

# A GENERALIZED EWMA CONTROL CHART AND ITS COMPARISON WITH THE OPTIMAL EWMA, CUSUM AND GLR SCHEMES<sup>1</sup>

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It is known that both the optimal exponentially weighted moving average (EWMA) and cumulative sum (CUSUM) control charts are based on a given reference value  $\delta$ , which, for the CUSUM chart, is the magnitude of a shift in the mean to be detected quickly. In this paper a generalized EWMA control chart (GEWMA) which does not depend on  $\delta$  is proposed for detecting the mean shift. We compare theoretically the GEWMA control chart with the optimal EWMA, CUSUM and the generalized likelihood ratio (GLR) control charts. The results of the comparison in which the in-control average run length approaches infinity show that the GEWMA control chart is better than the optimal EWMA control chart in detecting a mean shift of any size and is also better than the CUSUM control chart in detecting the mean shift which is not in the interval  $(0.7842\delta, 1.3798\delta)$ . Moreover, the GLR control chart has the best performance in detecting mean shift among the four control charts except when detecting a particular mean shift  $\delta$ , when the in-control average run length approaches infinity.

**1. Introduction.** Since the exponentially weighted moving average (EWMA) control chart was first introduced by Roberts (1959), a variety of EWMA methods, design strategies and enhancements have been developed to detect shifts in the process mean. Crowder (1987, 1989) and Lucas and Saccucci (1990) evaluated the average run length (ARL) properties of the EWMA chart and provided useful tables for the design of the EWMA chart. Lucas and Saccucci (1990) suggested various enhancements to the EWMA chart, such as the fast initial response feature that makes the control chart more sensitive at start-up, the combined Shewhart-EWMA, and the robust EWMA. Saccucci, Amin and Lucas (1992), Baxley (1995), Reynolds (1995, 1996a, b) and Jones, Champ and Rigdon (2001) studied the properties, performance and application of the EWMA chart with variable (adaptive) sampling intervals and estimated parameters. Wardell, Moskowitz and Plante (1994) and Jiang, Tsui and Woodall (2000) investigated the application of EWMA charts to autocorrelated processes. Montgomery and Mastrangelo (1991), Mastrangelo and Montgomery (1995) and Mastrangelo and Brown (2000) studied

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the application of the moving centerline EWMA to various underlying time series models. Another popular control chart is the cumulative sum (CUSUM) test proposed by Page (1954). Its properties have been thoroughly studied in the literature [see, e.g., Hawkins and Olwell (1998)]. A numerical comparison of the EWMA and CUSUM control charts was given by Lucas and Saccucci (1990) and Yashchin (1993). Srivastava and Wu (1993, 1997) and Wu (1994) considered design of the optimal EWMA control chart and compared it with the CUSUM and Shiryaev–Roberts control charts. From the papers cited above, we can see that the EWMA control chart is a powerful tool and can compete with the CUSUM control chart in detecting mean shifts.

However, it should be noted that both the optimal EWMA and CUSUM control charts are based on a given reference value  $\delta$  [see Srivastava and Wu (1993, 1997)], which for the CUSUM chart is the magnitude of a shift in the process mean to be detected quickly. In fact, we rarely know the exact shift value. Using the generalized likelihood ratio statistic, Siegmund and Venkatraman (1995) have proposed a CUSUM-like control chart called GLR (the generalized likelihood ratio) which does not depend on the value of  $\delta$ . Their simulation results show that the GLR is better than the CUSUM control chart in detecting a mean shift which is larger or smaller than  $\delta$  and is only slightly inferior in detecting the mean shift of size  $\delta$ . Can we extend the design of an EWMA control chart such that it does not need the value  $\delta$  and has good performance in detecting any mean shift? If one can obtain such an extended EWMA control chart, what would the relationship be between the extended EWMA, the optimal EWMA, CUSUM and GLR control charts?

The purpose of this paper is to study these two questions, with an emphasis on the second. In the next section, a generalized EWMA (GEWMA) control chart which does not depend on the value of  $\delta$  is presented. The theoretical comparison of the optimal EWMA, GEWMA, CUSUM and GLR control charts is shown in Section 3. Section 4 contains numerical results which compare the average run lengths (ARLs) of the four control charts. The paper concludes in Section 5, with the proofs of four lemmas given in the Appendix.

**2. The generalized EWMA control chart.** Let  $X_i$ ,  $i = 1, 2, \dots$ , be the  $i$ th observation on an i.i.d. process. Suppose that at some time period  $\tau$ , which is usually called a change point, the distribution of  $X_i$  changes from  $N(\mu_0, \sigma^2)$  to  $N(\mu, \sigma^2)$ ; in other words, from time period  $\tau$  onwards the mean of  $X_i$  undergoes a persistent shift of size  $\mu - \mu_0$ , where we assume that  $\mu$  and  $\tau$  are unknown,  $\mu_0$  and  $\sigma$  are known and without loss of generality,  $\mu_0 = 0$  and  $\sigma = 1$ . The first time (stopping time) outside the control limit  $c$  for the EWMA can be written as

$$T_E(c) = \inf\{n \geq 1 : |\bar{W}_n(r)| \geq c\},$$

where

$$\begin{aligned}
 \bar{W}_n(r) &= \frac{W_n(r)}{\sigma_{W_n}} = \frac{\sqrt{(2-r)}}{\sqrt{r[1-(1-r)^{2n}]}} \sum_{i=0}^{n-1} r(1-r)^i X_{n-i}, \\
 W_n(r) &= rX_n + (1-r)W_{n-1}(r), \\
 W_0(r) &= 0,
 \end{aligned}
 \tag{1}$$

$r$  is a weighting parameter ( $0 < r \leq 1$ ), and  $\sigma_{W_n}$  is the standard deviation of  $W_n(r)$ . Since the magnitude of the shift is unknown, it is natural to define the following control statistic and the first time (stopping time) outside the control limits by using a method like the maximum likelihood procedure to raise the sensitivity of the EWMA chart for detecting a change in the mean:

$$\begin{aligned}
 \tilde{W}_n &= \sup_{0 < r \leq 1} \{|\bar{W}_n(r)|\}, \\
 T(c) &= \inf\{n \geq 1 : \tilde{W}_n \geq c\}.
 \end{aligned}$$

Obviously, it is difficult to obtain  $\tilde{W}_n$  since there is an infinite choice of values of  $r$  in  $(0, 1]$ . Note that  $r(1-r)^i, 0 \leq i \leq n-1, 0 < r \leq 1$ , attains its maximum value when  $r = \frac{1}{i+1}$ . From this and (1) a feasible control statistic and its stopping times can be defined as follows:

$$(\bar{W}_n(1), \bar{W}_n(1/2), \dots, \bar{W}_n(1/n))$$

and

$$T_{GE}(c) = \inf \left\{ n \geq 1 : \max_{1 \leq k \leq n} \left| \bar{W}_n\left(\frac{1}{k}\right) \right| \geq c \right\},
 \tag{2}$$

where

$$\bar{W}_n\left(\frac{1}{k}\right) = \frac{\sqrt{(2-1/k)}}{\sqrt{1/k[1-(1-1/k)^{2n}]}} \sum_{i=0}^{n-1} \frac{1}{k} \left(1 - \frac{1}{k}\right)^i X_{n-i}.
 \tag{3}$$

The control statistic and its stopping time in (2) are the focus of this study, and can be called the generalized EWMA (GEWMA) control chart. The upward and downward stopping times of the GEWMA can be defined as

$$\begin{aligned}
 T_{GE}^+(c) &= \inf \left\{ n : \max_{1 \leq k \leq n} \bar{W}_n\left(\frac{1}{k}\right) \geq c \right\}, \\
 T_{GE}^-(c) &= \inf \left\{ n : \min_{1 \leq k \leq n} \bar{W}_n\left(\frac{1}{k}\right) \leq -c \right\}.
 \end{aligned}$$

It is obvious that  $T_{GE} = \min(T_{GE}^+, T_{GE}^-)$ .

To compare the GEWMA control chart with the other charts, the definitions of the optimal EWMA, CUSUM and GLR control charts are given in the following.

According to Wu (1994) the one-sided optimal EWMA control chart can be defined as

$$(4) \quad W_n^*(r^*) = \frac{\sqrt{(2-r^*)}}{\sqrt{r^*}} \sum_{i=0}^{n-1} r^*(1-r^*)^i X_{n-i},$$

$$T_E^*(c) = \inf\{n \geq 1 : W_n(r^*) \geq c\},$$

where  $r^* = 2a^*\delta^2/c^2$  is the optimum choice which minimizes the  $SADT_\delta$  (stationary average delay time). See Srivastava and Wu (1993) for a continuous time version of the EWMA scheme. For minimizing the  $ARL_\delta$  (average run length), Srivastava and Wu (1997) recommend  $a^* \approx 0.5117$  obtained by numerical search to minimize  $-\log(1 - \sqrt{a})/a$  for  $0 < a < 1$ . From (1) and (4) it follows that

$$W_n^*(r^*) = \bar{W}_n(r^*)\sqrt{1 - (1 - r^*)^{2n}}.$$

The two-sided stopping time of the CUSUM can be written as

$$T_C(c) = \min \{T_C^+(c), T_C^-(c)\},$$

where

$$(5) \quad T_C^+(c) = \inf \left\{ n : \max_{1 \leq k \leq n} \delta[S_n - S_{n-k} - \delta k/2] \geq c \right\}, \quad \delta > 0,$$

$$T_C^-(c) = \inf \left\{ n : \min_{1 \leq k \leq n} \delta[S_n - S_{n-k} + \delta k/2] \leq c \right\}, \quad \delta > 0,$$

and  $S_k = X_1 + \dots + X_k$ . When the mean shift  $\delta$  is unknown, Siegmund and Venkatraman (1995) use  $\hat{\delta} = (X_n + \dots + X_{n-k+1})/k$  to estimate  $\delta$ , giving the GLR (generalized likelihood ratio) chart; that is, the upward stopping time of the GLR is

$$(6) \quad T_{GL}^+(c) = \inf \left\{ n : \max_{1 \leq k \leq n} U_n(k) \geq c \right\}, \quad U_n(k) = (S_n - S_{n-k})/k^{1/2}.$$

In this paper we consider mainly the upward stopping times, that is,  $T_E^*(c)$ ,  $T_{GE}^+(c)$ ,  $T_C^+(c)$  and  $T_{GL}^+(c)$ .

**3. Comparison of the optimal EWMA, GEWMA, CUSUM and GLR.** For the convenience of discussion, we use standard quality control terminology. Let  $P(\cdot)$  and  $E(\cdot)$  denote the probability and expectation when there is no change in the mean. Denote  $P_\mu(\cdot)$  and  $E_\mu(\cdot)$  for the probability and expectation when the change point is at  $\tau = 1$ , and the true mean shift value is  $\mu$ . For a stopping time  $T$  as the alarm time with a detecting procedure, the two most frequently used operating characteristics are the in-control average run length ( $ARL_0$ ) and the out-of-control average run length ( $ARL_\mu$ ), defined by

$$ARL_0(T) = E(T),$$

$$ARL_\mu(T) = E_\mu(T).$$

Usually, comparisons of the control charts' performances are made by designing the common  $ARL_0$  and comparing the  $ARL_\mu$ s of the control charts for a given shift  $\mu$ . The chart with the smaller  $ARL_\mu$  is considered to have better performance. The comparisons of the optimal EWMA, GEWMA, CUSUM and GLR control charts are given in the following theorems and corollaries.

**THEOREM 1.** *Let  $c > 0$  be a common control limit for the optimal EWMA, GEWMA and GLR control charts. Then*

$$(7) \quad ARL_0(T_E^*) > ARL_0(T_{GE}^+)$$

and

$$(8) \quad ARL_0(T_E^*) > ARL_0(T_{GL}^+).$$

If  $c \rightarrow \infty$ , then there exists a constant  $M > 0$  such that

$$(9) \quad ARL_0(T_{GE}^+) \sim M \frac{(2\pi)^{1/2} e^{c^2/2}}{c}.$$

By Theorem 1 and the results of  $ARL_0$  for the optimal EWMA, CUSUM and GLR control charts given by Wu (1994), Siegmund (1985) and Siegmund and Venkatraman (1995), the following corollary can be obtained.

**COROLLARY 1.** *Let  $K = \int_0^\infty x\psi^2(x) dx$ , where*

$$\psi(x) = 2x^{-2} \exp\left\{-2 \sum_{n=1}^\infty \Phi\left(\frac{-x\sqrt{n}/2}{n}\right)\right\}$$

and  $\Phi$  is the distribution function of the standard normal distribution. Then

$$(10) \quad \begin{aligned} ARL_0(T_E^*) &\sim \frac{e^{0.834\delta} e^{c^2/2}}{0.408\delta^2 c} > ARL_0(T_{GL}^+) > \frac{(2\pi)^{1/2} e^{c^2/2}}{cK}, \\ ARL_0(T_{GE}^+) &\sim M \frac{(2\pi)^{1/2} e^{c^2/2}}{c} > ARL_0(T_C^+) \\ &\sim \frac{e^{(c+2\delta\rho)} - 1 - (c + 2\delta\rho)}{\delta^2/2} + O(\delta) \end{aligned}$$

for large  $c$ , where  $\rho \approx 0.583$ .

Two approximations for  $ARL_\mu(T_{GE}^+)$  and  $ARL_\mu(T_{GL}^+)$  are given in the following two theorems.

THEOREM 2. *If  $ARL_0(T_{GE}^+) \rightarrow \infty$  or  $c \rightarrow \infty$ , then*

$$\begin{aligned}
 (11) \quad ARL_\mu(T_{GE}^+) &= \frac{c^2}{b\mu^2} \left( 1 + o\left(\frac{\ln c}{c}\right) \right) \\
 &= \frac{2 \ln(ARL_0) + \ln(2 \ln(ARL_0)) - 2 \ln(\sqrt{2\pi} M)}{b\mu^2} \\
 &\quad \times \left( 1 + o\left(\frac{\ln(2 \ln(ARL_0))}{[\ln(ARL_0)]^{1/2}}\right) \right)
 \end{aligned}$$

for  $\mu > 0$ , where  $b = 2(1 - e^{-1})/(1 + e^{-1}) \approx 0.9242343$ .

THEOREM 3. *If  $ARL_0(T_{GL}^+) \rightarrow \infty$  or  $c \rightarrow \infty$ , then*

$$\begin{aligned}
 (12) \quad ARL_\mu(T_{GL}^+) &= \frac{c^2}{\mu^2} \left( 1 + o\left(\frac{\ln c}{c}\right) \right) \\
 &= \frac{2 \ln(ARL_0) + \ln(2 \ln(ARL_0)) - 2 \ln(\sqrt{2\pi}/K)}{\mu^2} \\
 &\quad \times \left( 1 + o\left(\frac{\ln(2 \ln(ARL_0))}{[\ln(ARL_0)]^{1/2}}\right) \right)
 \end{aligned}$$

for  $\mu > 0$ .

The following approximation for  $ARL_\mu(T_{GL})$  has been obtained by Siegmund and Venkatraman (1995), where  $T_{GL}$  is the two-sided stopping time of the GLR control chart: For  $\mu > 0$ , as  $c \rightarrow \infty$ ,

$$(13) \quad ARL_\mu(T_{GL}) = \frac{c^2 - 1}{\mu^2} + \frac{F(\mu)}{\mu} + o(1),$$

where  $F(\mu)$  is a function which does not depend on  $c$ . By Theorem 3, we have  $ARL_\mu(T_{GL}^+)/ARL_\mu(T_{GL}) \rightarrow 1$  as  $c \rightarrow \infty$ .

By Theorems 2 and 3, we have the following corollary.

COROLLARY 2. *If  $ARL_0(T_{GE}^+) = ARL_0(T_{GL}^+) \rightarrow \infty$ , then*

$$(14) \quad ARL_\mu(T_{GE}^+) > ARL_\mu(T_{GL}^+)$$

for  $\mu > 0$ .

The following theorems and corollary show the results of comparison of the four control charts.

THEOREM 4. For the optimal EWMA and GEWMA control charts, if  $ARL_0(T_E^*) = ARL_0(T_{GE}^+) \rightarrow \infty$ , then

$$(15) \quad ARL_\mu(T_E^*) > ARL_\mu(T_{GE}^+)$$

for  $\mu > 0$ . Furthermore, for  $\sqrt{a^*}\delta < \mu < 2\delta$ ,  $ARL_\mu(T_E^*) - ARL_\mu(T_{GE}^+)$  can attain its minimum value at  $\mu = \frac{2\sqrt{a^*}\delta}{1+\sqrt{1-b}} \approx 1.122\delta$ , where  $a^* = 0.5117$ .

THEOREM 5. For the CUSUM and GEWMA control charts, let  $ARL_0(T_C^+) = ARL_0(T_{GE}^+) \rightarrow \infty$ . Then

$$(16) \quad ARL_\mu(T_C^+) > ARL_\mu(T_{GE}^+)$$

if and only if  $0 < \mu < \delta/(1 + \sqrt{1-b})$  or  $\mu > \delta/(1 - \sqrt{1-b})$ , that is,  $0 < \mu < 0.7842\delta$  or  $\mu > 1.3798\delta$ .

THEOREM 6. For the CUSUM and GLR control charts, if  $ARL_0(T_C^+) = ARL_0(T_{GL}^+) \rightarrow \infty$ , then

$$(17) \quad ARL_\mu(T_C^+) > ARL_\mu(T_{GL}^+)$$

for  $0 < \mu \neq \delta$  and

$$(18) \quad \frac{ARL_\mu(T_{GL}^+)}{ARL_\mu(T_C^+)} \rightarrow 1$$

for  $\mu = \delta$ .

COROLLARY 3. For the optimal EWMA, GEWMA and the GLR control charts, if  $ARL_0(T_E^*) = ARL_0(T_{GE}^+) = ARL_0(T_{GL}^+) \rightarrow \infty$ , then

$$(19) \quad ARL_\mu(T_E^*) > ARL_\mu(T_{GE}^+) > ARL_\mu(T_{GL}^+)$$

for  $\mu > 0$ .

REMARK 1. It has been shown by Moustakides (1986) and Ritov (1990) that the performance in detecting the mean shift of the one-sided CUSUM control chart with the reference value  $\delta$  is optimal in the sense of Lorden (1971) if the real mean shift is  $\delta$ . Theorem 6 shows that when  $ARL_0 \rightarrow \infty$ , the GLR control chart is better than the CUSUM control chart in detecting any mean shift except the shift of size  $\delta$ . Theorems 4 and 5 prove that the GEWMA control chart becomes more efficient than the optimal EWMA control chart in detecting any mean shift and is better than the CUSUM control chart in detecting the mean shifts which are not in the interval  $[\delta/(1 + \sqrt{1-b}), \delta/(1 - \sqrt{1-b})]$  when  $ARL_0 \rightarrow \infty$ . It follows from Theorem 4 that the optimal EWMA has the best performance in detecting a mean shift when the true shift value  $\mu \approx 1.122\delta$ . Theorem 6 and Corollary 3 show that the GLR control chart has the best performance of detecting mean shift among the four control charts except detecting the mean shift  $\delta$  when  $ARL_0 \rightarrow \infty$ .

REMARK 2. The condition  $ARL_0 \rightarrow \infty$  in Theorems 2–6 means that  $c \rightarrow \infty$ .

Proofs of the theorems follow (for convenience of proof, four lemmas are given in the Appendix).

PROOF OF THEOREM 1. Since  $E(W_k(r^*)) = EX_k = 0$ ,  $Cov(W_i(r^*), W_j(r^*)) \geq Cov(X_i, X_j) = 0$ ,  $i \neq j$ , and  $Var(W_i(r^*)) = \sqrt{1 - (1 - r^*)^{2i}} \leq Var(X_i) = 1$ , it follows from Lemma 1 that

$$(20) \quad P(T_E^* > n) = P(W_k(r^*) < c, 1 \leq k \leq n) \geq P(X_k < c, 1 \leq k \leq n).$$

Obviously,

$$P(X_k < c, 1 \leq k \leq n) \geq P\left(\max_{1 \leq k \leq m} \bar{W}_m\left(\frac{1}{k}\right) < c, 1 \leq m \leq n\right) = P(T_{GE}^+ > n)$$

since  $\bar{W}_k(1) = X_k$ . By (20) we have  $P(T_E^* > n) \geq P(T_{GE}^+ > n)$ . Hence  $ET_E^* > E(T_{GE}^+)$ , that is,  $ARL_0(T_E^*(c)) > ARL_0(T_{GE}^+(c))$ . For the same reason, we have  $ARL_0(T_E^*(c)) > ARL_0(T_{GL}^+(c))$ .  $\square$

An approximate for  $ET_E^*(c)$  has been given by Wu (1994); that is,

$$ARL_0(T_E^*(c)) \sim \frac{e^{0.834\delta} e^{c^2/2}}{0.408\delta^2 c}$$

for large  $c$  and small  $\delta$ . By this and Lemma 4, we can choose a constant  $M > 0$  which satisfies

$$\frac{e^{0.834\delta}}{0.408\delta^2 \sqrt{2\pi}} \geq M > 0$$

such that

$$ARL_0(T_{GE}^+) \sim M \frac{(2\pi)^{1/2} e^{c^2/2}}{c}$$

for large  $c$ .

PROOF OF THEOREM 2. We first prove that

$$(21) \quad \frac{c^2 - 4c\sqrt{3\ln c}}{b\mu^2} - O\left(\frac{1}{\sqrt{\ln c}}\right) \leq ARL_\mu(T_{GE}^+(c)) \leq \frac{c^2 + 4c\sqrt{\ln c}}{b\mu^2} + o\left(\frac{1}{\ln c}\right)$$

for large  $c$ , where  $b$  is defined in Theorem 2. Let  $\mu_{nk} = E_{\mu}(\bar{W}_n(1/k))$ . It follows from (3) that

$$\mu_{nk} = \mu \sqrt{k} \frac{\sqrt{(2-1/k)}\sqrt{1-(1-1/k)^n}}{\sqrt{1+(1-1/k)^n}}.$$

Note that

$$\frac{9}{10} \leq b_n = \frac{(2-1/n)(1-(1-1/n)^n)}{1+(1-1/n)^n} \uparrow b = \frac{2(1-e^{-1})}{(1+e^{-1})}$$

for  $n \geq 2$  and

$$b - b_n = o\left(\frac{1}{n}\right).$$

Putting  $N = \frac{1}{\mu^2 b}(c^2 + 4c\sqrt{\ln c})$  and  $n = N + k$ , we have

$$\begin{aligned} c - \mu_{nn} &= -(\mu\sqrt{N+k}\sqrt{b_n} - c) \\ &= -\mu\sqrt{N+k}\sqrt{b_n} \left\{ 1 - \frac{1}{\sqrt{b_n/b + 4b_n\sqrt{\ln c}/(bc) + \mu^2 kb_n/c}} \right\} \\ &\leq -\mu B_N \sqrt{N+k} = -[2\sqrt{\ln c} + o(1/c)] \rightarrow -\infty \end{aligned}$$

as  $c \rightarrow \infty$ , where  $B_N = \sqrt{b_N}[1 - (b_N/b + 4b_N\sqrt{\ln c}/bc)^{-1/2}]$ . Let  $\varphi$  and  $\Phi$  be the standard normal density and distribution functions, respectively. Note that  $\mu^2 B_N^2 N = 4 \ln c + o(\sqrt{\ln c}/c)$  and  $b = b_N + o(1/c^2)$  for large  $c$ . It follows that

$$\begin{aligned} &\sum_{n=N+1}^{\infty} P_{\mu}(T_{GE}^+(c) > n) \\ &= \sum_{n=N+1}^{\infty} P\left(\bar{W}_l\left(\frac{1}{k}\right) < c - \mu_{lk}, 1 \leq k \leq l, 1 \leq l \leq n\right) \\ &\leq \sum_{n=N+1}^{\infty} \int_{-\infty}^{c-\mu_{nn}} \phi(x) dx \leq \sum_{n=N+1}^{\infty} \int_{\mu B_N \sqrt{n}}^{+\infty} \phi_1(x) dx \\ &\leq \sum_{k=1}^{\infty} \frac{\exp\{-\frac{1}{2}\mu^2 B_N^2(N+k)\}}{\mu\sqrt{2\pi} B_N \sqrt{N}} \\ &\leq \frac{\exp\{-\frac{1}{2}\mu^2 B_N^2 N\}}{\mu\sqrt{2\pi} B_N \sqrt{N}(1 - \exp\{-\frac{1}{2}\mu^2 B_N^2\})} \\ &\leq \frac{1}{4\sqrt{2\pi} b \mu^2 (\ln c)^{3/2}} \end{aligned}$$

for large  $c$ . Thus

$$\begin{aligned}
 (22) \quad ARL_{\mu}(T_{GE}^{+}(c)) &\leq \sum_{n=1}^N P_{\mu}(T_{GE}^{+}(c) > n) + \frac{1}{4\sqrt{2\pi}b\mu^2(\ln c)^{3/2}} \\
 &\leq N + \frac{1}{4\sqrt{2\pi}b_c\mu^2(\ln c)^{3/2}} \\
 &\leq \frac{1}{\mu^2b}(c^2 + 4c\sqrt{\ln c}) + o\left(\frac{1}{\ln c}\right)
 \end{aligned}$$

for large  $c$ . This proves the upward inequality of (21).

Suppose that  $Y_{lk}$ ,  $1 \leq k \leq l$ , are standard independent normal variables. By Lemmas 1 and 3, we have

$$\begin{aligned}
 (23) \quad \sum_{n=1}^m P_{\mu}(T_{GE}^{+}(c) > n) &= \sum_{n=1}^m P_{\mu}\left(\max_{1 \leq k \leq l} \bar{W}_l\left(\frac{1}{k}\right) < c, 1 \leq l \leq n\right) \\
 &= \sum_{n=1}^m P\left(\bar{W}_l\left(\frac{1}{k}\right) < c - \mu_{lk}, 1 \leq k \leq l, 1 \leq l \leq n\right) \\
 &\geq \sum_{n=1}^m P(Y_{lk} < c - \mu_{lk}, 1 \leq k \leq l, 1 \leq l \leq n) \\
 &= \sum_{n=1}^m \prod_{l=1}^n \prod_{k=1}^l P(Y_{lk} < c - \mu_{lk}) = \sum_{n=1}^m \prod_{l=1}^n \prod_{k=1}^l \Phi(c - \mu_{lk}) \\
 &\geq \sum_{n=1}^m ([\Phi(c - \mu_{mm})]^{(m+1)/2})^n \\
 &= \frac{[\Phi(c - \mu_{mm})]^{(m+1)/2}(1 - [\Phi(c - \mu_{mm})]^{(m+1)m/2})}{1 - [\Phi(c - \mu_{mm})]^{(m+1)/2}}.
 \end{aligned}$$

Let  $m = \frac{1}{b\mu^2}(c^2 - 4c\sqrt{3\ln c})$ . It follows that  $\Phi(c - \mu_{mm}) = 1 - (\sqrt{24\pi \ln cc^6})^{-1} \times (1 - O(1/\ln c))$  for large  $c$ . Thus

$$\begin{aligned}
 ARL_{\mu}(T_{GE}^{+}(c)) &\geq \frac{[\Phi(c - \mu_{mm})]^{(m+1)/2}(1 - [\Phi(c - \mu_{mm})]^{(m+1)m/2})}{1 - [\Phi(c - \mu_{mm})]^{(m+1)/2}} \\
 &= m - O\left(\frac{1}{\sqrt{\ln c}}\right)
 \end{aligned}$$

for large  $c$ . This is the downward inequality of (21). By Theorem 1, we have  $c^2 \approx 2 \ln(ARL_0) + \ln(2 \ln(ARL_0)) - 2 \ln(M\sqrt{2\pi})$  for large  $c$ . Thus, Theorem 2 can be obtained immediately.  $\square$

PROOF OF THEOREM 3. Let  $N = \frac{1}{\mu^2}(c^2 + 4c\sqrt{\ln c})$  and  $n = N + k$ . Since  $v_{nk} = E_\mu(U_n(k)) = \mu\sqrt{k}$ , it follows that

$$\begin{aligned} c - v_{nn} &= -(\mu\sqrt{N+k} - c) \\ &= -\mu\sqrt{N+k} \left\{ 1 - \frac{1}{\sqrt{1 + 4\sqrt{\ln c}/c + \mu^2 k/c}} \right\} \\ &\leq -\mu A_N \sqrt{N+k} = -[2\sqrt{\ln c} + o(1/c)] \end{aligned}$$

and  $\mu^2 A_N^2 N = 4 \ln c + o(\sqrt{\ln c}/c)$  for large  $c$ , where  $A_N = [1 - (1 + 4\sqrt{\ln c}/c)^{-1/2}]$ . As in (22) and (23) we can check

$$\begin{aligned} ARL_\mu(T_{GL}^+(c)) &\leq N + \frac{1}{4\sqrt{2\pi}\mu^2(\ln c)^{3/2}} \\ &\leq \frac{1}{\mu^2}(c^2 + 4c\sqrt{\ln c}) + o\left(\frac{1}{\ln c}\right) \end{aligned}$$

and

$$\begin{aligned} ARL_\mu(T_{GL}^+(c)) &\geq \frac{[\Phi(c - v_{mm})]^{(m+1)/2}(1 - [\Phi(c - v_{mm})]^{(m+1)m/2})}{1 - [\Phi(c - v_{mm})]^{(m+1)/2}} \\ &= m - O\left(\frac{1}{\sqrt{\ln c}}\right) \end{aligned}$$

for large  $c$ , where  $m = \frac{1}{\mu^2}(c^2 - 4c\sqrt{3\ln c})$ . This completes the proof of Theorem 3.  $\square$

PROOF OF THEOREM 4. From Corollary 1 and  $ARL_0(T_E^*) = ARL_0(T_{GE}^+) \rightarrow \infty$  it follows that there exists a positive increasing function  $f(c)$  such that  $f(c) = c - \varepsilon(c)$  and  $ET_E^*(f(c)) = ET_{GE}^+(c) \rightarrow \infty$  as  $c \rightarrow \infty$ , where  $0 < \varepsilon(c) < D/c$  and  $D$  is a constant. For  $\mu > \sqrt{a^*}\delta$ , the approximation for  $ARL_\mu(T_E^*(f(c)))$  is given by Wu (1994) as follows:

$$\begin{aligned} (24) \quad ARL_\mu(T_E^*(f(c))) &= \frac{1}{\delta^2} \left[ \frac{-\ln(1 - \sqrt{a^*}\delta/\mu)}{2a^*} (c - \varepsilon(c))^2 \right. \\ &\quad \left. - \frac{\delta^2}{4\mu^2} \frac{(1 - (1 - \sqrt{a^*}\delta/\mu)^2)}{(1 - \sqrt{a^*}\delta/\mu)^2} + o\left(\frac{1}{(c - \varepsilon(c))^2}\right) \right] \end{aligned}$$

for large  $c$ . In fact, a rough estimation for  $ARL_\mu(T_E^*(f(c)))$  can be obtained as follows. Let  $n = N + k$  and  $N = \frac{d}{2\delta^2 a^*}(c^2 + 2gc\sqrt{\ln c})$ , where  $d = -\ln(1 - \sqrt{a^*}\delta/\mu)$ ,  $g = e^d \sqrt{a^*}\delta/\mu d$ . Note that

$$\gamma_n = E_\mu(W_n^*(r^*)) = \frac{\mu\sqrt{(2-r^*)(1 - (1-r^*)^n)}}{\sqrt{r^*}},$$

$r^* = 2a^*\delta^2/c^2$  and  $(1 - e^{-d})\mu/(\sqrt{a^*}\delta) = 1$ . It follows that

$$\begin{aligned} c - \gamma_n &= -c \frac{\mu}{\sqrt{a^*}\delta} e^{-d} \left\{ \frac{2\delta^2 a^*}{c^2} (N+k) - d + O\left(\frac{\ln c}{c^2}\right) \right\} \\ &\leq -(N+k) \frac{\mu}{\sqrt{a^*}\delta} e^{-d} \left( \frac{2\delta^2 a^*}{c} - \frac{dc}{N} \right) \\ &= -(N+k) D_N = -[2\sqrt{\ln c} + o(1/c)] \end{aligned}$$

for large  $c$ , where  $D_N = \frac{\mu}{\sqrt{a^*}\delta} e^{-d} \left( \frac{2\delta^2 a^*}{c} - \frac{dc}{N} \right)$ . Thus

$$\begin{aligned} \sum_{n=N+1}^{\infty} P_{\mu}(T_E^*(c) > n) &= \sum_{n=N+1}^{\infty} P(W_k^*(r^*) < c - \gamma_k, 1 \leq k \leq n) \\ &\leq \sum_{n=N+1}^{\infty} \int_{-\infty}^{c-\gamma_n} \phi(x) dx \leq \sum_{n=N+1}^{\infty} \int_{(N+k)D_N}^{+\infty} \phi(x) dx \\ &\leq \sum_{k=1}^{\infty} \frac{\exp\{-\frac{1}{2}D_N^2(N+k)^2\}}{\sqrt{2\pi}D_N(N+k)} \leq \frac{N \exp\{-\frac{1}{2}D_N^2 N^2\}}{4\sqrt{2\pi}(\ln c)^{3/2}} \rightarrow 0 \end{aligned}$$

as  $c \rightarrow +\infty$ , that is,

$$\begin{aligned} ARL_{\mu}(T_E^*(c)) &\leq N + \frac{d}{2\sqrt{2\pi}\delta^2 a^*(\ln c)^{3/2}} \\ &\leq \frac{-\ln(1 - \sqrt{a^*}\delta/\mu)}{2\delta^2 a^*} (c^2 + 2gc\sqrt{\ln c}) + o\left(\frac{1}{\ln c}\right) \end{aligned}$$

for large  $c$  and  $\mu > \sqrt{a^*}\delta$ . As (23) we also have

$$\begin{aligned} ARL_{\mu}(T_E^*(c)) &\geq \frac{[\Phi(c - \gamma_m)](1 - [\Phi(c - \gamma_m)]^m)}{1 - [\Phi(c - \gamma_m)]} = m - O\left(\frac{1}{\sqrt{\ln c}}\right) \\ &= \frac{-\ln(1 - \sqrt{a^*}\delta/\mu)}{2\delta^2 a^*} (c^2 - 2gc\sqrt{\ln c}) - O\left(\frac{1}{\ln c}\right) \end{aligned}$$

for large  $c$  and  $\mu > \sqrt{a^*}\delta$ , where  $m = \frac{d}{2\delta^2 a^*} (c^2 - 2gc\sqrt{2\ln c})$ . Thus, a rough estimation for  $ARL_{\mu}(T_E^*(f(c)))$  is approximately

$$(25) \quad ARL_{\mu}(T_E^*(f(c))) = \frac{-\ln(1 - \sqrt{a^*}\delta/\mu)}{2a^*\delta^2} (c - \varepsilon(c))^2 \left( 1 + o\left(\frac{\ln c}{c}\right) \right).$$

It can be checked that

$$\frac{-\ln(1 - \sqrt{a^*}\delta/\mu)}{2a^*\delta^2} > \frac{1}{b\mu^2}$$

for  $\mu > \sqrt{a^*}\delta$ . By (25) and Theorem 2, we see that  $ARL_\mu(T_E^*(f(c))) > ARL_\mu(T_{GE}^+(c))$  for  $\mu > \sqrt{a^*}\delta$  as  $c \rightarrow \infty$  since

$$\frac{-\ln(1 - \sqrt{a^*}\delta/\mu)}{2a^*\delta^2}(c - \varepsilon(c))^2 > \frac{c^2}{b\mu^2}$$

for  $\mu > \sqrt{a^*}\delta$  as  $c \rightarrow +\infty$ . It can be checked that the following function for  $\sqrt{a^*}\delta < \mu < 2\delta$ ,

$$\frac{-\ln(1 - \sqrt{a^*}\delta/\mu)}{2a^*\delta^2} - \frac{1}{b\mu^2},$$

attains its minimum value at  $\mu = \frac{2\sqrt{a^*}\delta}{1+\sqrt{1-b}} \approx 1.122\delta$ . This means that the optimal EWMA control chart has relatively best performance in detecting mean shift ( $\sqrt{a^*}\delta < \mu < 2\delta$ ) when the true shift  $\mu = \frac{2\sqrt{a^*}\delta}{1+\sqrt{1-b}}$ .

Next we prove  $ARL_\mu(T_E^*(f(c))) > ARL_\mu(T_{GE}^+(c))$  for  $\mu \leq \sqrt{a^*}\delta$ . Note that, as  $c \rightarrow +\infty$ ,

$$\begin{aligned} f(c) - \gamma_m &= c - \varepsilon(c) - \frac{c\mu}{\sqrt{a^*}\delta} \frac{\sqrt{2-r^*}}{\sqrt{2}}(1 - (1-r^*)^m) \\ &\geq c\left(1 - \frac{\mu}{\sqrt{a^*}\delta} - o\left(\frac{1}{c^2}\right)\right) \rightarrow +\infty \end{aligned}$$

and

$$\begin{aligned} &1 - \Phi\left(c\left(1 - \frac{\mu}{\sqrt{a^*}\delta} - o\left(\frac{1}{c^2}\right)\right)\right) \\ &= \frac{\phi(c(1 - \mu/(\sqrt{a^*}\delta) - o(1/c^2)))}{c(1 - \mu/(\sqrt{a^*}\delta) - o(1/c^2))} \left(1 - O\left(\frac{1}{c^2}\right)\right) \end{aligned}$$

for  $\mu < \sqrt{a^*}\delta$ . As (23) we can obtain

$$\begin{aligned} ARL_\mu(T_E^*(f(c))) &= \sum_{m=1}^{\infty} P_\mu(T_E^*(f(c)) > m) \\ &\geq \sum_{m=1}^{\infty} [\Phi(f(c) - \gamma_m)]^m \\ &\geq \frac{\Phi(c(1 - \mu/(\sqrt{a^*}\delta) - o(1/c^2)))}{1 - \Phi(c(1 - \mu/(\sqrt{a^*}\delta) - o(1/c^2)))} \\ &= \sqrt{2\pi}c\left(1 - \frac{\mu}{\sqrt{a^*}\delta}\right) \exp\left\{\frac{c^2}{2}\left(1 - \frac{\mu}{\sqrt{a^*}\delta}\right)^2\right\} \\ &\quad \times \left(1 + O\left(\frac{1}{c^2}\right)\right) + O(1) \end{aligned}$$

for large  $c$ . Thus  $ARL_{\mu}(T_E^*(f(c))) > ARL_{\mu}(T_{GE}^+(c))$  for  $\mu < \sqrt{a^*}\delta$  as  $c \rightarrow +\infty$ .

Let  $\mu = \sqrt{a^*}\delta$  and  $m = \frac{c^2}{2a^*\delta^2}(\ln c - \ln \sqrt{4 \ln c})$ . It follows that

$$\begin{aligned} f(c) - \gamma_m &= c - \varepsilon(c) - c \frac{\sqrt{2-r^*}}{\sqrt{2}}(1 - (1-r^*)^m) \\ &\geq c \left(1 - \frac{2a^*\delta^2}{c^2}\right)^m + o\left(\frac{1}{c}\right) = 2\sqrt{\ln c} + o\left(\frac{1}{c}\right) \end{aligned}$$

and

$$\Phi(2\sqrt{\ln c}) = 1 - \frac{1}{2\sqrt{2\pi}c^2\sqrt{\ln c}} \left(1 - O\left(\frac{1}{\ln c}\right)\right)$$

for large  $c$ . Hence

$$\begin{aligned} ARL_{\mu}(T_E^*(f(c))) &\geq \sum_{k=1}^m P_{\mu}(T_E^*(f(c)) > k) \\ &\geq \sum_{k=1}^m [\Phi(f(c) - \gamma_m)]^k \\ &\geq \frac{\Phi(\sqrt{4 \ln c})(1 - [\Phi(\sqrt{4 \ln c})]^m)}{1 - \Phi(\sqrt{4 \ln c})} \\ &= 2\sqrt{2\pi}c^2\sqrt{\ln c} \left(1 - \exp\left\{\frac{-\ln c + \ln \sqrt{4 \ln c}}{2\sqrt{2\pi}\sqrt{\ln c}}\right\}\right) \\ &\quad \times \left(1 + O\left(\frac{1}{\ln c}\right)\right) + O(1) \end{aligned}$$

for large  $c$ . This means that  $ARL_{\mu}(T_E^*(f(c))) > ARL_{\mu}(T_{GE}^+(c))$  as  $c \rightarrow \infty$ . This completes the proof of Theorem 4.  $\square$

PROOF OF THEOREM 5. From Corollary 1 and  $ARL_0(T_{GE}^+) = ARL_0(T_C^+) \rightarrow \infty$  it follows that there exists a positive increasing function  $l(c)$  such that  $l(c) = \sqrt{2c + \ln 2c + \varepsilon(c)}$  and  $E(T_C^+(c)) = ET_{GE}^+(l(c)) \rightarrow \infty$  as  $c \rightarrow \infty$ , where  $|\varepsilon(c)| \leq M_0$  and  $M_0$  is a constant. For  $\mu < \delta/2$ , as (23) we can obtain

$$\begin{aligned} P_{\mu}(T_C^+(c) > m) &= P(U_n(k) < c/(\delta\sqrt{k}) + (\delta/2 - \mu)\sqrt{k}, 1 \leq k \leq n, 1 \leq n \leq m) \\ &\geq [\Phi(2\sqrt{c}\sqrt{(\delta/2 - \mu)/\delta})]^{m(m+1)/2} \end{aligned}$$

since  $c/(\delta\sqrt{k}) + (\delta/2 - \mu)\sqrt{k}$  attains its minimum value  $2\sqrt{c}\sqrt{(\delta/2 - \mu)/\delta}$  at  $k = c/(\delta/2 - \mu)\delta$ . Let  $m = 4\sqrt{(\delta/2 - \mu)/\delta}e^{c(\delta/2 - \mu)/\delta}$ . Then

$$\begin{aligned}
 ARL_\mu(T_C^+(c)) &\geq \sum_{n=1}^m \left( [\Phi(2\sqrt{c}\sqrt{(\delta/2 - \mu)/\delta})]^{(m+1)/2} \right)^n \\
 &= \frac{[\Phi(2\sqrt{c}\sqrt{(\delta/2 - \mu)/\delta})]^{(m+1)/2}}{1 - [\Phi(2\sqrt{c}\sqrt{(\delta/2 - \mu)/\delta})]^{(m+1)/2}} \\
 (26) \quad &\times \left( 1 - [\Phi(2\sqrt{c}\sqrt{(\delta/2 - \mu)/\delta})]^{m(m+1)/2} \right) \\
 &= m \left[ 1 - O\left(\frac{1}{\sqrt{c}}\right) \right] + O\left(\frac{1}{\sqrt{c}}\right)
 \end{aligned}$$

for large  $c$ . From this and (21) it follows that  $ARL_\mu(T_C^+) > ARL_\mu(T_{GE}^+)$  for  $\mu < \delta/2$  as  $c \rightarrow \infty$ . Let  $\mu = \delta/2$  and  $M = c^2/(8\delta^2 \ln c)$ . Similarly, we have

$$\begin{aligned}
 ARL_\mu(T_C^+(c)) &\geq \sum_{m=1}^M \left[ \Phi\left(\frac{c}{\delta\sqrt{m}}\right) \right]^{m(m+1)/2} \\
 (27) \quad &\geq \frac{[\Phi(\sqrt{8 \ln c})]^{(M+1)/2}}{1 - [\Phi(\sqrt{8 \ln c})]^{(M+1)/2}} \left( 1 - [\Phi(\sqrt{8 \ln c})]^{M(M+1)/2} \right) \\
 &= M \left[ 1 - O\left(\frac{1}{(\ln c)^{2+1/2}}\right) \right] + O\left(\frac{1}{(\ln c)^{2+1/2}}\right)
 \end{aligned}$$

for large  $c$ . Thus, by (21),

$$\begin{aligned}
 ARL_\mu(T_{GE}^+(l(c))) &\leq \frac{1}{b\mu^2} (l(c)^2 + 4l(c)\sqrt{\ln l(c)}) + o((\ln l(c))^{-1}) \\
 &= \frac{1}{b\mu^2} (2c + \ln 2c + \varepsilon(c) + 4l(c)\sqrt{\ln l(c)}) + o((\ln l(c))^{-1}) \\
 &< M \left[ 1 - O\left(\frac{1}{(\ln c)^{2+1/2}}\right) \right] + O\left(\frac{1}{(\ln c)^{2+1/2}}\right) \\
 &\leq ARL_\mu(T_C^+(c))
 \end{aligned}$$

for  $\mu = \delta/2$  as  $c \rightarrow \infty$ . For  $\mu > \delta/2$ , the approximation for  $ARL_\mu(T_C^+(c))$  is given by Wu (1994) as follows:

$$(28) \quad ARL_\mu(T_C^+(c)) \approx \frac{2(\mu - \delta/2)(c + 2\rho\delta)/\delta - 1 + e^{-2(\mu - \delta/2)(c + 2\rho\delta)/\delta}}{2(\mu - \delta/2)^2}.$$

Comparing this with (21) we can see that  $ARL_\mu(T_C^+(c)) > ARL_\mu(T_{GE}^+(l(c)))$  for  $\mu > \delta/2$  as  $c \rightarrow \infty$  if and only if

$$\frac{1}{(\mu - \delta/2)\delta} > \frac{2}{b\mu^2}.$$

This implies that  $0 < \mu < \delta/(1 + \sqrt{1 - b}) = 0.7842\delta$  or  $\mu > \delta/(1 - \sqrt{1 - b}) = 1.3798\delta$ . This completes the proof.  $\square$

PROOF OF THEOREM 6. From Corollary 1 and  $ARL_0(T_{GL}^+) = ARL_0(T_C^+) \rightarrow \infty$  it follows that there exists a positive increasing function  $h(c)$  such that  $h(c) = \sqrt{2c + \ln 2c + \varepsilon(c)}$  and  $E(T_C^+(c)) = ET_{GL}^+(h(c)) \rightarrow \infty$  as  $c \rightarrow \infty$ , where  $|\varepsilon(c)| \leq A$  and  $A$  is a constant. By (26), (27) and Theorem 3, we have  $ARL_\mu(T_C^+(c)) > ARL_\mu(T_{GL}^+(h(c)))$  as  $c \rightarrow \infty$  for  $\mu \leq \delta/2$ . Let  $\mu > \delta/2$ . Since

$$\frac{1}{(\mu - \delta/2)\delta} - \frac{2}{\mu^2} = \frac{1}{(\mu - \delta/2)\delta\mu^2}(\mu - \delta)^2,$$

it follows that  $ARL_\mu(T_C^+(c)) > ARL_\mu(T_{GL}^+(h(c)))$  as  $c \rightarrow \infty$  for  $\delta/2 < \mu \neq \delta$ . Furthermore, by (28) and Theorem 3 we have

$$\frac{ARL_\mu(T_{GL}^+(h(c)))}{ARL_\mu(T_C^+(c))} = \frac{(1/\mu^2)h(c)^2[1 + o(\sqrt{\ln h(c)}/h(c))]}{(1/((\mu - \delta/2)\delta)c)[1 + o(1/c)]} \rightarrow 1$$

as  $c \rightarrow +\infty$  for  $\mu = \delta$ .  $\square$

**4. Numerical illustration.** The purpose of this section is to present some simulation results of ARLs of the two-sided optimal EWMA, Shewhart-EWMA (a combination of a Shewhart chart and an optimal EWMA chart), GEWMA, GLR and CUSUM control charts, that is, the ARLs of the two-sided stopping times. The numerical results of the ARLs were obtained based on 10,000 repetitions. Tables 1 and 2 compare the simulation results for various values of the mean shift  $\mu$  with change point  $\tau = 1$ . The values in the parentheses in every column of Tables 1 and 2 are the standard deviations of the simulation results of the stopping times. In the last two rows of Tables 1 and 2,  $r^*$  is the weighted parameter of the optimal EWMA which satisfies  $r^* = 2a^*\delta^2/c^2$  and  $c$  denotes various values of the width of the control limits. In the third column and last row of Tables 1 and 2,  $c$  and  $L$  denote values of the width of the control limits of the optimal EWMA chart and Shewhart chart, respectively. The reference value for the optimal EWMA and CUSUM is taken to be 1, that is,  $\delta = 1$ . Table 1 illustrates that the GEWMA control chart is better than the optimal EWMA and Shewhart-EWMA control charts except in detecting the mean shifts of size around 1.122 (at least from 0.5 to 1.25) and better than the CUSUM except in detecting the mean shifts of size around 1 (at least from 0.75 to 1.25), and also better than the GLR control chart except in detecting small shifts, that is,  $\mu < 0.25$ . We consider the  $ARL_0 \approx 865$  in Table 2. Although  $ARL_0$  is relatively large in Table 2, the increase of  $c$  is small. In this case, we have the same conclusions as from Table 1. As can be seen from Tables 1 and 2, the Shewhart-EWMA control chart is better than the optimal EWMA, especially in detecting a large mean shift; the standard deviation of the stopping time for the GEWMA

TABLE 1  
*Comparison of ARLs of the five control charts with  $ARL_0 \approx 435$  and  $N = 10,000$  independent simulation trials*

Shifts ( $\mu$ )	Optimal EWMA	Shewhart- EWMA	GEWMA	GLR	CUSUM
0	437 (434)	430 (428)	438 (424)	439 (435)	434 (436)
0.1	297 (288)	294 (285)	304 (275)	295 (267)	326 (323)
0.25	110 (102)	109 (102)	105 (78.8)	108 (80.4)	132 (123)
0.5	32.4 (25)	32.4 (25)	34.9 (22.7)	36.2 (23.3)	37.2 (30.4)
0.75	15.7 (9.63)	15.7 (9.63)	17.4 (10.3)	18.1 (10.7)	16.7 (10.8)
1	9.95 (5.01)	9.92 (5.03)	10.7 (5.92)	11.1 (6.18)	10.3 (5.45)
1.25	7.24 (3.11)	7.19 (3.14)	7.36 (3.91)	7.58 (3.98)	7.34 (3.32)
1.5	5.73 (2.18)	5.67 (2.23)	5.41 (2.75)	5.59 (2.8)	5.70 (2.26)
2	4.03 (1.24)	3.91 (1.35)	3.41 (1.64)	3.54 (1.65)	3.98 (1.28)
3	2.63 (0.65)	2.29 (0.86)	1.85 (0.83)	1.91 (0.81)	2.55 (0.65)
4	2.06 (0.37)	1.47 (0.57)	1.25 (0.47)	1.3 (0.49)	2.00 (0.38)
$r^*$	0.12869	0.12869			
$c$	2.82	$c = 2.82, L = 3.9$	3.29	3.45	4.94

chart is slightly smaller than for the GLR chart, and the standard deviations for the optimal EWMA, Shewhart-EWMA and CUSUM charts are larger than for the GEWMA and GLR charts when  $\mu < 0.5$  and smaller when  $\mu > 0.75$ .

TABLE 2  
*Comparison of ARLs of the five control charts with  $ARL_0 \approx 865$  and  $N = 10,000$  independent simulation trials*

Shifts ( $\mu$ )	Optimal EWMA	Shewhart- EWMA	GEWMA	GLR	CUSUM
0	867 (868)	863 (864)	866 (853)	862 (840)	868 (877)
0.1	524 (507)	521 (506)	481 (401)	477 (406)	592 (593)
0.25	155 (144)	155 (144)	137 (94.2)	139 (95.8)	200 (188)
0.5	39.9 (30.7)	39.9 (30.7)	41.6 (25.6)	42.9 (25.9)	46.1 (37.4)
0.75	18.3 (10.9)	18.3 (10.9)	20.2 (11.55)	20.9 (11.6)	19.2 (12.1)
1	11.5 (5.53)	11.5 (5.54)	12.3 (6.59)	12.7 (6.68)	11.6 (5.90)
1.25	8.29 (3.38)	8.28 (3.39)	8.35 (4.31)	8.63 (4.36)	8.25 (3.54)
1.5	6.50 (2.23)	6.48 (2.35)	6.11 (3.08)	6.31 (3.08)	6.38 (2.40)
2	4.58 (1.32)	4.53 (1.36)	3.76 (1.75)	3.89 (1.75)	4.42 (1.35)
3	2.96 (0.69)	2.77 (0.88)	2.01 (0.88)	2.07 (0.85)	2.82 (0.69)
4	2.24 (0.45)	1.80 (0.68)	1.32 (0.51)	1.38 (0.53)	2.15 (0.41)
$r^*$	0.11125	0.11125			
$c$	3.033	$c = 3.033, L = 4.4$	3.50	3.67	5.62

**5. Conclusion and discussion.** By taking the maximum weighting parameter in the EWMA control chart, a generalized EWMA is proposed in this paper. The idea is to raise the sensitivity of the EWMA chart in detecting large and small mean shifts. From the theoretical study and numerical simulation it is clear that when the control average run length goes to infinity, the GEWMA control chart is better than the optimal EWMA control chart in detecting any mean shift. It is also more efficient than the CUSUM control chart in detecting the mean shift which is not in the interval  $[\delta/(1 + \sqrt{1-b}), \delta/(1 - \sqrt{1-b})]$ . However, the numerical calculation of ARL for the optimal EWMA control chart or for the CUSUM control chart is faster than for the GEWMA and GRL control charts. We also prove that when the control average run length goes to infinity, the GLR control chart has the best performance in detecting the mean shifts among the four control charts except in detecting the mean shift  $\delta$ . Although the GLR control chart is better than the GEWMA control chart in detecting the mean shift when the control average run length goes to infinity, the GEWMA has better performance than GLR in detecting the mean shift which is not small, that is,  $\mu \geq 0.25$ , when the control limit is not large enough, and the time for numerical calculation of ARL for the GEWMA control chart is less than for the GLR control chart.

In order to use the GEWMA control chart in practice, one must deal with the computational issue. Here a feasible approach is suggested to lighten the computational burden. As it is reasonable to neglect a very small mean shift in practice, we can suppose the mean shift has a minimal value,  $\delta_0 > 0$ , that is, the mean shift which is less than  $\delta_0$  is ignored. Since the optimal weighting parameter  $r^*$  that minimizes the  $SADT_{\delta_0}$  [see Srivastava and Wu (1993)] and  $ARL_{\delta_0}$  [average run length for a two-sided EWMA scheme; see Srivastava and Wu (1997)] is approximately

$$\begin{aligned}
 r^* &= \frac{\delta_0^2 a^*}{\log(\delta_0^2 ARL_0)} (1 + o(1)) \\
 &= \frac{2\delta_0^2 a^*}{c^2} (1 + o(1)),
 \end{aligned}$$

we can take a natural number  $n_0$  such that  $n_0 = \lceil \frac{1}{r^*} \rceil$ , where  $\lceil x \rceil$  denotes the smallest integer greater than or equal to  $x$ . Thus, the stopping time  $T_{GE}^+$  for the GEWMA control chart can be modified as follows:  $T_{GE_0}^+ = \inf\{n \geq 1 : \max_{1 \leq k \leq \min(n_0, n)} \{\bar{W}_n(\frac{1}{k}) \geq c\}\}$ . This revised definition not only lightens the computational burden but also keeps the effectiveness in detecting the mean shift,  $\mu \geq \delta_0$ .

APPENDIX

We first mention a known result in Slepian (1962) and Gupta (1963) that will be used in proving the theorems.

LEMMA 1. *If  $(X_1, \dots, X_n)$  and  $(Y_1, \dots, Y_n)$  are two multivariate normal vectors with  $E(X_k) = E(Y_k)$  and  $\text{Var}(X_k) \leq \text{Var}(Y_k)$  for all  $k$ , and  $\text{Cov}(X_i, X_j) \geq \text{Cov}(Y_i, Y_j)$  for all  $i \neq j$ , then given any real numbers  $c_1, \dots, c_n$ ,*

$$(29) \quad P(X_1 < c_1, \dots, X_n < c_n) \geq P(Y_1 < c_1, \dots, Y_n < c_n).$$

LEMMA 2. *Let  $\beta_{ij}(m, n) = \text{Cov}(U_m(i), U_n(j))$  and  $\alpha_{ij}(m, n) = \text{Cov}(\bar{W}_m(1/i), \bar{W}_n(1/j))$ . Then*

$$(30) \quad \alpha_{ij}(n, n) \geq \beta_{ij}(n, n).$$

*Furthermore, let  $b \gg 0$ ,  $m_0 = bc^2$ ,  $m_1 = \sqrt{b}m_0$ ,  $m \geq i \geq m_1$ ,  $n \geq j \geq m_1$  and  $0 < m - n < m_0$ ; then, for large  $b$ ,*

$$(31) \quad \alpha_{ij}(m, n) > \beta_{ij}(m, n)$$

*as  $c \rightarrow +\infty$ .*

PROOF. Let  $\alpha_{ij} = \alpha_{ij}(n, n)$  and  $\beta_{ij} = \beta_{ij}(n, n)$ . Since  $(\bar{W}_n(1), \bar{W}_n(1/2), \dots, \bar{W}_n(1/n))$  and  $(U_n(1), U_n(2), \dots, U_n(n))$  are two  $n$ -dimensional multivariate normal distributions with  $E(\bar{W}_n(1/k)) = E(U_n(k)) = 0$  and  $\text{Var}(\bar{W}_n(1/k)) = \text{Var}(U_n(k)) = 1$  for  $1 \leq k \leq n$ , it follows from (3) and (6) that

$$\alpha_{ij} = \frac{\sqrt{(2-1/i)}\sqrt{1/i} \sqrt{(2-1/j)}\sqrt{1/j}}{\sqrt{1-(1-1/i)^{2n}} \sqrt{1-(1-1/j)^{2n}}} \frac{1 - (1-1/i)^n(1-1/j)^n}{1 - (1-1/i)(1-1/j)}$$

and

$$\beta_{ij} = \sqrt{\frac{i}{j}}.$$

Since

$$\left(\frac{\alpha_{ij}}{\beta_{ij}}\right)^2 = \frac{(2-1/i)(2-1/j)}{i^2(1-(1-1/i)(1-1/j))^2} \frac{[1-(1-1/i)^n(1-1/j)^n]^2}{[1-(1-1/i)^{2n}][1-(1-1/j)^{2n}]},$$

we can define a function as follows:

$$h(x, y) = \frac{(1+x)(1+y)}{(1-x^{2n})(y+i(1-y))^2} - \frac{(1-y^{2n})}{(1-x^ny^n)^2}, \quad 1 > y \geq x \geq 0,$$

for  $j > i$ , where  $x = 1 - 1/i$ ,  $y = 1 - 1/j$ . It is easy to check that  $dh(x, y)/dy > 0$  for  $y > x$ . Thus  $h(x, y) > 0$  for  $y > x$ , since  $h(x, x) = 0$ , so that  $(\alpha_{ij}/\beta_{ij})^2 > 1$

for  $j > i$ , that is,  $\alpha_{ij} > \beta_{ij}$ , for  $j > i$ . By the same reasoning, we have  $\alpha_{ij} > \beta_{ij}$ , for  $j < i$ .

Let  $\theta = m/i$ ,  $\omega = n/j$ . Then  $(1 - 1/i)^{m-n} \rightarrow 1$ ,  $(1 - 1/i)^n \rightarrow e^{-\theta}$  and  $(1 - 1/j)^n \rightarrow e^{-\omega}$  as  $c \rightarrow +\infty$ ,  $b \rightarrow +\infty$ . Since

$$\alpha_{ij}(m, n) = \frac{(1 - 1/i)^{m-n} \sqrt{(2 - 1/i)} \sqrt{1/i}}{\sqrt{1 - (1 - 1/i)^{2m}}} \times \frac{\sqrt{(2 - 1/j)} \sqrt{1/j}}{\sqrt{1 - (1 - 1/j)^{2n}}} \frac{1 - (1 - 1/i)^n (1 - 1/j)^n}{1 - (1 - 1/i)(1 - 1/j)}$$

and

$$\beta_{ij}(m, n) = \begin{cases} \sqrt{\frac{j}{i}}, & \text{for } i \geq m - n + j, \\ \frac{i - (m - n)}{\sqrt{i} \sqrt{j}} < \min \left\{ \sqrt{\frac{i}{j}}, \sqrt{\frac{j}{i}} \right\}, & \text{for } i < m - n + j, \end{cases}$$

it follows that

$$\alpha_{ij}(m, n) \rightarrow \frac{\sqrt{(2 - 1/i)} \sqrt{(2 - 1/j)} \sqrt{ij}}{j + i - 1} \frac{1 - e^{-\theta} e^{-\omega}}{\sqrt{1 - e^{-2\theta}} \sqrt{1 - e^{-2\omega}}}$$

as  $c \rightarrow +\infty$ ,  $b \rightarrow +\infty$ . It can be checked that

$$\frac{\sqrt{(2 - 1/i)} \sqrt{(2 - 1/j)} \sqrt{ij}}{j + i - 1} > \sqrt{\frac{j}{i}}$$

for  $i \geq m - n + j$ ,

$$\frac{\sqrt{(2 - 1/i)} \sqrt{(2 - 1/j)} \sqrt{ij}}{j + i - 1} > \sqrt{\frac{i}{j}}$$

for  $i < m - n + j$  and  $j > i$ ,

$$\frac{\sqrt{(2 - 1/i)} \sqrt{(2 - 1/j)} \sqrt{ij}}{j + i - 1} > \sqrt{\frac{j}{i}} > \frac{i - (m - n)}{\sqrt{i} \sqrt{j}}$$

for  $i < m - n + j$  and  $i > j$ , and

$$\frac{1 - e^{-\theta} e^{-\omega}}{\sqrt{1 - e^{-2\theta}} \sqrt{1 - e^{-2\omega}}} > 1.$$

Thus  $\alpha_{ij}(m, n) > \beta_{ij}(m, n)$  as  $c \rightarrow +\infty$ ,  $b \rightarrow +\infty$ .  $\square$

LEMMA 3. Let  $E_\mu(\bar{W}_n(1/k)) = \mu_{nk}$  and  $E_\mu(U_n(k)) = v_{nk}$ . Then:

- (i)  $\mu_{nk} > \mu_{n(k-1)}$ ,  $\mu_{n(k-1)} \geq \mu_{(n-1)(k-1)}$ ;
- (ii)  $\mu_{n1} = v_{n1}$ .

And there exists  $n'$  such that  $\mu_{nk} \geq v_{nk}$  for  $2 \leq k \leq n'$ ,  $v_{nk} > \mu_{nk}$  for  $k > n'$  and  $3 \leq n' < n$ .

PROOF. From (3) and (6) it follows that

$$\mu_{nk} = \mu\sqrt{k} \frac{\sqrt{(2-1/k)}\sqrt{1-(1-1/k)^n}}{\sqrt{1+(1-1/k)^n}},$$

$$v_{nk} = \mu\sqrt{k}.$$

Obviously,  $\mu_{n1} = v_{n1}$ . Let  $1 - 1/k = x$  and set

$$(\mu_{nk})^2 = f(x) = \mu^2 \frac{(1+x)(1-x^n)}{(1-x)(1+x^n)}.$$

Then  $f'(x) > 0$  and  $f(0) = \mu^2$ , so that  $\mu_{nk} > \mu_{n(k-1)}$ . It is obvious that  $\mu_{n(k-1)} \geq \mu_{(n-1)(k-1)}$ . Hence we have  $\mu_{nk} > \mu_{(n-1)(k-1)}$ . This proves (i). Since  $(1 - 1/k)[1 - 2(1 - 1/k)^{n-1} - (1 - 1/k)^n] \geq 0$ , it implies that  $\mu_{nj} \geq v_{nj}$  for  $j \leq k$ . Let  $g(y) = 1 - 2y^{n-1} - y^n$ ,  $0 \leq y \leq 1$ . It follows that  $g'(y) < 0$ ,  $g(0) > 0$  and  $g(1) < 0$ , so that there exists a number  $y_n$  such that  $g(y_n) = 0$ ,  $g(y) \geq 0$  for  $y \leq y_n$  and  $g(y) < 0$  for  $y > y_n$ . Setting  $y_n = 1 - a_n/n$ , by  $g(1 - a_n/n) = 0$  we can get

$$n \approx \frac{a_n(1 - e^{-a_n})}{1 - 3e^{-a_n}}$$

and  $\ln 4 \geq a_n \geq a_{n+1} \downarrow \ln 3$ . Taking  $n' = \lceil n/a_n \rceil$  we get the required results, where  $\lceil x \rceil$  denotes the smallest integer greater than or equal to  $x$ .  $\square$

LEMMA 4. For large  $b$ , as  $c \rightarrow \infty$ ,

$$(32) \quad ARL_0(T_{GE}^+(c)) \geq \frac{(2\pi)^{1/2}e^{c^2/2}}{b^{3/2}c}.$$

PROOF. Denote by  $\varphi$  and  $\Phi$  the standard normal density and distribution functions, respectively. Let  $b \gg 0$ ,  $m_0 = bc^2$  and  $m_1 = \sqrt{b}m_0$ . Let  $m$ ,  $c \rightarrow +\infty$  such that  $m[b^{3/2}c\varphi(c)] \rightarrow t \in (0, +\infty)$ . Then, it follows from Lemmas 1 and 2 that

$$P(T_{GE}^+(c) > m) = P\left(\max_{1 \leq k \leq n} \bar{W}_n\left(\frac{1}{k}\right) < c, 1 \leq n \leq m\right)$$

$$= P\left(\bar{W}_n\left(\frac{1}{k}\right) < c, 1 \leq k \leq n, 1 \leq n \leq m\right)$$

$$\begin{aligned}
 &\geq P\left(\bar{W}_n\left(\frac{1}{k}\right) < c, m_1 \leq k \leq n, m_1 \leq n \leq m\right) \\
 &\quad \times P\left(\bar{W}_n\left(\frac{1}{k}\right) < c, 1 \leq k < m_1, m_1 \leq n \leq m\right) \\
 &\quad \times P\left(\bar{W}_n\left(\frac{1}{k}\right) < c, 1 \leq k < n, 1 \leq n < m_1\right) \\
 &\geq \prod_{l=1}^{m/m_0} P\left(\bar{W}_n\left(\frac{1}{k}\right) < c, m_1 \leq k \leq n, \right. \\
 &\quad \left. [(l-1)m_0] \vee m_1 < n \leq lm_0\right) \\
 &\quad \times [\Phi(c)]^{m_1(m-m_1)} \times [\Phi(c)]^{m_1(m_1-1)/2} \\
 &\geq \prod_{l=1}^{m/m_0} P(U_n(k) < c, m_1 \leq k \leq n, [(l-1)m_0] \vee m_1 < n \leq lm_0) \\
 &\quad \times [\Phi(c)]^{m_1 m} \times [\Phi(c)]^{-m_1(m_1+1)/2}
 \end{aligned}$$

as  $c \rightarrow +\infty$  for large  $b$ . By Lemma 3 in Siegmund and Venkatraman (1995), there exists  $\delta(b) \rightarrow 0$  as  $b \rightarrow +\infty$  such that, for all large  $c$  and  $1 \leq l \leq m/m_0$ ,

$$\begin{aligned}
 &P(U_n(k) < c, m_1 \leq k \leq n, [(l-1)m_0] \vee m_1 < n \leq lm_0) \\
 &\geq \left[1 - \delta(b) \frac{m_0}{m}\right].
 \end{aligned}$$

Thus, for large  $b$ ,

$$\begin{aligned}
 P(T_{GE}^+(c) > m) &\geq \left[1 - \delta(b) \frac{m_0}{m}\right]^{m/m_0} [\Phi(c)]^{m_1 m} \times [\Phi(c)]^{-m_1(m_1+1)/2} \\
 &\sim e^{-t} = \exp\{-m[b^{3/2}c\varphi(c)]\}
 \end{aligned}$$

as  $c \rightarrow +\infty$ . By the properties of the exponential distribution, we have  $E(T_{GE}^+(c)) \geq [b^{3/2}c\varphi(c)]^{-1}$  as  $c \rightarrow +\infty$ . This completes the proof.  $\square$

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