and the covariance matrix

$$
\mathscr{D} := \begin{bmatrix} \mathscr{D}_{11} & \mathscr{D}_{12} \\ \mathscr{D}_{12} & \mathscr{D}_{22} \end{bmatrix},
$$

where

$$
\begin{aligned}
\mathscr{D}_{11} &:= \big[\alpha(1-\alpha)/q^2(\alpha)\big]\boldsymbol{\Sigma}^{-1},\\
\mathscr{D}_{22} &:= \operatorname*{plim}_n n^{-1}\big[\mathscr{G}' - \mathscr{W}_n\boldsymbol{\Sigma}_n^{-1}\mathscr{X}_n'\big]\big[\mathscr{G}' - \mathscr{W}_n\boldsymbol{\Sigma}_n^{-1}\mathscr{X}_n'\big]',\\
\mathscr{D}_{12} &:= \big[\alpha(1-\alpha)/q(\alpha)\big]\operatorname*{plim}_n n^{-1}\boldsymbol{\Sigma}_n^{-1}\mathscr{X}_n'\big[\mathscr{G}' - \mathscr{W}_n\boldsymbol{\Sigma}_n^{-1}\mathscr{X}_n'\big]'.
\end{aligned}
$$

However, by definition, $n^{-1}\boldsymbol{\Sigma}_n^{-1}\mathscr{X}_n'[\mathscr{G}' - \mathscr{W}_n\boldsymbol{\Sigma}_n^{-1}\mathscr{X}_n']' \equiv \boldsymbol{\Sigma}_n^{-1}\mathscr{W}_n' - \boldsymbol{\Sigma}_n^{-1}\mathscr{W}_n' \equiv \mathbf{0}$, for all $n \ge 1$, w.p.1. This proves the claim of independence for each $\alpha \in (0,1)$. The result is proved similarly for any finite-dimensional joint distribution. □

3. L-Estimators of $\boldsymbol{\rho}$ and regression rank-scores statistics. In this section we derive the asymptotic distributions of two types of L-estimators based on ARQ and ARRS processes. For a finite signed measure ν with compact support on $(0,1)$ an L-estimator of the *first type* of $\boldsymbol{\rho}$ is defined to be

$$\mathscr{T}_n^\nu = \int_0^1 \hat{\boldsymbol{\rho}}_n(\alpha)\, d\nu(\alpha). \tag{3.1}$$

For the second type of L-estimator, assume that ν is a probability measure, absolutely continuous with respect to the Lebesgue measure on $(0,1)$, with density J of bounded variation on $(0,1)$ and vanishing outside compacts in $(0,1)$. Then the *second type* of L-estimator is defined to be

$$\mathscr{L}_n^\nu := \left(\mathscr{X}_n'\hat{\mathscr{T}}_n\mathscr{X}_n\right)^{-1}\mathscr{X}_n'\hat{\mathscr{T}}_n\mathbf{X}_n, \tag{3.2}$$

where $\hat{\mathscr{T}}_n := \operatorname{diag}\{\hat{\mathscr{T}}_{ni},\ 1 \le i \le n\}$, with $\hat{\mathscr{T}}_{ni} := -\int_0^1 J(t)\, d\hat{a}_{ni}(t) = \int_0^1 \hat{a}_{ni}(t)\, dJ(t)$, $1 \le i \le n$. Observe that because $\int_0^1 J(\alpha)\, d\alpha = 1 = \int_0^1 (1-\alpha)\, dJ(\alpha)$ and because by (1.5),

$$\sum_i \mathbf{Y}_{i-1}\{\hat{a}_{ni}(\alpha) - (1-\alpha)\} = \mathbf{0}, \qquad n \ge 1,\ \alpha \in (0,1), \tag{3.3}$$

we obtain

$$\begin{aligned} n^{-1}\left(\mathscr{X}_n'\hat{\mathscr{T}}_n\mathscr{X}_n\right) &= n^{-1}\sum_i \mathbf{Y}_{i-1}\mathbf{Y}_{i-1}'\hat{\mathscr{T}}_{ni} \\ &= n^{-1}\sum_i \mathbf{Y}_{i-1}Y_{i-1}'\int_0^1 \hat{a}_{ni}(\alpha)\, dJ(\alpha) \\ &= n^{-1}\int_0^1 \sum_i \mathbf{Y}_{i-1}\{\hat{a}_{ni}(\alpha) - (1-\alpha)\}\, dJ(\alpha)\mathbf{Y}_{i-1}' \\ &\quad + n^{-1}\sum_i \mathbf{Y}_{i-1}\mathbf{Y}_{i-1}' \\ &= \boldsymbol{\Sigma}_n \quad \text{for all } n \ge 1, \text{ w.p.1.} \end{aligned} \tag{3.4}$$

Observations (3.3) and (3.4) will be used repeatedly in the sequel.

Estimators in (3.1) and (3.2) are analogs of their regression counterparts as defined at (4.15) and (4.21) in GJ. Various comments appearing in GJ with regard to the connection between trimmed mean in the one-sample location model and these estimators equally apply to the current setup.

The following theorem is an immediate consequence of Theorem 2.1.

THEOREM 3.1. *Assume that* (1.1), (a1), (a2) *and* (F1) *hold. Let* ν *be a finite signed measure with compact support in* $(0,1)$. *Then*

$$\begin{aligned} &n^{1/2}\{\mathscr{T}_n^\nu - \boldsymbol{\rho}(\nu, F)\} \\ &\qquad = -\boldsymbol{\Sigma}_n^{-1}n^{1/2}\int_0^1 \{\mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha)/q(\alpha)\}\, d\nu(\alpha) + o_p(1), \end{aligned} \tag{3.5}$$

where $m := \nu(0,1)$ *and* $\boldsymbol{\rho}(\nu, F) := \boldsymbol{\rho}m + \mathbf{e}_1\int_0^1 F^{-1}(\alpha)\, d\nu(\alpha)$. *Consequently,*

$$n^{1/2}\{\mathscr{T}_n^\nu - \boldsymbol{\rho}(\nu, F)\} \Rightarrow N(0, \boldsymbol{\Sigma}^{-1}\tau^2), \tag{3.6}$$

where $\tau^2 := \int_0^1\int_0^1\{[s \wedge t - st]/q(s)q(t)\}\, d\nu(s)\, d\nu(t)$.

REMARK. 3.1. As in the one-sample and linear regression models, there is an asymptotic equivalence between these L-estimators and the M-estimators of $\boldsymbol{\rho}$. More precisely, let ψ be a nondecreasing bounded function from $\mathbb{R}$ to $\mathbb{R}$ with a compact support in $\mathbb{R}$ and such that $\int \psi\, dF = 0$. An M-estimator $\tilde{\boldsymbol{\rho}}$ of $\boldsymbol{\rho}$ corresponding to the given ψ is defined as a solution $\mathbf{t}$ of the equation

$$\mathbf{W}_n(\mathbf{t}) := n^{-1} \sum_i \mathbf{Y}_{i-1}\psi(X_i - \mathbf{t}'\mathbf{Y}_{i-1}) = 0.$$

Observe that

$$\mathbf{W}_n(\mathbf{t}) \equiv \psi(\infty) n^{-1} \sum_i \mathbf{Y}_{i-1} - \int n^{-1} \sum_i \mathbf{Y}_{i-1} I(X_i - \mathbf{t}'\mathbf{Y}_{i-1} \le x)\, d\psi(x).$$

This and an application of (2.3) to the entities given at (2.13) and the monotonicity of the indicator function readily yield, under (a1), (a2) and (F1), that $\forall\, L < \infty$,

$$\sup_{\|\mathbf{s}\| \le L} \left\| n^{1/2}\left[\mathbf{W}_n(\boldsymbol{\rho} + n^{-1/2}\mathbf{s}) - \mathbf{W}_n(\boldsymbol{\rho})\right] + \boldsymbol{\Sigma}_n \mathbf{s} \int f(x)\, d\psi(x) \right\| = o_p(1).$$

Furthermore, an argument similar to the one used in proving (2.14) above shows that $\|n^{1/2}(\tilde{\boldsymbol{\rho}} - \boldsymbol{\rho})\| = O_p(1)$. Consequently one obtains that if ψ is as above and if (a1), (a2) and (F1) hold, then

$$n^{1/2}(\tilde{\boldsymbol{\rho}} - \boldsymbol{\rho}) = \boldsymbol{\Sigma}_n^{-1}\left\{\int f\, d\psi\right\}^{-1} n^{1/2}\mathbf{W}_n(\boldsymbol{\rho}) + o_p(1).$$

This fact and simple algebra show that if ψ is related to ν by the relation

$$\left\{\int f\, d\psi\right\}^{-1} d\psi(F^{-1}(\alpha)) \equiv \{q(\alpha)\}^{-1}\, d\nu(\alpha),$$

then the corresponding M-estimator $\tilde{\boldsymbol{\rho}}$ satisfies the relation

$$n^{1/2}(\tilde{\boldsymbol{\rho}} - \boldsymbol{\rho}) = -\boldsymbol{\Sigma}_n^{-1} n^{1/2} \int_0^1 \{\mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha)/q(\alpha)\}\, d\nu(\alpha) + o_p(1).$$

Put this observation together with (3.5) to readily obtain that

$$n^{1/2}\left\{\mathscr{T}_n^{\nu} - \boldsymbol{\rho} - \mathbf{e}_1 \int x f(x)\, d\psi(x) \Big/ \int f\, d\psi\right\}$$
$$= \left\{\boldsymbol{\Sigma}_n \int f\, d\psi\right\}^{-1} \cdot n^{-1/2} \sum_i \mathbf{Y}_{i-1}\psi(\varepsilon_i) + o_p(1).$$

The next theorem establishes the asymptotic equivalence of $\mathscr{L}_n^{\nu}$ and $\mathscr{T}_n^{\nu}$.

THEOREM 3.2. *Assume that* (1.1), (a1), (a2) *and* (F1) *hold. Let* ν *be a probability measure with density* J, *with respect to the Lebesgue measure on* $(0, 1)$, *such that* J *is of bounded variation and vanishes outside a compact* *subinterval in* $(0,1)$. *Then*

$$n^{1/2}(\mathscr{L}_n^{\nu} - \boldsymbol{\rho}(\nu, F)) = n^{1/2}(\mathscr{T}_n^{\nu} - \boldsymbol{\rho}(\nu, F)) + o_p(1). \tag{3.7}$$

PROOF. Many algebraic calculations are similar to those in GJ, so we shall be brief whenever possible. Note that now $m = 1$. Let $\mu = \mu(\nu, F) := \int_0^1 F^{-1}(\alpha)\, d\nu(\alpha)$. Then from the definition (3.2) and relation (3.4),

$$\begin{aligned} \boldsymbol{\Sigma}_n n^{1/2}(\mathscr{L}_n^{\nu} - \boldsymbol{\rho}(\nu, F)) &= n^{-1/2}\mathscr{X}_n'\hat{\mathscr{T}}_n(\mathbf{X}_n - \mathscr{X}_n\boldsymbol{\rho} - \mu\mathscr{X}_n\mathbf{e}_1) \\ &= n^{-1/2}\sum_i \mathbf{Y}_{i-1}\hat{\mathscr{T}}_{ni}(\varepsilon_i - \mu). \end{aligned} \tag{3.8}$$

Next, define $\psi_F^{\nu}(t) := \int_0^1[\alpha - I(t \le F^{-1}(\alpha))]\{q(\alpha)\}^{-1}\, d\nu(\alpha)$, $t \in \mathbb{R}$. An integration by parts and the fact that $\int_0^1 \alpha\, dJ(\alpha) + 1 = \int_0^1 \alpha\, dJ(\alpha) + \int_0^1 J(\alpha)\, d\alpha = \int_0^1 \alpha\, dJ(\alpha) + \int_0^1 (1-\alpha)\, dJ(\alpha) = \int_0^1 dJ(\alpha) = 0$, shows that $\forall\ t \in \mathbb{R}$,

$$\begin{aligned} \psi_F^{\nu} &= \int_0^1 [\alpha - I(t \le F^{-1}(\alpha))] J(\alpha)\, dF^{-1}(\alpha) \\ &= \int_0^1 [F^{-1}(\alpha) I(t \le F^{-1}(\alpha)) + tI(t \ge F^{-1}(\alpha)) - \alpha\{F^{-1}(\alpha) - \mu\}]\, dJ(\alpha). \end{aligned}$$

This and the continuity of F imply that w.p.1, $\forall\ 1 \le i \le n$,

$$\begin{aligned} \psi_F^{\nu}(\varepsilon_i) = \int_0^1 [F^{-1}(\alpha) I(\varepsilon_i \le F^{-1}(\alpha)) \\ + \varepsilon_i I(\varepsilon_i > F^{-1}(\alpha)) - \alpha\{F^{-1}(\alpha) - \mu\}]\, dJ(\alpha). \end{aligned} \tag{3.9}$$

Also, note that

$$-n^{1/2}\int_0^1 \{\mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha)/q(\alpha)\}\, d\nu(\alpha) \equiv n^{-1/2}\sum_i \mathbf{Y}_{i-1}\psi_F^{\nu}(\varepsilon_i). \tag{3.10}$$

We also need to define

$$\begin{aligned} \hat{\varepsilon}_{ni}(\alpha) &:= \hat{a}_{ni}(\alpha)\varepsilon_i + (1 - \hat{a}_{ni}(\alpha))F^{-1}(\alpha), \\ \varepsilon_i^*(\alpha) &:= I(\varepsilon_i > F^{-1}(\alpha))\varepsilon_i + I(\varepsilon_i \le F^{-1}(\alpha))F^{-1}(\alpha), \\ &\qquad 1 \le i \le n, 0 < \alpha < 1. \end{aligned}$$

Then, w.p.1, for all $1 \le i \le n$, $0 < \alpha < 1$, we obtain

$$\hat{\varepsilon}_{ni}(\alpha) - \varepsilon_i^*(\alpha) = \{\hat{a}_{ni}(\alpha) - I(\varepsilon_i > F^{-1}(\alpha))\}\{\varepsilon_i - F^{-1}(\alpha)\}.$$

Moreover, from the definition of $\hat{\mathscr{T}}_{ni}$, (3.3) and (3.9), we obtain

$$\begin{aligned} &n^{-1/2}\sum_i \mathbf{Y}_{i-1}\{\hat{\mathscr{T}}_{ni}(\varepsilon_i - \mu) - \psi_F^{\nu}(\varepsilon_i)\} \\ &\quad\equiv n^{-1/2}\sum_i \mathbf{Y}_{i-1}\int_0^1 [(\hat{\varepsilon}_{ni}(\alpha) - \varepsilon_i^*(\alpha)) \\ &\qquad\qquad + \{\hat{a}_{ni}(\alpha) - (1-\alpha)\}\{F^{-1}(\alpha) - \mu\}]\, dJ(\alpha) \\ &\quad\equiv n^{-1/2}\sum_i \mathbf{Y}_{i-1}\int_0^1 \{\hat{a}_{ni}(\alpha) - I(\varepsilon_i > F^{-1}(\alpha))\}\{\varepsilon_i - F^{-1}(\alpha)\}\, dJ(\alpha). \end{aligned} \tag{3.11}$$

Now fix a $\delta > 0$ and $0 < k < \infty$. Let $A_n^{\delta} := [\sup\{\|\hat{\boldsymbol{\Delta}}_n(\alpha)\|;\ \delta \le \alpha \le 1 - \delta\} \le$

$k]$ and $\xi_i := \varepsilon_i - F^{-1}(\alpha)$, $1 \le i \le n$, where $\hat{\boldsymbol{\Delta}}_n(\alpha)$ is as in (2.24). Then, using (2.27) and the fact that $\hat{a}_{ni}(\alpha) \in [0,1]$ for all $1 \le i \le n$ and all $\alpha \in (0,1)$, we obtain that on A_n^δ,

$$\begin{aligned}\big|\big[\hat{a}_{ni}(\alpha) - I(\varepsilon_i > F^{-1}(\alpha))\big]\{\varepsilon_i - F^{-1}(\alpha)\}\big| \\ \le 2kn^{-1/2}\|\mathbf{Y}_{i-1}\| I\big(|\xi_i| \le kn^{-1/2}\|\mathbf{Y}_{i-1}\|\big),\end{aligned}$$

for all $1 \le i \le n$ and all $\alpha \in (0,1)$. Hence, from (3.11), a conditioning argument and from the stationarity of $\{\|\mathbf{Y}_{i-1}\|\}$, we obtain

$$\begin{aligned}&EI(A_n^\delta)\left\| n^{-1/2}\sum_i \mathbf{Y}_{i-1}\{\hat{\mathscr{T}}_{ni}(\varepsilon_i - \mu) - \psi_F^\nu(\varepsilon_i)\}\right\| \\ &\quad \le 2kE\left\{n^{-1}\sum_i \|\mathbf{Y}_{i-1}\|^2 \int_0^1 I(|\varepsilon_i - F^{-1}(\alpha)| \le n^{-1/2}\|\mathbf{Y}_{i-1}\|)\, dJ(\alpha)\right\} \\ &\quad = 2kE\left\{n^{-1}\sum_i \|\mathbf{Y}_{i-1}\|^2 \int_0^1 \big[F\big(F^{-1}(\alpha) + n^{-1/2}\|\mathbf{Y}_{i-1}\|\big)\right. \\ &\qquad\qquad \left. -F\big(F^{-1}(\alpha) - n^{-1/2}\|\mathbf{Y}_{i-1}\|\big)\big]\, dJ(\alpha)\right\} \\ &\quad = 2kE\left\{\|\mathbf{Y}_0\|^2 \int_0^1 \big[F\big(F^{-1}(\alpha) + n^{-1/2}\|\mathbf{Y}_0\|\big)\right. \\ &\qquad\qquad \left. -F\big(F^{-1}(\alpha) - n^{-1/2}\|\mathbf{Y}_0\|\big)\big]\, dJ(\alpha)\right\}.\end{aligned}$$

The continuity of F and the dominated convergence theorem imply that the last expression converges to zero. Now combine this with (3.11), (3.10), (3.8) and (3.5) to conclude (3.7) in a routine fashion. □

Next, we turn to the *autoregression rank-scores statistics* (ARRS's). To define these let φ be a d.f. on $(0,1)$ and define

$$\text{(3.12)}\quad \hat{b}_{ni} = -\int_0^1 \varphi(\alpha)\, d\hat{a}_{ni}(\alpha), \qquad 1 \le i \le n; \qquad \mathbf{V}_{n\mathbf{g}} = n^{-1}\sum_i \mathbf{g}_{ni}\hat{b}_{ni}.$$

Observe that integration by parts yields that $\hat{b}_{ni} = \varphi(0) + \int_0^1 \hat{a}_{ni}(\alpha)\, d\varphi(\alpha)$. Hence,

$$\begin{aligned}\int_0^1 \hat{\mathbf{U}}_{n\mathbf{g}}(\alpha)\, d\varphi(\alpha) &= n^{-1}\sum_i \mathbf{g}_{ni}\hat{b}_{ni} - n^{-1}\sum_i \mathbf{g}_{ni}\left[\varphi(0) + \int_0^1 (1-\alpha)\, d\varphi(\alpha)\right] \\ &= \mathbf{V}_{n\mathbf{g}} - \bar{\mathbf{g}}_n\bar{\varphi} \quad \text{where } \bar{\varphi} := \int_0^1 \varphi(\alpha)\, d\alpha,\ \bar{\mathbf{g}}_n := n^{-1}\sum_i \mathbf{g}_{ni}.\end{aligned}$$

Thus,

$$\mathbf{V}_{n\mathbf{g}} - \bar{\mathbf{g}}_n \bar{\varphi} = \int_0^1 n^{-1} \sum_i \mathbf{g}_{ni} \{\hat{a}_{ni}(\alpha) - (1-\alpha)\}\, d\varphi(\alpha). \tag{3.13}$$

Consequently, from (2.26) and Corollary 2.1 we readily obtain the following theorem.

THEOREM 3.3. *Under the assumptions of Theorem* 2.2,

$$\begin{aligned} &n^{1/2}(\mathbf{V}_{n\mathbf{g}} - \bar{\mathbf{g}}_n \bar{\varphi}) \\ &\quad = n^{1/2} \int_0^1 [\mathbf{U}_{n\mathbf{g}}(\alpha) - n^{-1} \mathscr{G}' \mathscr{X}_n \mathbf{\Sigma}_n^{-1} \mathbf{U}_{n\mathbf{x}}(\alpha)]\, d\varphi(\alpha) + o_p(1). \end{aligned} \tag{3.14}$$

Consequently, the asymptotic distribution of $n^{1/2}(\mathbf{V}_{n\mathbf{g}} - \bar{\mathbf{g}}_n \bar{\varphi})$ *is q-variate normal with the mean vector* $\mathbf{0}$ *and the covariance matrix*

$$\mathscr{D}_{22} \int_0^1 [\varphi(\alpha) - \bar{\varphi}]^2\, d\alpha.$$

REMARK 3.2. Note that the leading r.v.'s in the right-hand side of (3.14) are equal to

$$n^{-1/2} \sum_i \{\mathbf{g}_{ni} - n^{-1} \mathscr{G}' \mathscr{X}_n \mathbf{\Sigma}_n^{-1} \mathbf{Y}_{i-1}\} [\varphi(F(\varepsilon_i)) - E\varphi(F(\varepsilon_1))].$$

Thus, unlike the linear regression setup, $n^{1/2}(\mathbf{V}_{n\mathbf{g}} - \bar{\mathbf{g}}\bar{\varphi})$ is not asymptotically distribution-free (a.d.f.) in general, However, if the components of $\mathbf{g}_{ni}$ are stationary and ergodic, then by the ergodic theorem, the sequence of r.v.'s $\mathscr{D}_{22n}^{-1/2} n^{1/2}(\mathbf{V}_{n\mathbf{g}} - \bar{\mathbf{g}}\bar{\varphi})$ is a.d.f., where $\mathscr{D}_{22n} = n^{-1}\mathscr{G}'[\mathbf{I} - \mathscr{X}_n(\mathscr{X}_n'\mathscr{X}_n)^{-1}\mathscr{X}_n']\mathscr{G}$.

4. Asymptotic uniform linearity of autoregression rank-scores statistics. This section proves that the ARRS processes and statistics based on residuals are a.u.l. These results are similar to those obtained by Jurečková (1992a) in the linear regression setup. They are useful for testing subhypotheses in AR(p) models in the presence of a trend. Accordingly, let $q \geq 1$ be a fixed integer and $\{\mathbf{w}_{ni}, 1 \leq i \leq n\}$ be a triangular array of $q \times 1$ vectors. We shall state the following theorem for the two cases.

CASE 1. $\{\mathbf{w}_{ni}, 1 \leq i \leq n\}$ are random vectors such that $\mathbf{w}_{ni}$ is $\mathscr{F}_{i-1}$-measurable and independent of ε_i, $1 \leq i \leq n$, and such that the following hold:

$$\max_{1 \leq i \leq n} n^{-1/2} \|\mathbf{w}_{ni}\| = o_p(1), \tag{Wr1}$$

$$n^{-1} \mathbf{\Sigma} \|\mathbf{Y}_{i-1} \mathbf{w}_{ni}'\| = O_p(1). \tag{Wr2}$$

CASE 2. $\{\mathbf{w}_{ni}, 1 \leq i \leq n\}$ are nonrandom vectors such that the following hold:

(Wc1) $(\mathbf{W}'\mathbf{W})^{-1}$ exists for all $n \ge p$,

(Wc2) $\max_{1\le i\le n} \|\mathbf{w}_{ni}'(\mathbf{W}'\mathbf{W})^{-1/2}\| = o(1)$,

where $\mathbf{W}$ is the $n \times q$ matrix with rows $\mathbf{w}_{ni}'$, $1 \le i \le n$.

Now let, for $\mathbf{t} \in \mathbb{R}^q$,

$$
\begin{aligned}
X_{nit} &:= X_i - n^{-1/2}\mathbf{w}_{ni}'\mathbf{t}, \qquad 1 \le i \le n, \text{ in Case 1},\\
&:= X_i - \mathbf{w}_{ni}'\mathbf{A}_{\mathbf{w}}^{-1}\mathbf{t}, \qquad 1 \le i \le n, \text{ in Case 2}
\end{aligned}
\tag{4.1}
$$

where $\mathbf{A}_{\mathbf{w}}^{-1} := (\mathbf{W}'\mathbf{W})^{-1/2}$. In either case, let $\mathbf{X}_{nt} := [X_{n1t}, \dots, X_{nnt}]'$. Consider an analog of the dual programming problem (1.5) based on the residuals $\mathbf{X}_{nt}$, namely

$$
\begin{aligned}
&\text{maximize } \mathbf{X}_{nt}'\mathbf{a}, \text{ with respect to } \mathbf{a},\\
&\text{subject to } \mathscr{X}_n'\mathbf{a} = (1-\alpha)\mathscr{X}_n'\mathbf{1}_n,\ \mathbf{a} \in [0,1]^n.
\end{aligned}
\tag{4.2}
$$

Let $\{\hat{a}_{ni}(\alpha, \mathbf{t})\}$, $\mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha, \mathbf{t})$, $\hat{\boldsymbol{\rho}}_{\mathrm{md}}(\alpha, \mathbf{t})$, $\hat{\boldsymbol{\rho}}_n(\alpha, \mathbf{t}), \dots,$ denote the respective analogs of $\{\hat{a}_{ni}(\alpha)\}$, $\mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha)$, $\hat{\boldsymbol{\rho}}_{\mathrm{md}}(\alpha)$, $\hat{\boldsymbol{\rho}}_n(\alpha), \dots$. The following lemma is similar in spirit to Lemma 2.2 and Theorem 2.1. It gives the asymptotic representations of the autoregression quantile processes obtained from (4.2).

LEMMA 4.1. *Assume that* (1.1), (a1), (a2) *and* (F1) *hold. Then the following hold.*

In either Case 1 *or Case* 2, *for every* $0 < a \le 1/2$, $0 < K < \infty$ *and* $0 < L < \infty$,

(a) $$\sup_{\substack{a\le\alpha\le 1-a\\ \|\mathbf{t}\|\le K,\, \|\mathbf{s}\|\le L}} \|n^{1/2}[\mathbf{T}_n(\boldsymbol{\rho}(\alpha) + n^{-1/2}\mathbf{s}, \alpha, \mathbf{t}) - \mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha, \mathbf{t})] - \boldsymbol{\Sigma}_n\mathbf{s}q(\alpha)\| = o_p(1),$$

(b) $$\sup_{\substack{a\le\alpha\le 1-a\\ \|\mathbf{t}\|\le K}} \|n^{1/2}[\mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha, \mathbf{t}) - \mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha)] - \mathscr{Z}_n\mathbf{t}q(\alpha)\| = o_p(1),$$

where $\mathscr{Z}_n := n^{-1}\mathscr{X}_n'\mathbf{W}$ *in Case* 1 *and* $\mathscr{Z}_n := n^{-1/2}\mathscr{X}_n'\mathbf{W}\mathbf{A}_{\mathbf{w}}^{-1}$ *in Case* 2.

In addition, for every $0 < a \le 1/2$ *and* $0 < L < \infty$,

(c) $$n^{1/2}(\hat{\boldsymbol{\rho}}_{\mathrm{md}}(\alpha, \mathbf{t}) - \boldsymbol{\rho}(\alpha)) = -\{q(\alpha)\}^{-1} n^{1/2}\mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha) + \mathscr{Z}_n\mathbf{t} + \bar{o}_p(1),$$

(d) $$n^{1/2}\{\hat{\boldsymbol{\rho}}_{\mathrm{md}}(\alpha, \mathbf{t}) - \hat{\boldsymbol{\rho}}_n(\alpha, \mathbf{t})\} = \bar{o}_p(1),$$

(e) $$n^{1/2}\{\hat{\boldsymbol{\rho}}_n(\alpha, \mathbf{t}) - \boldsymbol{\rho}(\alpha)\} = \overline{O}_p(1),$$

where $\bar{o}_p(1)$ $\{\overline{O}_p(1)\}$ *is a sequence of processes that converge to zero* {*are bounded*}, *uniformly over* $a \le \alpha \le 1 - a$, $\|\mathbf{t}\| \le K$, *in probability.*

PROOF. (a) Follows from Lemma 2.1, by an argument similar to the one used in the proof of (2.10) of Lemma 2.2. Note that Lemma 2.1 is general enough to cover both cases mentioned above.

(b) First, consider Case 1. Apply (2.3) with $\xi_{ni} \equiv n^{-1/2}\mathbf{w}'_{ni}\mathbf{t}$ and the rest of the entities as in (2.13), to obtain, in view of (F1), that $\forall\ \mathbf{t} \in \mathbb{R}^q$,

$$\sup_{a \le \alpha \le 1-a} \left\| n^{1/2}\mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha, \mathbf{t}) - n^{1/2}\mathbf{T}_n(\boldsymbol{\rho}(\alpha), \alpha) - n^{-1}\mathscr{X}'_n \mathbf{W}\mathbf{t}q(\alpha) \right\| = o_p(1).$$

Note that these $\{\xi_{ni}\}$ satisfy all conditions of Lemma 2.1, for each $\mathbf{t} \in \mathbb{R}^q$. The uniformity w.r.t. $\mathbf{t}$ is achieved as in the proof of Theorem 1.2 of Koul (1991). The proof for the Case 2 is obtained exactly similarly by taking $\xi_{ni} \equiv \mathbf{w}'_{ni}\mathbf{A}_{\mathbf{w}}^{-1}\mathbf{t}$ in (2.3).

The proof of (c) is similar to the proof of (2.11)(i) of Lemma 2.2. By an argument similar to one used in (2.18), it follows that $\|n^{1/2}\mathbf{T}_n(\hat{\boldsymbol{\rho}}_n(\alpha, \mathbf{t}), \alpha, \mathbf{t})\| = \bar{o}_p(1)$ and hence (d) and (e) follow. □

The following theorem gives the main result of this section.

THEOREM 4.1. *In addition to* (1.1), (a1) *and* (a2), *suppose that* (F1), (G1) *and* (G2) *hold. Then, in either Case* 1 *or Case* 2, *with* $\hat{\boldsymbol{\Delta}}_n(\alpha, \mathbf{t}) := n^{1/2}(\hat{\boldsymbol{\rho}}_n(\alpha, \mathbf{t}) - \boldsymbol{\rho}(\alpha))$,

$$\begin{aligned} n^{1/2}\hat{\mathbf{U}}_{n\mathbf{g}}(\alpha, \mathbf{t}) &= n^{1/2}\mathbf{U}_{n\mathbf{g}}(\alpha) + n^{-1}\mathscr{G}'\mathscr{X}_n n^{1/2}\{\hat{\boldsymbol{\Delta}}_n(\alpha, \mathbf{t}) - \hat{\boldsymbol{\Delta}}_n(\alpha)\} \\ &\quad + n^{-1/2}\mathscr{G}'\mathbf{W}^*\mathbf{t}q(\alpha) + \bar{o}_p(1), \end{aligned} \tag{4.3}$$

where $\mathbf{W}^* := n^{-1/2}\mathbf{W}$ *in Case* 1 *and* $\mathbf{W}^* := \mathbf{W}\mathbf{A}_{\mathbf{w}}^{-1}$ *in Case* 2.

Moreover, if the score function φ *is of bounded variation and constant outside a compact subinterval of* (0, 1), *then in either Case* 1 *or Case* 2, $\forall\ 0 < L < \infty$,

$$\begin{aligned} \sup_{\|\mathbf{t}\| \le L} \Big\| & n^{1/2}[\mathbf{V}_{n\mathbf{g}}(\mathbf{t}) - \mathbf{V}_{n\mathbf{g}}] \\ & - n^{-1}\mathscr{G}'\mathscr{X}_n n^{1/2} \int \{\hat{\boldsymbol{\Delta}}_n(\alpha, \mathbf{t}) - \hat{\boldsymbol{\Delta}}_n(\alpha)\}\, d\varphi(\alpha) \\ & - n^{-1/2}\mathscr{G}'\mathbf{W}^*\mathbf{t} \int q(\alpha)\, d\varphi(\alpha) \Big\| = o_p(1). \end{aligned} \tag{4.4}$$

PROOF. We shall carry out details only for the Case 2 because they are similar for Case 1. Now, akin to (2.27), we obtain that w.p.1, $\forall\ \alpha \in (0, 1)$, $\mathbf{t} \in \mathbb{R}^q$, $1 \le i \le n$,

$$\begin{aligned} \hat{a}_{ni}(\alpha, \mathbf{t}) &= I(X_{nit} > \mathbf{Y}'_{i-1}\hat{\boldsymbol{\rho}}_n(\alpha, \mathbf{t})) + \hat{a}_{ni}(\alpha, \mathbf{t}) I(X_{nit} = \mathbf{Y}'_{i-1}\hat{\boldsymbol{\rho}}_n(\alpha, \mathbf{t})) \\ &= 1 - I\big(\varepsilon_i \le F^{-1}(\alpha) + n^{-1/2}\mathbf{Y}'_{i-1}\hat{\boldsymbol{\Delta}}_n(\alpha, \mathbf{t}) + \mathbf{w}'_{ni}\mathbf{A}_{\mathbf{w}}^{-1}\mathbf{t}\big) \\ &\quad + \hat{a}_{ni}(\alpha, \mathbf{t}) I(X_{nit} = \mathbf{Y}'_{i-1}\hat{\boldsymbol{\rho}}_n(\alpha, \mathbf{t})). \end{aligned}$$

Hence, w.p.1, $\forall\ \alpha \in (0, 1)$, $t \in \mathbb{R}^q$,

$$n^{1/2}\big[\hat{\mathbf{U}}_{n\mathbf{g}}(\alpha,\mathbf{t}) - \hat{\mathbf{U}}_{n\mathbf{g}}(\alpha)\big]$$
$$= -n^{-1/2}\sum_i \mathbf{g}_{ni}\Big\{I\big(\varepsilon_i \le F^{-1}(\alpha) + n^{-1/2}\mathbf{Y}_{i-1}'\hat{\mathbf{\Delta}}_n(\alpha,\mathbf{t}) + \mathbf{w}_{ni}'\mathbf{A}_{\mathbf{w}}^{-1}\mathbf{t}\big)$$
$$-I\big(\varepsilon_i \le F^{-1}(\alpha) + n^{-1/2}\mathbf{Y}_{i-1}'\hat{\mathbf{\Delta}}_n(\alpha)\big)\Big\}$$
$$+ n^{-1/2}\sum_i \mathbf{g}_{ni}\hat{a}_{ni}(\alpha,\mathbf{t})I\big(X_{nit} = \mathbf{Y}_{i-1}'\hat{\boldsymbol{\rho}}_n(\alpha,\mathbf{t})\big)$$
$$- n^{-1/2}\sum_i \mathbf{g}_{ni}\hat{a}_{ni}(\alpha)I\big(X_{nit} = \mathbf{Y}_{i-1}'\hat{\boldsymbol{\rho}}_n(\alpha)\big)$$
$$= -R_1(\alpha,\mathbf{t}) + R_2(\alpha,\mathbf{t}) - R_2(\alpha), \quad \text{say.}$$

By (G2), $R_2(\alpha,\mathbf{t}) = \bar{o}_p(1)$ and $R_2(\alpha) = \bar{o}_p(1)$. See also the proof of (2.25). To handle $R_1(\alpha,\mathbf{t})$, let

$$\mathbf{T}(\alpha,\mathbf{s},\mathbf{t}) := n^{-1/2}\sum_i \mathbf{g}_{ni}\big\{I\big(\varepsilon_i \le F^{-1}(\alpha) + n^{-1/2}\mathbf{Y}_{i-1}'\mathbf{s} + \mathbf{w}_{ni}'\mathbf{A}_{\mathbf{w}}^{-1}\mathbf{t}\big)\big\}; \quad \mathbf{s},\mathbf{t} \in \mathbb{R}^p.$$

Write $\mathbf{T}(\alpha)$ for $\mathbf{T}(\alpha,\mathbf{0},\mathbf{0})$. Apply (2.3) to $\xi_{ni} \equiv n^{-1/2}\mathbf{Y}_{i-1}'\mathbf{s} + \mathbf{w}_{ni}'\mathbf{A}_{\mathbf{w}}^{-1}\mathbf{t}$ and the rest of the entities as in (2.13) to conclude that $\forall\ \mathbf{s},\mathbf{t} \in \mathbb{R}^p$,

$$\sup\Big\{\Big\|\mathbf{T}(\alpha,\mathbf{s},\mathbf{t}) - \mathbf{T}(\alpha) - n^{-1/2}\Sigma\mathbf{g}_{ni}\big(n^{-1/2}\mathbf{Y}_{i-1}'\mathbf{s} + \mathbf{w}_{ni}'\mathbf{A}_{\mathbf{w}}^{-1}\mathbf{t}\big)q(\alpha)\Big\|; \quad 0 \le \alpha \le 1\Big\} = o_p(1).$$

This and an argument similar to the one used in the proof of (2.10) yield that $\forall\ 0 < a \le 1/2$, $0 < K$, $L < \infty$,

$$\sup\Big\|\mathbf{T}(\alpha,\mathbf{s},\mathbf{t}) - \mathbf{T}(\alpha) - n^{-1/2}\Sigma_i\mathbf{g}_{ni}\big(n^{-1/2}\mathbf{Y}_{i-1}'\mathbf{s} + \mathbf{w}_{ni}'\mathbf{A}_{\mathbf{w}}^{-1}\mathbf{t}\big)q(\alpha)\Big\| = o_p(1), \tag{4.5}$$

where the supremum is over $a \le \alpha \le 1 - a$, $\|\mathbf{s}\| \le L$, $\|\mathbf{t}\| \le K$. Similarly, we also obtain

$$\sup_{\substack{a \le \alpha \le 1-a \\ \|\mathbf{s}\| \le L}} \big\|\mathbf{T}(\alpha,\mathbf{s},\mathbf{0}) - \mathbf{T}(\alpha) - n^{-1}\Sigma\mathbf{g}_{ni}\mathbf{Y}_{i-1}'\mathbf{s}q(\alpha)\big\| = o_p(1). \tag{4.6}$$

Now note that $R_1(\alpha,\mathbf{t}) = \mathbf{T}(\alpha, \hat{\mathbf{\Delta}}_n(\alpha,\mathbf{t}),\mathbf{t}) - \mathbf{T}(\alpha, \hat{\mathbf{\Delta}}_n(\alpha), 0)$. Therefore, from (4.5), (4.6), (2.11)(i) of Lemma 2.2 and Lemma 4.1(e), it readily follows that

$$R_1(\mathbf{t},\alpha) = n^{-1}\mathscr{G}'\mathscr{X}_n n^{1/2}\big\{\hat{\mathbf{\Delta}}_n(\alpha,\mathbf{t}) - \hat{\mathbf{\Delta}}_n(\alpha)\big\} + n^{-1/2}\mathscr{G}'\mathbf{W}\mathbf{A}_{\mathbf{w}}^{-1}\mathbf{t}q(\alpha) + \bar{o}_p(1).$$

Hence (4.3) follows.

The assertion about (4.4) follows from (4.3) by using integration by parts as in the proof of Theorem 3.2. □

Acknowledgment. The authors would like to thank the three referees whose critical reading of the paper helped to improve the original presentation.

REFERENCES

BROCKWELL, P. J. and DAVIS, R. A. (1987). *Time Series: Theory and Methods*. Springer, New York.

GUTENBRUNNER, C. and JUREČKOVÁ, J. (1992). Regression rank scores and regression quantiles. *Ann. Statist.* **20** 305–329.

GUTENBRUNNER, C., JUREČKOVÁ, J., KOENKER, R. and PORTNOY, S. (1993). Tests of linear hypotheses based on regression rank scores. *Journal of Nonparametric Statistics* **2** 307–331.

HAJÉK, J. and ŠIDÁK, Z. (1967). *Theory of Rank Tests*. Academic Press, New York.

HALL, P. and HEYDE, C. C. (1980). *Martingale Limit Theory and its Applications*. Academic Press, New York.

HANNAN, E. J. (1970). *Multiple Time Series*. Wiley, New York.

JUREČKOVÁ, J. (1971). Nonparametric estimation of regression coefficients. *Ann. Math. Statist.* **42** 1328–1338.

JUREČKOVÁ, J. (1992a). Uniform asymptotic linearity of regression rank scores process. In *Proceedings of the Meeting on Nonparametric Statistics and Related Topics* (A. K. Md. E. Saleh, ed.) 217–228. North-Holland, Amsterdam.

JUREČKOVÁ, J. (1992b). Estimation in a linear model based on regression rank scores. *Journal of Nonparametric Statistics* **1** 197–204.

KOENKER, R. and BASSETT, G. (1978). Regression quantiles. *Econometrica* **46** 33–50.

KOENKER, R. and D'OREY, V. (1987). Algorithm AS 229: Computing regression quantiles. *J. Roy. Statist. Soc. Ser. C* **36** 383–393.

KOENKER, R. and D'OREY, V. (1993). Remark AS R92. A remark on Algorithm AS 229: Computing dual regression quantiles and regression rank scores. *J. Roy. Statist. Soc. Ser. C* **43** 410–414.

KOENKER, R. W. and PORTNOY, S. (1987). *L*-estimation for linear models. *J. Amer. Statist. Assoc.* **82** 851–857.

KOUL, H. L. (1991). A weak convergence result useful in robust autoregression. *J. Statist. Plann. Inference* **29** 291–308.

KOUL, H. L. and OSSIANDER, M. (1994). Weak convergence of randomly weighted dependent residual empiricals with applications to autoregression. *Ann. Statist.* **22** 540–562.

PORTNOY, S. and KOENKER, R. W. (1989). Adaptive *L*-estimation for linear models. *Ann. Statist.* **17** 362–381.

RUPPERT, D. and CARROLL, R. J. (1980). Trimmed least squares estimation in linear models. *J. Amer. Statist. Assoc.* **75** 828–838.

DEPARTMENT OF STATISTICS AND PROBABILITY
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824-1027

DEPARTMENT OF MATHEMATICS AND STATISTICS
DUNTON TOWERS
CARLETON UNIVERSITY
OTTAWA, ONTARIO
CANADA K1S 5B6