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ON THE RATE OF UNIFORM CONVERGENCE OF THE PRODUCT–LIMIT ESTIMATOR: STRONG AND WEAK LAWS

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By approximating the classical product–limit estimator of a distribution function with an average of iid random variables, we derive sufficient and necessary conditions for the rate of (both strong and weak) uniform convergence of the product–limit estimator over the whole line. These findings somehow fill a longstanding gap in the asymptotic theory of survival analysis. The result suggests a natural way of estimating the rate of convergence. We also prove a related conjecture raised by Gill and discuss its application to the construction of a confidence interval for a survival function near the endpoint.

1. Introduction. Let $\{X_i,\ i \geq 1\}$ and $\{U_i,\ 1 \geq 1\}$ be two independent sequences of nonnegative iid random variables. Let F and G be the common distributions of $\{X_i\}$ and $\{U_i\}$, respectively, and assume F is continuous. The maximum likelihood estimator of F based on $\{X_1, \dots, X_n\}$ is its empirical distribution. However, in many cases, the complete observations of X_i are not available. In random censorship models, we only observe $\{Z_i,\ \delta_i,\ 1 \leq i \leq n\}$, where $Z_i = X_i \wedge U_i$ and $\delta_i = I_{\{X_i \leq U_i\}}$. Based on these observations, the analogue of the empirical distribution is the celebrated product–limit estimator, introduced by Kaplan and Meier (1958) and defined by

$$\hat{F}_n(t) = 1 - \prod_{0 \leq s \leq t} \left(1 - \frac{dN_n(s)}{Y_n(s)}\right), \tag{1.1}$$

where

$$N_n(s) = \sum_{i=1}^{n} I_{\{Z_i \leq s,\ \delta_i = 1\}}, \quad dN_n(s) = N_n(s) - N_n(s-)$$

and

$$Y_n(s) = \sum_{i=1}^{n} I_{\{Z_i \geq s\}}.$$

We also introduce the notation $H(t) = P(Z_1 \leq t) = 1 - (1 - F(t))(1 - G(t))$, $H_1(t) = P(Z_1 \leq t,\ \delta_1 = 1) = \int_0^t (1 - G)\, dF$ and $\tau_H = \sup\{t\colon H(t) < 1\}$.

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The large-sample properties of the product–limit estimator have drawn considerable attention from many authors. In summary, the relevant results appearing in the literature may be classified into two cases: (1) restricting to a time interval $[0, t_0]$ with $t_0 < \tau_H$ and (2) extending to the time interval $[0, \tau_H]$, essentially the whole line. For the rate of strong uniform convergence of the first case, Foldes and Rejto (1981a, 1981b) proved that

$$\limsup_n n^{1/2}(\log\log n)^{-1/2} \sup_{t \le t_0} \left|\hat{F}_n(t) - F(t)\right| = C \quad \text{a.s.,} \tag{1.2}$$

where C is some constant depending on t_0. (Hereafter, a.s. is the abbreviated "almost surely," and all limits with n are taken as $n \to \infty$.) In particular, two types of strong approximations provided new insights to understand the asymptotic behavior of the product–limit estimator. Lo and Singh (1986) showed a representation of $\hat{F}_n - F$ by an average of iid random variables plus a small remainder term. They divided the time interval $[0, t_0]$ into small intervals in a proper manner and proved that

$$\begin{aligned} &\sup_{t \le t_0} \left| \hat{F}_n(t) - F(t) - \frac{1 - F(t)}{n} \sum_{j=1}^{n} \eta_j(t) \right| \\ &\qquad = O\left(n^{-3/4}(\log n)^{3/4}\right) \quad \text{a.s.,} \end{aligned} \tag{1.3}$$

where

$$\eta_j(t) = \frac{\delta_j}{1 - H(Z_j -)} I_{\{Z_j \le t\}} - \int_0^{t \wedge Z_j} \frac{dH_1}{(1 - H)^2} \tag{1.4}$$

is the influence curve of the Kaplan–Meier estimator. The order in (1.3) was later much improved. Burk, Csorgo and Horvath (1988) and Major and Rejto (1988) used the strong approximation of the empirical process and obtained the following results:

$$\sup_{t \le t_0} \left| \hat{F}_n(t) - F(t) - n^{-1/2} W_n(t) \right| = O(\log n / n) \quad \text{a.s.,} \tag{1.5}$$

where W_n is a Gaussian process with covariance satisfying

$$\lim_n E(W_n(s) W_n(t)) = (1 - F(s))(1 - F(t)) \int_0^{s \wedge t} \frac{dH_1}{(1 - H)^2} \tag{1.6}$$

for $s \le t_0$ and $t \le t_0$. It is easy to see (1.6) is also the covariance of $(1 - F(s))\eta_1(s)$ and $(1 - F(t))\eta_1(t)$. Applying the functional law of the iterated logarithm, one can see that both (1.3) and (1.5) imply (1.2) with

$$C = \sup_{t \le t_0} \sqrt{2}\,(1 - F(t)) \left(\int_0^t (1 - H)^{-2}\, dH_1 \right)^{1/2}.$$

Therefore the result of weak convergence of $n^{1/2}(\hat{F}_n - F)$ to a Gaussian process in space $D[0, t_0]$, first shown by Breslow and Crowley (1974), can be regarded as a simple corollary of either (1.3) or (1.5).

However, results on the rate of uniform convergence on the interval $[0, \tau_H]$ are far less complete. Although the approximations of (1.3) and (1.5) indicate that the behavior of $\hat{F}_n(t) - F(t)$ is close to a sample average or a Gaussian process, it cannot be extended to the interval $[0, \tau_H]$ without additional conditions. The purpose of this paper is to give a full analysis of the behavior of $\hat{F}_n(t)$, especially near τ_H. By developing some useful representations that are slightly different from (1.3), we characterize the rate of uniform convergence by some sufficient and necessary conditions on the distributions of lifetime and censoring time. These developments lead to some important findings on the asymptotic behavior of $\hat{F}_n$. For example, under certain conditions, a confidence interval can be derived for the distribution function near the endpoint where the ordinary confidence interval based on the Greenwood formula may be inaccurate. To better understand the aim of this paper, we give a brief review of the existing results in the literature.

Under the strongest condition

$$G(\tau_H) < 1 = F(\tau_H),$$

Foldes and Reijto (1981b) and Gu and Lai (1990) proved (1.2) holds over $[0, \tau_H]$. The more important and challenging part of the problem is the case when

$$F(\tau_H) < 1. \tag{1.7}$$

(This assumption implies $\tau_H < \infty$.) By assuming

$$\int_0^{\tau_H} \frac{1}{1-G}\, dF < \infty \tag{1.8}$$

in addition to (1.7), Gu and Lai (1990) proved that (1.2) can be extended to the interval $[0, \tau_H]$. In fact, the above conditions guarantee the weak convergence of $n^{1/2}(\hat{F}_n(t) - F(t))$ to a Gaussian process in the space $D[0, \tau_H]$ and hence the associated law of iterated logarithm can, in turn, be derived. Without assuming (1.8), Gill (1983) and Ying (1989) considered the normalized process

$$\frac{n^{1/2}\big(\hat{F}_n(t) - F(t)\big)}{(1 - F(t))\big(1 + \int_0^t dH_1/(1-H)^2\big)} \tag{1.9}$$

and proved that it converges weakly to a Brownian bridge in the space $D[0, \tau_H]$. However, their result does not render a rate of uniform convergence of $\hat{F}_n - F$ over $[0, \tau_H]$. In fact, the Studentized process corresponding to (1.9) may not even be a tight sequence in $D[0, \tau_H]$ without (1.8) [see Chen and Ying (1994)]. To the best of our knowledge, the only result regarding uniform convergence of $\hat{F}_n - F$ over $[0, \tau_H]$, without assuming (1.8), is the strong uniform consistency proved by Stute and Wang (1993). In general, the problem of the rate of strong or weak uniform convergence over the whole line remains unanswered.

In practice, inference about the survival function $1 - F$ relies on the rate of convergence of $\hat{F}_n - F$ over $[0, \tau_H]$, particularly near the endpoint τ_H.

Since the estimator $\hat{F}_n$ is rather unstable near τ_H, several approaches were proposed to amend the problem. In general, it is commonly accepted that $\hat{F}_n(t)$ should be regarded as unreliable when t is bigger than the last uncensored observation [see, e.g., Hall and Wellner (1980)]. In other words, instead of considering $\hat{F}_n(t) - F(t)$ one should consider a stopped process $\hat{F}_n(t \wedge X_n^*) - F(t \wedge X_n^*)$, where $X_n^* = \max\{\delta_j Z_j: 1 \le j \le n\}$ is the largest uncensored observation. More generally, one can consider the stopped process $\hat{F}_n(t \wedge t_n) - F(t \wedge t_n)$, where $t_n \to \tau_H$. However, there has been no theoretical justification of whether, or to what extent, the stopped process is more stable than the nonstopped process.

To avoid this difficulty, Lai and Ying (1988) proposed a modified product–limit estimator by smoothly decreasing $1 - \hat{F}_n$ to 0 in an interval which is chosen so that the number of observations larger than the two endpoints of the interval are at the same order $n^{-\lambda}$, where $0 < \lambda < 1$. Essentially, the product–limit estimator is truncated up to this interval. This modified estimator was successfully applied to linear regression and rank regression models with censored data [see, e.g., Lai and Ying (1988, 1991)]. However, it is practically hard to determine an appropriate λ. The essence of the problem goes back to the rate of convergence of $\hat{F}_n(t) - F(t)$ when t is close to τ_H.

To tackle the same problem, Gill (1993) proposed a rather simple method to estimate $F(\tau_H)$ in a slightly more specific setup. Suppose $F(\tau_H) < 1$ and both F and G have left-continuous positive densities at τ_H, say $f(\tau_H)$ and $g(\tau_H)$, respectively. Since $\hat{F}_n$ is unstable near τ_H, it was suggested therein to use $\hat{F}_n(\tau_H - n^{-1/2})$ instead of $\hat{F}_n(\tau_H)$ to estimate $F(\tau_H)$. The open question that arose there is whether this estimator is accurate to order $O_P((\log n/n)^{1/2})$. This conjecture is intuitively appealing. First, one can see

$$F(\tau_H) - F\big(\tau_H - n^{-1/2}\big) \sim f(\tau_H) n^{-1/2} = o\big((\log n/n)^{1/2}\big),$$

since F has positive left-continuous density at τ_H. Second, suppose the representation given in (1.3) can be extended to $[0, \tau_H]$ with a negligible remainder term, then $\hat{F}_n(\tau_H - n^{-1/2}) - F(\tau_H - n^{-1/2})$ can be approximated by $(1 - F(\tau_H - n^{-1/2}))/n\sum_{i=1}^n \eta_i(\tau_H - n^{-1/2})$ with variance

$$\frac{\big(1 - F\big(\tau_H - n^{-1/2}\big)\big)^2}{n} \int_0^{\tau_H - n^{-1/2}} \frac{dH_1}{(1-H)^2}$$
$$\sim \frac{1}{n} \int_0^{\tau_H - n^{-1/2}} \frac{f(t)\,dt}{g(\tau_H)(\tau_H - t)} \sim \left(\frac{f(\tau_H)\log n}{2g(\tau_H)n}\right).$$

Therefore it is natural to conjecture that estimation of $F(\tau_H)$ by $\hat{F}_n(\tau_H - n^{-1/2})$ is accurate to order $O_P((\log n/n)^{1/2})$.

In this paper, we provide a rather general analysis of the rate of convergence of $\hat{F}_n - F$ over the interval $[0, \tau_H]$, especially near the endpoint τ_H. Focusing our attention on the case when (1.7) holds, we present sufficient and necessary conditions for both strong and weak uniform convergence at the rate n^{-p}, where $0 \le p < 1/2$. We also derive a representation of $\hat{F}_n - F$ by a sample average plus a remainder term which is negligible over $[0, \tau_H]$. It is shown that the asymptotic behavior of $\hat{F}_n(t) - F(t)$ over $[0, \tau_H]$ still coincides with that of an average of iid random variables. Our findings indicate that one can detect the rate of uniform convergence of $\hat{F}_n$ by comparing the number of uncensored observations with that of censored observations near the endpoint. We then point out a way to consistently estimate the rate of convergence. In particular, we also pay attention to the asymptotic behavior of the stopped process and $\hat{F}_n(\tau_H)$ which are of special interest. The limiting distributions of $\hat{F}_n(\tau_H)$ are also clarified. As a byproduct, we give a solution to the aforementioned conjecture in Gill (1993). We prove that $\hat{F}_n(\tau_H - n^{-1/2})$ and $\hat{F}_n(\tau_H)$ converge at the same rate $(\log n/n)^{1/2}$ to the normal distribution with mean 0 and variance $f(\tau_H)/(2g(\tau_H))$, where $f(\tau_H)$ and $g(\tau_H)$ are the positive left-continuous density functions of $F(t)$ and $G(t)$ at $t = \tau_H$, respectively. This result provides a way to construct a confidence interval for $F(t)$ with t near τ_H and hence complements the usual approach based on the Greenwood formula that may not be reliable for t near τ_H. Sections 2 and 3 contain respectively the results on strong and weak convergence. The lemmas are proved in the appendices.

Before we move on to the next section, we note that it is not possible to find a universal rate of uniform convergence without adding any conditions. In other words, $\hat{F}_n$ may converge to F at an arbitrarily slow rate near the endpoint τ_H when the censoring is arbitrarily heavy [see, e.g., Chen and Ying (1994)].

2. Strong convergence. The following theorem gives a sufficient and necessary condition for the rate of strong uniform convergence of $\hat{F}_n(t) - F(t)$ over $[0, \tau_H]$.

THEOREM 2.1. *Assume $F(\tau_H) < 1$. Then, for $0 < p < 1/2$,*

$$\sup_{t \le \tau_H} \left| \hat{F}_n(t) - F(t) \right| = o(n^{-p}) \quad a.s. \tag{2.1}$$

if and only if

$$\int_0^{\tau_H} (1 - G)^{-p/(1-p)}\, dF < \infty. \tag{2.2}$$

Furthermore, if (2.2) *does not hold, then*

$$\begin{aligned} n^p \limsup_n \sup_{t \le X_n^*} \left| \hat{F}_n(t) - F(t) \right| &= n^p \limsup_n \left| \hat{F}_n(\tau_H) - F(\tau_H) \right| \\ &= \infty \quad a.s. \end{aligned} \tag{2.3}$$

where $X_n^* = \max\{\delta_i X_i\colon 1 \le i \le n\}$ is the largest uncensored observation.

REMARK. The heaviness of the censoring near the endpoint is reflected by the size of p in condition (2.2) [and also in conditions (3.2), (3.4) and (3.10) in the next section]. The smaller the p is, the fewer the uncensored observations are near the endpoint. This, in turn, is reflected in the convergence rate of $\hat{F}_n$: the smaller the p, the slower the rate of convergence. (The same remark can be made for the weak convergence discussed in Section 3.) When $p = 0$, (2.2) holds trivially, and (2.1) becomes the strong uniform consistency of $\hat{F}_n$ [see Stute and Wang (1993)]. In fact, the above theorem can be viewed as a Marcinkiewicz–Zigmund type of strong law of large numbers. When $p = 1/2$, condition (2.2) is identical to (1.8), under which the law of the iterated logarithm was proved by Gu and Lai (1990).

REMARK. As stated previously, we assume the continuity of F throughout the paper. For discontinuous F, our main results can still be derived with some attention paid to the discontinuity points of F. One can also appeal to an argument involving smoothing. The idea is to move the probability mass at each discontinuity point of F evenly onto a small interval. Specifically, let $A = \{a_i,\ i = 1, 2, \dots\}$ be the collection of discontinuity points of F with probability $p_i = P(X = a_i) > 0$. Let $d_i > 0$ be such that $\sum_{i=1}^{\infty} d_i < \infty$. One can choose to consider

$$\tilde{X} = X + \sum_{i=1}^{\infty} d_i I_{\{a_i < X\}} + \xi \sum_{i=1}^{\infty} d_i I_{\{a_i = X\}},$$

$$\tilde{U} = U + \sum_{i=1}^{\infty} d_i I_{\{a_i \le U\}},$$

where ξ is independent of (X, U) and follows the uniform distribution on $[0, 1]$. Let $\tilde{F}$ and $\tilde{G}$ be the distribution functions of $\tilde{X}$ and $\tilde{U}$, respectively. Then $\tilde{F}$ is a distribution which smooths the probability mass of F at its discontinuity points to the associated small intervals and $\tilde{G}$ assigns probability 0 to these intervals. Then it can be verified that the rate of uniform convergence for the product–limit estimator based on samples from $\tilde{F}$ and $\tilde{G}$ is the same as that of the product–limit estimator $\hat{F}_n$. And condition (2.2) is also equivalent to its tilde version. Therefore Theorem 2.1 also holds for discontinuous F. We note that, for discontinuous F, $F(t)$ is estimable only for $t < \tau_H$. Furthermore, $F(\tau_H)$ may not be estimable if $P(X = \tau_H) > 0$ and $P(U = \tau_H) = 0$. In general, the rate of uniform convergence for discontinuous F should be considered over the interval $[0, \tau_H)$ instead of $[0, \tau_H]$.

In this paper, we assume $F(\tau_H) < 1$, which implies $\tau_H < \infty$. The case $\tau_H = \infty$ is also of interest. In fact, we may classify all possible situations into three cases: (1) $F(\tau_H) < 1$ and $G(\tau_H) = 1$; (2) $F(\tau_H) = G(\tau_H) = 1$; and (3) $F(\tau_H) = 1$ and $G(\tau_H) < 1$. The first case is the subject of this paper. The third case is relatively easy to study, and all classical results on large-sample properties (the strong law of large numbers, the central limit theorem and the law of the iterated logarithm) can be proved via the representation in (1.5) which holds over $[0, \tau_H]$ in this case. The second case is a rather difficult one which also includes $\tau_H = \infty$. With some other assumption in addition to (1.8), Lai and Gu (1990) proved the law of the iterated logarithm. We do not know any other results on the rate of convergence for case 2. A full analysis of this case might be technically much harder. Meanwhile, this case is practically of secondary interest since we know $1 - F(t)$ is close to 0 when t is close to τ_H.

We may conclude from Theorem 2.1 that

$$n^p \limsup_n \sup_{t \le \tau_H} \left| \hat{F}_n(t) - F(t) \right| = n^p \limsup_n \sup_{t \le X_n^*} \left| \hat{F}_n(t) - F(t) \right|$$
$$= n^p \limsup_n \left| \hat{F}_n(\tau_H) - F(\tau_H) \right| \quad \text{a.s.},$$

which takes the value 0 or ∞ according to whether or not (2.2) holds. Hence the process $\hat{F}_n(t) - F(t)$ and the stopped process $\hat{F}_n(t \wedge X_n^*) - F(t \wedge X_n^*)$ have the same rate of strong uniform convergence. The process $\hat{F}_n(t) - F(t)$ achieves this rate at the endpoint τ_H. In Section 3 we shall show similar results for weak convergence.

By a direct application of Theorem 2.1, one can easily prove the following corollary.

COROLLARY 2.2. *Suppose* $F(\tau_H) < 1$ *and*

$$(2.4) \quad 0 < \liminf_{t \to \tau_H} \frac{(F(\tau_H) - F(t))^\alpha}{1 - G(t)} \le \limsup_{t \to \tau_H} \frac{(F(\tau_H) - F(t))^\alpha}{1 - G(t)} < \infty$$

for some $\alpha > 0$. *Then* (2.1) *holds for* $0 < p < 1/2$ *if and only if* $p < 1/(1 + \alpha)$. *In particular*, *if* $\alpha \le 1$, (2.1) *holds for all* $p < 1/2$.

The proof is straightforward and is omitted.

REMARK. The critical case is $p = 1/(1 + \alpha) < 1/2$, and we shall return to this case in Corollary 3.2.

We also present in the following proposition a strong approximation of $\hat{F}_n - F$ by an average of iid random variables. This representation plays an important role in the proof of our main results.

PROPOSITION 2.3. *Assume* $F(\tau_H) < 1$ *and* (2.2) *holds for some* $0 < p < 1/2$. *Then*

$$(2.5) \quad \begin{aligned} &\sup_{t \le t_n} \left| \hat{F}_n(t) - F(t) - \frac{1 - F(t)}{n} \sum_{i=1}^n \left(\frac{\delta_i I_{\{Z_i \le t\}}}{1 - H(Z_i -)} - \int_0^t \frac{dH_1}{1 - H} \right) \right| \\ &\quad = O\Big(n^{-1/2} (\log n)^{1/2} \left[(1 - H(t_n -))^{(3p-1)/2(1-p)} \vee 1 \right] \Big) \quad a.s., \end{aligned}$$

where t_n is such that $(1 - H(t_n -)) n^\lambda \to \infty$ for some $0 < \lambda < 1$. Furthermore,

$$(2.6) \quad \begin{aligned} &\sup_{t \le \tau_H} \left| \hat{F}_n(t) - F(t) - \frac{1 - F(t)}{n} \sum_{i=1}^n \left(\frac{\delta_i I_{\{Z_i \le t\}}}{1 - H(Z_i -)} - \int_0^t \frac{dH_1}{1 - H} \right) \right| \\ &\quad = O\big((n^{-1/2} \vee n^{-3p/2}) (\log n)^{1/2} \big) \quad a.s. \end{aligned}$$

REMARK. The above approximations of $\hat{F}_n - F$ by an average of iid variables are different from (1.3). The representation in (1.3) achieves a better order but is restricted to some interval $[0, t_0]$ with $t_0 < \tau_H$. It is reasonable that a representation of $\hat{F}_n - F$ by an average of iid variables over $[0, \tau_H]$ may produce a larger error term. It should be noted that the iid random variables in (1.3) are different from those in (2.5) and (2.6). To see their relations, we notice that both terms in the expression of $\eta_j(t)$ in (1.4) have the same mean $\int_0^t dH_1/(1-H)$. So we can write

$$\sum_{j=1}^{n} \eta_j(t) = \sum_{j=1}^{n} \left(\frac{\delta_j}{1 - H(Z_j -)} I_{\{Z_j \le t\}} - \int_0^t \frac{dH_1}{1-H} \right) - \sum_{j=1}^{n} \left(\int_0^{t \wedge Z_j} \frac{dH_1}{(1-H)^2} - \int_0^t \frac{dH_1}{1-H} \right).$$

Roughly speaking, when t is close to τ_H, the oscillation of the second term in the above expression of $\sum_{j=1}^n \eta_j(t)$ is dominated by that of the first term. So the order of $\sum_{i=1}^n \eta_i(t)$ is the same as that of the first term in the above expression. To obtain an approximation for $\hat{F}_n - F$ over $[0, \tau_H]$, it suffices to use the first term of the above expression. This, in fact, greatly simplifies the technical analysis. We remind the readers that the validity of the above representations depends on certain conditions such as (2.2).

For an illustration of the above results, we give the following example.

EXAMPLE. Let $G(t) = 1 - (1-t)^\beta$, $0 \le t \le 1$, where $\beta > 0$. Suppose $F(1) < 1$ and $F(t)$ has positive continuous density at $t = 1$ [e.g., $F(t) = 1 - e^{-t}$, $t \ge 0$]. The uniform convergence of $\hat{F}_n - F$ over $[0,1]$ can be classified into three cases:

(i) $\beta < 1$. Equation (2.1) holds for all $p < 1/2$. In this case, (1.8) and hence the law of the iterated logarithm hold;
(ii) $\beta = 1$. Equation (2.1) is also true for all $p < 1/2$. This is now an example of the conjecture raised in Gill (1993) which shall be addressed in Theorem 3.6;
(iii) $\beta > 1$. Equation (2.1) holds only for $p < 1/(\beta + 1)$.

One can understand that the order of $\hat{F}_n - F$ over $[0, \tau_H]$ is dominated by that of the average of the iid random variables presented in (2.6). For instance, when $\beta = 2$,

$$\limsup_n n^{1/3} \sup_{t \le \tau_H} \left| \hat{F}_n(t) - F(t) \right| = \infty \quad \text{a.s.}$$

And the order on the right-hand side of (2.6) is $(\log n/n)^{-1/2}$, which is substantially smaller than $n^{-1/3}$. Thus (2.6) indeed provides a proper approximation.

In relation to the estimation of the distribution function, an equally important problem is the estimation of the cumulative hazard function which is defined as

$$\Lambda(t) = -\log(1 - F(t)) = \int_0^t \frac{dH_1}{1 - H}$$

and estimated by

$$\hat{\Lambda}_n(t) = \int_0^t \frac{dN_n(s)}{Y_n(s)}.$$

Parallel to Theorem 2.1 and Proposition 2.3, similar results also hold for the rate of convergence of $\hat{\Lambda}_n$.

PROPOSITION 2.4. *Assume* $F(\tau_H) < 1$ *and* $0 < p < 1/2$. *Then*:

(i) *Equation* (2.2) *implies*

$$\begin{aligned} &\sup_{t \le t_n} \left| \hat{\Lambda}_n(t) - \Lambda(t) - \frac{1}{n} \sum_{i=1}^n \left(\frac{\delta_i I_{\{Z_i \le t\}}}{1 - H(Z_i -)} - \Lambda(t) \right) \right| \\ &\quad = O\Big(n^{-1/2} (\log n)^{1/2} \Big[(1 - H(t_n -))^{(3p-1)/2(1-p)} \vee 1 \Big] \Big), \end{aligned} \tag{2.7}$$

where t_n *is such that* $n^{\lambda}(1 - H(t_n -)) \to \infty$ *for some* $0 < \lambda < 1$. *Furthermore*,

$$\begin{aligned} &\sup_{t \le \tau_H} \left| \hat{\Lambda}_n(t) - \Lambda(t) - \frac{1}{n} \sum_{i=1}^n \left(\frac{\delta_i I_{\{Z_i \le t\}}}{1 - H(Z_i -)} - \Lambda(t) \right) \right| \\ &\quad = O\Big((n^{-1/2} \vee n^{-3p/2}) (\log n)^{1/2} \Big) \quad a.s. \end{aligned} \tag{2.8}$$

(ii) *We have*

$$\sup_{t \le \tau_H} \left| \hat{\Lambda}_n(t) - \Lambda(t) \right| = o(n^{-p}) \quad a.s. \tag{2.9}$$

if and only if (2.2) *holds*.

(iii) *If* (2.2) *does not hold*, *then*

$$\begin{aligned} &n^p \limsup_n \left| \hat{\Lambda}_n(\tau_H) - \Lambda(\tau_H) \right| \\ &\quad = n^p \limsup_n \sup_{t \le X_n^*} \left| \hat{\Lambda}_n(t) - \Lambda(t) \right| = \infty \quad a.s. \end{aligned} \tag{2.10}$$

In the following we show the proof of the above proposition. We first use the empirical approximation demonstrated in Lemma A.1 in Appendix A to show (2.7). To prove (2.8), we divide the interval $[0, \tau_H]$ into two parts: $[0, \tau_n]$ and $[\tau_n, \tau_H]$, where τ_n defined in (A.1) in Appendix A is appropriately chosen. Since (2.7) also holds for $t_n = \tau_n$, we then argue that the variation of $\hat{\Lambda}_n - \Lambda$ over the interval $[\tau_n, \tau_H]$ is negligible to show (2.8). The proof of (2.9) and (2.10) uses the representation results in (2.8) and some martingale properties of the counting process. Proposition 2.4 shall rely on some lemmas that are deferred to Appendix A. With the help of Proposition 2.4, the proofs of Theorem 2.1 and Proposition 2.3 essentially only utilize the Taylor expansion [see (2.24)].

PROOF OF PROPOSITION 2.4. (i) Suppose (2.2) holds. Write

$$\begin{aligned}\hat{\Lambda}_n(t) - \Lambda(t) &= \frac{1}{n}\left(\int_0^t \frac{dN_n}{1-H} - n\int_0^t \frac{dH_1}{(1-H)}\right)\\ &\quad - \frac{1}{n}\int_0^t \frac{Y_n - n(1-H)}{Y_n(1-H)}\,dN_n.\end{aligned} \tag{2.11}$$

Let t_n be such that $n^{\lambda}(1 - H(t_n -)) \to \infty$ for some $0 < \lambda < 1$. Observe that

$$\begin{aligned}&\frac{1}{n}\int_0^{t_n} \frac{dN_n}{(1-H)^{1/(1-p)}}\\ &\quad \le \frac{1}{n}\int_0^{\tau_H} \frac{dN_n}{(1-H)^{1/(1-p)}}\\ &\quad = \frac{1}{n}\sum_{i=1}^{n} \frac{\delta_i}{(1-H(Z_i -))^{1/(1-p)}}\\ &\quad \to \int_0^{\tau_H} \frac{dH_1}{(1-H)^{1/(1-p)}} \quad \text{a.s. by the law of large numbers}\\ &\quad < \infty\end{aligned}$$

by condition (2.2). Now apply Lemma A.1 and write

$$\begin{aligned}&\sup_{t \le t_n}\left|\frac{1}{n}\int_0^t \frac{Y_n - n(1-H)}{Y_n(1-H)}\,dN_n\right|\\ &\quad \le \int_0^{t_n} \frac{|Y_n - n(1-H)|}{(1-H)Y_n}\,\frac{dN_n}{n}\\ &\quad \le O(1)n^{-1/2}(\log n)^{1/2}\int_0^{t_n}(1-H)^{-3/2}\,\frac{dN_n}{n} \quad \text{a.s. by Lemma A.1}\\ &\quad \le O(1)n^{-1/2}(\log n)^{1/2}\Big[(1-H(t_n -))^{-3/2+1/(1-p)} \vee 1\Big]\\ &\qquad \times \int_0^{t_n} \frac{dN_n/n}{(1-H)^{1/(1-p)}} \quad \text{a.s.}\\ &\quad \le O(1)n^{-1/2}(\log n)^{1/2}\Big[(1-H(t_n -))^{(3p-1)/2(1-p)} \vee 1\Big] \quad \text{a.s.}\end{aligned} \tag{2.12}$$

Hence (2.7) holds by (2.11) and (2.12). Now replace t_n in (2.12) by τ_n as defined in (A.1). We have

$$\sup_{t\le\tau_n}\left|\frac{1}{n}\int_0^t \frac{Y_n-n(1-H)}{Y_n(1-H)}\,dN_n\right| \le O\big((n^{-1/2}\vee n^{-3p/2})(\log n)^{1/2}\big) \quad \text{a.s.} \tag{2.13}$$

Observe that (A.4) in Lemma A.2 implies

$$P\left(\sup_{t>\tau_n}\left|\frac{1}{n}\int_0^t \frac{Y_n-n(1-H)}{Y_n(1-H)}\,dN_n\right|\ne 0 \text{ i.o.}\right)=0.$$

So (2.13) also holds over the interval $[0,\tau_H]$. Therefore (2.8) follows.

(ii) Suppose (2.2) holds. Similarly to (2.11), we can write

$$\begin{aligned}\hat\Lambda_n(t)-\Lambda(t) &= \frac{1}{n}\left(\int_0^t\frac{dN_n}{1-H}-\int_0^t\frac{Y_n}{(1-H)^2}\,dH_1\right)\\ &\quad -\frac{1}{n}\int_0^t\frac{Y_n-n(1-H)}{Y_n(1-H)}\,dN_n+\frac{1}{n}\int_0^t\frac{Y_n-n(1-H)}{(1-H)^2}\,dH_1,\end{aligned} \tag{2.14}$$

where the first term is $(1/n)\sum_{j=1}^n\eta_j(t)$. We can compute the last term in (2.14) in a similar fashion to (2.12) and find that its order is the same as that in (2.13). It then follows from Lemma A.5 and (2.14) that

$$\sup_{t\le\tau_n} n^p\left|\hat\Lambda_n(t)-\Lambda(t)\right|\to 0 \quad \text{a.s.}$$

In order to show (2.9), it then suffices to show

$$\sup_{\tau_n\le t\le\tau_H} n^p\left|\hat\Lambda_n(t)-\Lambda(t)\right|\to 0 \quad \text{a.s.} \tag{2.15}$$

To this end, we observe the monotonicity of $\Lambda(t)$ and $\hat\Lambda_n(t)$ and use the triangle inequality to write

$$\begin{aligned}&\sup_{\tau_n\le t\le\tau_H} n^p\left|\hat\Lambda_n(t)-\Lambda(t)\right|\\ &\quad\le \sup_{\tau_n\le t\le\tau_H} n^p\left|\hat\Lambda_n(t)-\hat\Lambda_n(\tau_n)-(\Lambda(t)-\Lambda(\tau_n))\right|+n^p\left|\hat\Lambda_n(\tau_n)-\Lambda(\tau_n)\right|\\ &\quad\le n^p\big(\hat\Lambda_n(\tau_H)-\hat\Lambda_n(\tau_n)\big)+n^p(\Lambda(\tau_H)-\Lambda(\tau_n))+n^p\left|\hat\Lambda_n(\tau_n)-\Lambda(\tau_n)\right|.\end{aligned}$$

The third term in the above expression clearly converges to 0 a.s. by the preceding arguments. For the second term, we have

$$\begin{aligned}n^p(\Lambda(\tau_H)-\Lambda(\tau_n)) &= n^p\int_{\tau_n}^{\tau_H}\frac{dH_1}{1-H}\\ &\le n^p(1-H(\tau_n))^{p/(1-p)}\int_{\tau_n}^{\tau_H}\frac{dH_1}{(1-H)^{1/(1-p)}}\to 0,\end{aligned}$$

since $n^p(1 - H(\tau_n))^{p/(1-p)} \le M^{p/(1-p)}$ by the definition of τ_n in (A.1). For the first term, we notice

$$P\Big(n^p\big(\hat{\Lambda}_n(\tau_H) - \hat{\Lambda}_n(\tau_n)\big) \neq 0 \text{ i.o.}\Big) = P\big(N_n(\tau_H) - N_n(\tau_n) \neq 0 \text{ i.o.}\big) = 0$$

by (A.4) in Lemma A.2. So (2.15) follows. The sufficiency is thus proved. The necessity is implied in the next part.

(iii) Suppose (2.2) does not hold. By the continuity of $\Lambda(t)$, we can write

$$\sup_{1 \le j \le n} \frac{\delta_j}{Y_n(Z_j)} = \sup_{t \le X_n^*} \big|\hat{\Lambda}_n(t) - \hat{\Lambda}_n(t-)\big| \le 2 \sup_{t \le X_n^*} \big|\hat{\Lambda}_n(t) - \Lambda(t)\big|, \tag{2.16}$$

where X_n^* is the largest uncensored observation among the first n observations. Hence (A.5) in Lemma A.2 implies that

$$\limsup_n \sup_{t \le X_n^*} \big|\hat{\Lambda}_n(t) - \Lambda(t)\big| = \infty \quad \text{a.s.}$$

To show

$$\limsup_n n^p \big|\hat{\Lambda}_n(\tau_H) - \Lambda(\tau_H)\big| = \infty \quad \text{a.s.},$$

we realize that

$$\begin{aligned}
&\limsup_n n^p \big|\hat{\Lambda}_n(\tau_H) - \hat{\Lambda}_{n+1}(\tau_H)\big| \\
&\quad \le \limsup_n n^p \Big|\big(\hat{\Lambda}_n(\tau_H) - \Lambda(\tau_H)\big) - \big(\hat{\Lambda}_{n+1}(\tau_H) - \Lambda(\tau_H)\big)\Big| \\
&\quad \le 2 \limsup_n n^p \big|\hat{\Lambda}_n(\tau_H) - \Lambda(\tau_H)\big|.
\end{aligned}$$

So it suffices to show

$$\limsup_n n^p \big|\hat{\Lambda}_n(\tau_H) - \hat{\Lambda}_{n+1}(\tau_H)\big| = \infty \quad \text{a.s.} \tag{2.17}$$

Now we write

$$\begin{aligned}
&\hat{\Lambda}_n(\tau_H) - \hat{\Lambda}_{n+1}(\tau_H) \\
&\quad = \int_0^{\tau_H} \frac{dN_n(t)}{Y_n(t)} - \int_0^{\tau_H} \frac{dN_{n+1}(t)}{Y_{n+1}(t)} \\
&\quad = (1 - \delta_{n+1}) \int_0^{Z_{n+1}} \frac{dN_n(t)}{Y_n(t)(Y_n(t) + 1)} \\
&\qquad - \delta_{n+1}\left(\frac{1}{Y_{n+1}(Z_{n+1})} - \int_0^{Z_{n+1}} \frac{dN_n(t)}{Y_n(t)(Y_n(t) + 1)}\right).
\end{aligned} \tag{2.18}$$

By (A.5) in Lemma A.2 and the Kolmogorov zero–one law, we know either

$$\limsup_n n^p \delta_{n+1}\Big(\frac{1}{Y_{n+1}(Z_{n+1})} - \int_0^{Z_{n+1}} \frac{dN_n(t)}{Y_n(t)(Y_n(t)+1)}\Big) = \infty \quad \text{a.s.} \tag{2.19}$$

or

$$\limsup_n n^p \delta_{n+1} \int_0^{Z_{n+1}} \frac{dN_n(t)}{Y_n(t)(Y_n(t)+1)} = \infty \quad \text{a.s.} \tag{2.20}$$

If we can show (2.20) implies

$$\limsup_n n^p (1-\delta_{n+1}) \int_0^{Z_{n+1}} \frac{dN_n}{Y_n(Y_n+1)} = \infty \quad \text{a.s.} \tag{2.21}$$

then, by combining (2.21) and (2.19), we know (2.17) holds since at least one of (2.21) and (2.19) is true. Now assume (2.20). To show (2.21), let m be an arbitrary positive number and define

$$T_n = \inf\left\{ t > 0, \int_0^t \frac{dN_n}{Y_n(Y_n+1)} \geq m n^{-p} \right\} \tag{2.22}$$

($T_n = \infty$ if the set is empty). Let $\mathscr{F}_n = \sigma\{(Z_1, \delta_1), \dots, (Z_n, \delta_n)\}$. Let A_n be the event $\{Z_n \geq T_{n-1}\}$ and let B_n be the event $\{\delta_n = 0\}$. It is easy to see

$$\begin{aligned} P(A_n \cap B_n \mid \mathscr{F}_{n-1}) &= P(\delta_n = 0, Z_n \geq T_{n-1} \mid \mathscr{F}_{n-1}) \\ &\geq \inf_{t < \tau_H} \frac{P(\delta_n = 0, Z_n \geq t)}{P(Z_n \geq t)} P(Z_n \geq T_{n-1} \mid \mathscr{F}_{n-1}) \\ &\leq \inf_{t<\tau_H} P(\delta_1 = 0 \mid Z_1 \geq t) P(A_n \mid \mathscr{F}_{n-1}) \quad \text{a.s.}, \end{aligned}$$

where the first inequality is due to the independence of (Z_n, δ_n) and $\mathscr{F}_{n-1}$. Since (2.20) implies $P(A_n \text{ i.o.}) = 1$, it follows from Lemma A.3 that

$$\begin{aligned} P(\delta_{n+1} = 0, Z_{n+1} \geq T_n \text{ i.o.}) &= P(B_n \cap A_n \text{ i.o.}) \\ &\geq \inf_{0 \leq t < \tau_H} P(\delta_1 = 0 \mid Z_1 \geq t). \end{aligned} \tag{2.23}$$

Now one notices $\lim_{t \to \tau_H} P(\delta_1 = 0 \mid Z_1 \geq t) = 1$. Hence

$$\inf_{0 \leq t < \tau_H} P(\delta_1 = 0 \mid Z_1 \geq t) > 0.$$

Again, by the Kolmogorov zero–one law, we know from (2.23) that

$$P(\delta_{n+1} = 0, Z_{n+1} \geq T_n \text{ i.o.}) = 1,$$

which implies

$$\limsup_n n^p (1-\delta_{n+1}) \int_0^{Z_{n+1}} \frac{dN_n}{Y_n(Y_n+1)} \geq m \quad \text{a.s.}$$

Let $m \to \infty$; (2.21) and hence (2.17) follow. The proof is complete. □

PROOF OF THEOREM 2.1. Write $1 - \hat{F}_n(t) = \exp(-\tilde{\Lambda}_n(t))$, where

$$\tilde{\Lambda}_n(t) = -\int_0^t \log\left(1 - \frac{1}{Y_n}\right) dN_n.$$

By the strong uniform consistency of $\hat{F}_n$ over $[0, \tau_H]$ [cf. Stute and Wang (1993)], we can write, via the triangle inequality and the Taylor expansion,

$$\begin{aligned} &\left| \sup_{t\in D} \left| \hat{F}_n(t) - F(t) \right| - \sup_{t \in D} (1 - F(t)) \left| \tilde{\Lambda}_n(t) - \Lambda(t) \right| \right| \\ &\qquad \le \sup_{t\in D} \left| \left| \hat{F}_n(t) - F(t) \right| - (1 - F(t)) \left| \tilde{\Lambda}_n(t) - \Lambda(t) \right| \right| \\ &\qquad \le \sup_{t\in D} \left| \hat{F}_n(t) - F(t) - (1 - F(t)) \big(\tilde{\Lambda}_n(t) - \Lambda(t) \big) \right| \\ &\qquad \le \sup_{t\in D} \left| \tilde{\Lambda}_n(t) - \Lambda(t) \right|^2 = o(1) \sup_{t\in D} \left| \tilde{\Lambda}_n(t) - \Lambda(t) \right| \end{aligned} \tag{2.24}$$

for large n, where D can be any possibly random set in $[0, \tau_H]$. Therefore

$$\begin{aligned} \tfrac{1}{2}(1 - F(\tau_H)) \sup_{t\in D} \left| \tilde{\Lambda}_n(t) - \Lambda(t) \right| &\le \sup_{t\in D} \left| \hat{F}_n(t) - F(t) \right| \\ &\le 2 \sup_{t\in D} \left| \tilde{\Lambda}_n(t) - \Lambda(t) \right| \end{aligned} \tag{2.25}$$

for all large n and $D \subseteq [0, \tau_H]$. We note that the strong consistency of $\hat{\Lambda}_n(\tau_H)$ (not depending on the assumption) implies

$$\min\{Y_n(Z_i) \colon \delta_i = 1, 1 \le i \le n\} \to \infty \quad \text{a.s.} \tag{2.26}$$

Thus we can employ the Taylor expansion on $\log(1 - 1/(Y_n(Z_j))$ for every $1 \le j \le n$ and show

$$\sup_{t\in D} \left| \tilde{\Lambda}_n(t) - \hat{\Lambda}_n(t) \right| \le \sup_{t\in D} \int_0^t \frac{dN_n}{Y_n^2} \tag{2.27}$$

for large n and $D \subseteq [0, \tau_H]$.

We first show (2.2) implies (2.1). Suppose (2.2) holds. From (A.3) in Appendix A, we have

$$\begin{aligned} \int_0^{\tau_n} \frac{dN_n}{Y_n^2} &= O(1) \int_0^{\tau_n} \frac{dN_n}{n^2 (1 - H)^2} \quad \text{a.s.} \\ &= O(1) n^{-1} (1 - H(\tau_n -))^{-2 + 1/(1-p)} \int_0^{\tau_n} \frac{d(N_n/n)}{(1 - H)^{1/(1-p)}} \\ &= O(n^{-2p}) \quad \text{a.s.} \end{aligned} \tag{2.28}$$

Since (A.4) in Lemma A.2 implies $P(\int_{\tau_n}^{\tau_H} dN_n / Y_n^2 \ne 0 \text{ i.o.}) = 0$, it follows from (2.27) and (2.28) that

$$\sup_{t \le \tau_H} \left| \tilde{\Lambda}_n(t) - \hat{\Lambda}_n(t) \right| \le \int_0^{\tau_H} \frac{dN_n}{Y_n^2} = O(n^{-2p}) \quad \text{a.s.} \tag{2.29}$$

Now (2.1) follows from (2.25) and Proposition 2.4(ii). The necessity is implied in the following proof.

It suffices to show (2.3) is true if (2.2) does not hold. Now assume (2.2) does not hold. Again, by the Taylor expansion and Lemma A.2, we have

$$\limsup_n \left(-n^p \delta_n \log\left(1 - \frac{1}{Y_n(Z_n)}\right)\right) = \limsup_n \frac{n^p \delta_n}{Y_n(Z_n)} = \infty \quad \text{a.s.}$$

for $0 < p < 1/2$. Now we can carry out a proof similar to the proof of part (iii) of Proposition 2.4 to show

$$\limsup_n n^p \left|\tilde{\Lambda}_n(\tau_H) - \Lambda(\tau_H)\right| = \limsup_n \sup_{t \le X_n^*} n^p \left|\tilde{\Lambda}_n(t) - \Lambda(t)\right| = \infty \quad \text{a.s.}$$

Then (2.3) follows from (2.25) with $D = [0, X_n^*]$ and $\{\tau_H\}$. The proof is complete. □

PROOF OF PROPOSITION 2.3. Similarly to (2.27) and (2.28), we can show

$$\begin{aligned}
&\sup_{t \le t_n} \left|\tilde{\Lambda}_n(t) - \hat{\Lambda}_n(t)\right| \\
&\quad = O(1) \int_0^{t_n} \frac{dN_n}{Y_n^2} \qquad (2.30)\\
&\quad = O(1) n^{-1} (1 - H(t_n -))^{-2 + 1/(1-p)} \\
&\quad = o(1) n^{-1/2} (\log n)^{1/2} \left[(1 - H(t_n -))^{(3p-1)/2(1-p)} \vee 1\right] \quad \text{a.s.,}
\end{aligned}$$

because $n^\lambda(1 - H(t_n -)) \to \infty$ for some $0 < \lambda < 1$. Replace $\tilde{\Lambda}_n$ by $\hat{\Lambda}_n$ in (2.24). Then (2.5) follows from (2.30) and Proposition 2.4(i). In view of (2.29) and $2p > (3p/2 \wedge 1/2)$ for $0 < p < 1/2$, (2.6) can be shown similarly. □

3. Weak convergence.

The following theorem presents sufficient and necessary conditions for the weak convergence of $\hat{F}_n - F$.

THEOREM 3.1. *Suppose* $F(\tau_H) < 1$ *and* $0 < p < 1/2$. *Then*:

(i)

$$\sup_{t \le \tau_H} \left|\hat{F}_n(t) - F(t)\right| = O_P(n^{-p}) \tag{3.1}$$

if and only if

$$\limsup_{t \to \tau_H} \frac{\left(\int_t^{\tau_H} (1 - G)\, dF\right)^{1-p}}{1 - G(t)} < \infty; \tag{3.2}$$

(ii)

$$\sup_{t \le \tau_H} \left|\hat{F}_n(t) - F(t)\right| = o_P(n^{-p}) \tag{3.3}$$

if and only if

$$\lim_{t \to \tau_H} \frac{\left(\int_t^{\tau_H} (1-G)\, dF\right)^{1-p}}{1-G(t)} = 0. \tag{3.4}$$

With τ_H *replaced by* X_n^* *in* (3.1) *and* (3.3), *the above assertions still hold.*

Applying the above theorem, we can determine the exact order of the weak uniform convergence of $\hat{F}_n - F$.

COROLLARY 3.2. *Suppose* $F(\tau_H) < 1$, $0 < p < 1/2$ *and*

$$0 < \liminf_{t \to \tau_H} \frac{\left(\int_t^{\tau_H} (1-G)\, dF\right)^{1-p}}{1-G(t)} \le \limsup_{t \to \tau_H} \frac{\left(\int_t^{\tau_H} (1-G)\, dF\right)^{1-p}}{1-G(t)} < \infty. \tag{3.5}$$

Assume t_n *is such that* $n^{1-p}(1 - H(t_n -)) = O(1)$ *or* $n^{-p} Y_n(t_n) = O_p(1)$. *Then*

$$\lim_{m \to \infty} \lim_n P\left(\frac{n^{-p}}{m} \le \sup_{t \le t_n} \left| \hat{F}_n(t) - F(t) \right| < m n^{-p} \right) = 1. \tag{3.6}$$

In particular, (3.6) *holds with* $t_n = \tau_H$.

Consider the uniform convergence of $\hat{F}_n - F$ on $[0, t_n]$, where t_n is chosen such that $Y_n(t_n) \sim n^{\lambda}$ for some $0 < \lambda < 1$. From the above corollary, we know that $\sup_{t \le t_n} |\hat{F}_n(t) - F(t)|$ may converge at the same order as $\sup_{t \le \tau_H} |\hat{F}_n(t) - F(t)|$ does, depending on the heaviness of the censoring near τ_H measured by the size of p. Thus we may conclude that modifying the product–limit estimator by truncating it up to the n^{λ}th largest observation, regardless of the censoring near τ_H, may not provide a more stable estimator in the sense of the rate of uniform convergence. Hence it is important to determine how heavy the censoring is near τ_H.

The above theorems suggest a natural approach to estimating p from the data and then determining the rate of convergence. Since $\int_0^t (1-G)\, dF$ is the distribution function of the uncensored observations, it can be estimated by its empirical distribution $N_n(t)/n$. Because $(1 - G(t))/(1 - H(t)) \to 1/(1 - F(\tau_H)) > 0$ as $t \to \tau_H$, $1 - G(t)$ in condition (3.5) can be replaced by $1 - H(t)$. Therefore (3.5) can be rewritten as

$$\begin{aligned} -\infty &< \liminf_{t \to \tau_H} \left(\log(1 - H(t)) - (1-p) \log\left(\int_t^{\tau_H} (1-G)\, dF \right) \right) \\ &\le \limsup_{t \to \tau_H} \left(\log(1 - H(t)) - (1-p) \log\left(\int_t^{\tau_H} (1-G)\, dF \right) \right) < \infty. \end{aligned}$$

The above boundedness condition determines a unique p. By the empirical approximation given (A.3), we can show

$$\sup_{n^{\lambda_1} \le Y_n(t) \le n^{\lambda_2}} \left| \frac{Y_n(t)/n}{1 - H(t-)} - 1 \right| \to 0 \quad \text{a.s.}$$

and

$$\sup_{n^{\lambda_1} \le Y_n(t) \le n^{\lambda_2}} \left| \frac{(N_n(\tau_H) - N_n(t))/n}{\int_t^{\tau_H} (1 - G)\, dF} - 1 \right| \to 0 \quad \text{a.s.}$$

for any $p < \lambda_1 < \lambda_2 < 1$ under condition (3.5). Thus (3.5) implies, with probability 1,

$$\begin{aligned}
-\infty < \liminf_n \inf_{n^{\lambda_1} \le Y_n(t) \le n^{\lambda_2}} &\big(\log(Y_n(t)/n) \\
&-(1-p)\log\big((N_n(\tau_H) - N_n(t))/n\big)\big) \\
\le \limsup_n \sup_{n^{\lambda_1} \le Y_n(t) \le n^{\lambda_2}} &\big(\log(Y_n(t)/n) \\
&-(1-p)\log\big((N_n(\tau_H) - N_n(t))/n\big)\big) < \infty,
\end{aligned} \tag{3.7}$$

which also determines a unique p. Therefore p can be consistently estimated. For example, we can choose to consider a simple one-dimensional linear regression problem with observations:

$$\Big(\log(j/n), \log\Big(\big(N_n(Z_{(n)}) - N_n(Z_{(n-j)})\big)/n\Big)\Big), \qquad n^{\lambda_1} \le j \le n^{\lambda_2},$$

where $p < \lambda_1 < \lambda_2 < 1$ and $\{Z_{(j)}: 1 \le j \le n\}$ are the ordered $\{Z_j: 1 \le j \le n\}$. Consider $\log(j/n)$ as the covariate and $\log((N_n(Z_n) - N_n(Z_{(n-j)}))/n)$ as the response in the linear regression model without intercept. Let $\hat{p}$ be the estimated p via, for example, the least squares method. If (3.5) holds, then

$$\begin{aligned}
&\left| \frac{1}{1-\hat{p}} - \frac{1}{1-p} \right| \\
&\quad = \left| \frac{\sum_{n^{\lambda_1} \le j \le n^{\lambda_2}} \log(j/n) \log\big(\big[N_n(Z_{(n)}) - N_n(Z_{(n-j)})\big]/n\big)}{\sum_{n^{\lambda_1} \le j \le n^{\lambda_2}} (\log(j/n))^2} - \frac{1}{1-p} \right| \\
&\quad = \left| \frac{\sum_{n^{\lambda_1} \le j \le n^{\lambda_2}} \log(j/n) O(1)}{\sum_{n^{\lambda_1} \le j \le n^{\lambda_2}} (\log(j/n))^2} \right| \quad \text{a.s. by (3.7)} \\
&\quad = O\left(\frac{1}{\log n} \right) \quad \text{a.s.}
\end{aligned}$$

Then clearly $|\hat{p} - p| = O(1/\log n)$ a.s. and $n^{-\hat{p}}$ is a consistent estimator of the rate of uniform convergence of $\hat{F}_n - F$ over $[0, \tau_H]$. The properties of this type of estimator remain to be studied.

Although there might not be an optimal way to determine appropriate λ_1 and λ_2 in the above estimation, it is different from the seemingly same problem of finding an appropriate λ in truncating the product–limit estimator up to the n^λth largest observation. Modifying the product–limit estimator by disregarding it near τ_H may lose information and may still not provide better inference about F due to the aforementioned reason. We propose to estimate the rate of convergence and hence utilize the information of the observations near τ_H.

Parallel to Proposition 2.3, we also present a weak approximation of $\hat{F}_n - F$ by an average of iid random variables.

PROPOSITION 3.3. *Suppose* $F(\tau_H) < 1$ *and* (3.2) *holds for some* $0 < p < 1/2$. *Then we have*

$$\sup_{t \le t_n} \left| \hat{F}_n(t) - F(t) - \frac{1 - F(t)}{n} \sum_{i=1}^{n} \left(\frac{\delta_i I_{\{Z_i \le t\}}}{1 - H(Z_i -)} - \Lambda(t) \right) \right| = O_P\Big(n^{-1/2} \Big((1 - H(t_n -))^{(3p-1)/2(1-p)} \vee 1 \Big) (\log n)^2 \Big), \tag{3.8}$$

where $t_n \to \tau_H$ *is such that* $n^\lambda(1 - H(t_n -)) \to \infty$ *for some* $0 < \lambda < 1$. *Furthermore*,

$$\sup_{t \le \tau_H} \left| \hat{F}_n(t) - F(t) - \frac{1 - F(t)}{n} \sum_{i=1}^{n} \left(\frac{\delta_i I_{\{Z_i \le t\}}}{1 - H(Z_i -)} - \Lambda(t) \right) \right| = O_P\Big((n^{-1/2} \vee n^{-3p/2}) (\log n)^2 \Big). \tag{3.9}$$

The following proposition shows that under certain proper conditions $n^p(\hat{F}_n(\tau_H) - F(\tau_H))$ converges to a stable distribution.

PROPOSITION 3.4. *Suppose* $F(\tau_H) < 1$ *and* $0 < p < 1/2$. *Then*

$$0 < \lim_{t \to \tau_H} \frac{\left(\int_t^{\tau_H} (1 - G)\, dF \right)^{1-p}}{1 - G(t)} < \infty \tag{3.10}$$

implies

$$n^p \Big(\hat{F}_n(\tau_H) - F(\tau_H) \Big) \to \mathscr{D} \quad \text{in distribution}, \tag{3.11}$$

where $\mathscr{D}$ *denotes a stable distribution.*

REMARK. The above proposition can be presented in a slightly more general fashion involving a slowly varying function. Namely, (3.11) is equivalent to

$$\int_t^{\tau_H} (1 - G)\, dF \sim (1 - G(t))^{1/(1-p)} L\left(\frac{1}{1 - G(t)} \right),$$

where L is a slowly varying function at ∞. We note that when $p = 1/(1+\alpha) < 1/2$ and the upper and lower limits in (2.4) are equal in Corollary 2.2, $n^p(\hat{F}_n(\tau_H) - F(\tau_H))$ converges to a stable law.

We now proceed to investigate the asymptotic distribution of $\hat{F}_n(t)$ for t near τ_H when the censoring near τ_H is relatively moderate. Specifically, we are interested in the case when both F and G have positive left-continuous density at $t = \tau_H$. We can see that (3.4) is satisfied for all $p < 1/2$ in this case. Thus we expect $\hat{F}_n(\tau_H)$ converges to $F(\tau_H)$ at a rate faster than n^{-p} for all $p < 1/2$. In Theorem 3.6, we shall not only show the rate of convergence is $(\log n/n)^{1/2}$ but also prove its asymptotic normality. The following proposition presents a slightly more general result.

PROPOSITION 3.5. *Suppose $F(\tau_H) < 1$ and*

$$\lim_{t\to\tau_H} \frac{\int_t^{\tau_H}(1-G)\,dF}{(1-G(t))^2\int_0^t dF/(1-G)} = 0. \tag{3.12}$$

Let t_n be such that

$$n(1-G(t_n))^2\int_0^{t_n}\frac{dF}{1-G} \to 1 \tag{3.13}$$

and

$$\frac{n(F(\tau_H)-F(t_n))^2}{\int_0^{t_n}(1-G)^{-1}\,dF} \to 0.$$

Then

$$\left(\frac{n}{\int_0^{t_n} dF/(1-G)}\right)^{1/2}\left(\hat{F}_n(t_n') - F(\tau_H)\right) \to N(0,1), \tag{3.14}$$

where $N(0,1)$ is the standard normal distribution and t_n' is such that

$$\liminf_n \frac{\int_0^{t_n'}(1-G)^{-1}\,dF}{\int_0^{t_n}(1-G)^{-1}\,dF} \geq 1 \tag{3.15}$$

and

$$\lim_n \frac{n(F(\tau_H)-F(t_n'))^2}{\int_0^{t_n}(1-G)^{-1}\,dF} = 0. \tag{3.16}$$

In particular, (3.14) *holds for $t_n' = \tau_H$ and $t_n' = t_n$.*

REMARK. Equation (3.12) implies

$$\int_0^t \frac{dF}{1-G} = L\left(\frac{1}{1-G(t)}\right), \tag{3.17}$$

where $L(\cdot)$ is a slowly varying function [see, e.g., Feller (1966)]. It is also easy to show that (1.8) is a special case of (3.12). Since (3.13) implies $t_n \to \tau_H$, we know

$$\frac{\int_0^{t_n} dF/(1-G)}{\int_0^{t_n} dH_1/(1-H)^2} \to \big(1 - F(\tau_H)\big)^2.$$

Thus (3.14) can also be presented as

$$\left(\frac{n}{\big(1-F(t_n)\big)^2 \int_0^{t_n} dH_1/(1-H)^2}\right)^{1/2} \big(\hat{F}_n(t'_n) - F(\tau_H)\big) \to N(0,1)$$

for t'_n satisfying (3.15) and (3.16).

Using the above proposition, we can prove the conjecture by Gill (1993) mentioned in Section 1.

THEOREM 3.6. *Suppose $F(\tau_H) < 1$. Let $f(\tau_H)$ and $g(\tau_H)$ be the positive left-continuous density functions of $F(t)$ and $G(t)$ at $t = \tau_H$, respectively. Then*

$$\left(\frac{2g(\tau_H)n}{f(\tau_H)\log n}\right)^{1/2} \big(\hat{F}_n(t'_n) - F(\tau_H)\big) \to N(0,1) \tag{3.18}$$

for all t'_n such that

$$t'_n = \tau_H + o\left(\frac{\log n}{n}\right)^{1/2} \le \tau_H. \tag{3.19}$$

In particular, (3.18) *holds with $t'_n = \tau_H - n^{-1/2}$ and $t'_n = \tau_H$.*

Hence we conclude that when both F and G have positive left-continuous densities at τ_H, both $\hat{F}_n(\tau_H)$ and the modified estimator $\hat{F}_n(\tau_H - n^{-1/2})$ converge to $F(\tau_H)$ at the same rate $(\log n/n)^{1/2}$ with the same limiting normal distribution. It is interesting to see that in this case it is not necessary to modify the estimator $\hat{F}_n(\tau_H)$.

EXAMPLE. Recall the example given in Section 2 following Proposition 2.3. We know that when $p = 1/(\beta + 1) < 1/2$, Proposition 3.4 applies. When $\beta = 1$, it becomes an example of Theorem 3.6.

Proposition 3.5 and Theorem 3.6 contain an important application to the inference about $F(t)$ with t near τ_H. Normally, the confidence interval for $F(t)$ at a *fixed* $t < \tau_H$ is built based on the fact that $n^{1/2}(\hat{F}_n(t) - F(t)) \to N(0, (1 - F(t))^2\sigma^2(t))$, where $\sigma^2(t) = \int_0^t dH_1/(1-H)^2$ [see (1.6)]. So $\sigma^2(t)$ is naturally estimated by the Greenwood formula:

$$\hat{\sigma}^2(t) = \int_0^t n\, dN_n(s)/\big(Y_n(s)Y_n(s+)\big),$$

which is basically the empirical analogue of $\sigma^2(t)$. Then the level $100 \times (1 - \alpha)\%$ confidence interval for $F(t)$ is given by

$$\Big(\hat{F}_n(t) - z(\alpha/2)n^{-1/2}\big(1 - \hat{F}_n(t)\big)\hat{\sigma}(t),\\ \hat{F}_n(t) + z(\alpha/2)n^{-1/2}\big(1 - \hat{F}_n(t)\big)\hat{\sigma}(t)\Big),$$

where $z(\cdot)$ is the upper quantile of the standard normal distribution. However, when t is close to τ_H, the normal approximation tends to be less accurate if $\sigma(\tau_H) = \infty$. This is indeed the case when $F(\tau_H) < 1$ and both F and G have positive left-continuous densities at $t = \tau_H$. Thus the aforementioned confidence interval for $F(t)$ with t near τ_H may be invalid. Theorem 3.6 provides theoretical justification for a confidence interval for $F(\tau_H)$ as

$$\left(\hat{F}_n(\tau_H) - z\left(\frac{\alpha}{2}\right)\left(\frac{\hat{f}_n(\tau_H)\log n}{2n\hat{g}_n(\tau_H)}\right)^{1/2},\right.\\ \left.\hat{F}_n(\tau_H) + z\left(\frac{\alpha}{2}\right)\left(\frac{\hat{f}_n(\tau_H)\log n}{2n\hat{g}_n(\tau_H)}\right)^{1/2}\right), \tag{3.20}$$

where $\hat{f}_n(\tau_H)$ and $\hat{g}_n(\tau_H)$ are estimators of $f(\tau_H)$ and $g(\tau_H)$, respectively. Since $F(t)$ and $G(t)$ can be consistently estimated for $t \le \tau_H$, $f(\tau_H)$ and $g(\tau_H)$ can also be consistently estimated by the assumption of the left continuity of f and g at $t = \tau_H$. Thus the confidence interval given in (3.20) is valid.

We see from the above corollaries that the asymptotic behavior of $\hat{F}_n(t)$ at the endpoint $t = \tau_H$ is drastically different under different conditions. Under the strongest condition (1.8), $\hat{F}_n(\tau_H) - F(\tau_H)$ converges to the normal distribution at the order of $n^{-1/2}$. In Theorem 3.6, the censoring near τ_H is relatively moderate and the rate of convergence is $O_P((\log n/n)^{1/2})$. When the censoring near τ_H tends to be heavier [e.g., condition (3.6)], $\hat{F}_n(\tau_H) - F(\tau_H)$ converges at a slower rate (n^{-p} with $p < 1/2$).

To prove the above results, we shall first prove the following proposition.

PROPOSITION 3.7. *Suppose* $F(\tau_H) < 1$ *and* $0 < p < 1/2$. *Then, for any* $\varepsilon > 0$:

(i) *equation* (3.2) *implies*

$$\sup_{t \le t_n}\left|\hat{\Lambda}_n(t) - \Lambda(t) - \frac{1}{n}\sum_{i=1}^n\left(\frac{\delta_i I_{\{Z_i \le t\}}}{1 - H(Z_i-)} - \Lambda(t)\right)\right|\\ = O_P\Big(n^{-1/2}\Big(\big(1 - H(t_n-)\big)^{(3p-1)/2(1-p)} \vee 1\Big)(\log n)^2\Big), \tag{3.21}$$

where $t_n \to \tau_H$ *is such that* $n^{\lambda}(1 - H(t_n -)) \to \infty$ *for some* $\lambda > 0$. *Furthermore,*

$$\sup_{t \le \tau_H} \left| \hat{\Lambda}_n(t) - \Lambda(t) - \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\delta_i I_{\{Z_i \le t\}}}{1 - H(Z_i -)} - \Lambda(t) \right) \right| \tag{3.22}$$
$$= O_P\big((n^{-1/2} \vee n^{-3p/2})(\log n)^2\big);$$

(ii)

$$\sup_{t \le \tau_H} \left| \hat{\Lambda}_n(t) - \Lambda(t) \right| = O_P(n^{-p}) \tag{3.23}$$

if and only if (3.2) *holds*;

(iii)

$$\sup_{t \le \tau_H} \left| \hat{\Lambda}_n(t) - \Lambda(t) \right| = o_P(n^{-p}) \tag{3.24}$$

if and only if (3.4) *holds. With* τ_H *replaced by* X_n^* *in* (3.23) *and* (3.24), *the above assertions still hold.*

PROOF. The proof is along the lines of the proof of Proposition 2.4.

(i) Suppose (3.2) holds. To show (3.21), recall the expression of $\hat{\Lambda}_n(t) - \Lambda(t)$ in (2.11). We only need to show

$$\sup_{t \le t_n} \frac{1}{n} \left| \int_0^t \frac{Y_n - n(1-H)}{Y_n(1-H)} \, dN_n \right| \tag{3.25}$$
$$= O_P\Big(n^{-1/2}\big[(1 - H(t_n -))^{(3p-1)/2(1-p)} \vee 1\big](\log n)^2\Big).$$

In view of (2.12), it suffices to show

$$\int_0^{t_n} \frac{dN_n/n}{(1-H)^{3/2}} = O_P\Big(\big((1 - H(t_n -))^{(3p-1)/2(1-p)} \vee 1\big)(\log n)^{3/2}\Big).$$

To this end, we realize that (3.2) implies

$$\int_\varepsilon^{\tau_H} \frac{dH_1}{(1-H)^{1/(1-p)}(-\log(1-H))^{3/2}} < \infty$$

for any $0 < \varepsilon < \tau_H$. Since $-\log(1 - H(t_n -)) = O(\log n)$, similarly to (2.12), we can write

$$\begin{aligned} E\int_0^{t_n} \frac{dN_n/n}{(1-H)^{3/2}} &= \int_0^{t_n} \frac{dH_1}{(1-H)^{3/2}} \\ &= O(1) \int_\varepsilon^{t_n} \frac{dH_1}{(1-H)^{1/(1-p)}(-\log(1-H))^{3/2}} \\ &\quad \times \big[(1 - H(t_n -))^{-3/2 + 1/(1-p)} \vee 1\big](-\log(1 - H(t_n)))^{3/2} \\ &= O\Big(\big[(1 - H(t_n -))^{(3p-1)/2(1-p)} \vee 1\big](\log n)^{3/2}\Big). \end{aligned}$$

Thus (3.21) holds. Then (3.22) follows from (B.4) and (3.21) with $t_n = \tau_n$.

(ii) Assume (3.2) holds. The above proof can also be carried out to show

$$\sup_{t\le\tau_n}\left|\int_0^t \frac{Y_n - n(1-H)}{Y_n(1-H)}\,dH_1\right| = O_P\big((n^{-1/2}\vee n^{-3p/2})(\log n)^2\big).$$

By Lemma B.3 and the martingale property of $\eta_n(t)$, $0 \le t \le \tau_H$, we can write

$$P\left(n^p \sup_{t\le\tau_H}\left|\frac{1}{n}\sum_{i=1}^n \eta_i(t)\right| > m\right) \le n^p E\left|\frac{1}{n}\sum_{j=1}^n \eta_j(\tau_H)\right|\Big/ m \to 0 \tag{3.26}$$

uniformly over n when $m\to\infty$. Thus $\sup_{t\le\tau_H}|(1/n)\sum_{j=1}^n \eta_j(t)| = O_P(n^{-p})$. Now recall the expression of $\hat\Lambda_n(t) - \Lambda(t)$ in (2.14). We have

$$\sup_{t\le\tau_n}\left|\hat\Lambda_n(t) - \Lambda(t)\right| = O_P(n^{-p}). \tag{3.27}$$

To show (3.27) for $t > \tau_n$, it suffices to show $n^p(\Lambda(\tau_H) - \Lambda(\tau_n)) = O(1)$ in view of (B.4) in Lemma B.2. Now write

$$\begin{aligned} n^p(\Lambda(\tau_H) - \Lambda(\tau_n)) &= n^p\int_{\tau_n}^{\tau_H}\frac{dH_1}{1-H} \\ &= O(1)n^p\int_{\tau_n}^{\tau_H}\frac{dH_1(t)}{(H_1(\tau_H) - H_1(t))^{1-p}} \quad \text{by condition (3.2)} \\ &= O(1)n^p(H_1(\tau_H) - H_1(\tau_n))^p \\ &= O(1)n^p(1 - H(\tau_n))^{p/(1-p)} = O(1). \end{aligned} \tag{3.28}$$

Thus we have proved that (3.2) implies (3.23). On the other hand, suppose (3.23) holds with τ_H replaced by X_n^*. By (2.16) we know

$$\sup_{1\le i\le n}\frac{\delta_i}{Y_n(Z_i)} = O_P(n^{-p}), \tag{3.29}$$

which, by (A.3), is equivalent to

$$\sup_{1\le i\le n}\frac{\delta_i}{1 - H(Z_i-)} = O_P(n^{1-p}).$$

This is equivalent to (B.2) [see, e.g., Chow and Teicher (1988)], which is also equivalent to (3.2) by Lemma B.2.

(iii) The proof is similar to part (ii). We omit the details. □

We now apply Proposition 3.7 to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Similarly to the proof of Theorem 2.1, the proof of this theorem essentially depends on the Taylor expansion and Proposition 3.7.

(i) Assume (3.2) holds. In view of (2.25), to show that (3.2) implies (3.1), it is enough to show

$$\sup_{t \le \tau_H} \left| \tilde{\Lambda}_n(t) - \hat{\Lambda}_n(t) \right| = O_P\left(n^{-2p} (\log n)^2\right) \tag{3.30}$$

by Proposition 3.7. With condition (3.2), one can mimic (2.28) and prove $\int_0^{\tau_n} dN_n / Y_n^2 = O_P(n^{-2p}(\log n)^2)$. Hence (3.30) follows from (B.4) in Lemma B.2 and the inequality in (2.29). On the other hand, if (3.1) holds with τ_H replaced by X_n^*, we have

$$\sup_{t \le X_n^*} \left| \tilde{\Lambda}_n(t) - \Lambda(t) \right| = O_P(n^{-p}),$$

which entails

$$\max_{1 \le i \le n} \left(-\delta_i \log\left(1 - \frac{1}{Y_n(Z_i)}\right)\right) = O_P(n^{-p}).$$

This is equivalent to (3.29). Hence (3.2) follows.

(ii) The proof is similar to part (i) and is omitted. □

PROOF OF COROLLARY 3.2. We shall only consider t_n such that $n^{1-p}(1 - H(t_n))$ is bounded. Similar arguments can be applied for t_n satisfying $n^{-p} Y_n(t) = O_P(1)$. Now assume $n^{1-p}(1 - H(t_n))$ is bounded. Choose M large enough in the definition of τ_n in (A.1). Then $t_n \ge \tau_n$ for all large n. From Theorem 3.1, (2.24) and (3.30), we know it suffices to prove that (3.5) implies

$$\lim_{m \to \infty} \lim_n P\left(\sup_{t \le \tau_n} \left| \hat{\Lambda}_n(t) - \Lambda(t) \right| \ge \frac{n^{-p}}{m} \right) = 1. \tag{3.31}$$

Now observe that

$$\begin{aligned} &\liminf_n nP\left(\frac{\delta_1}{1 - H(Z_1 -)} > \frac{n^{1-p}}{m} \right) \\ &\quad \ge \liminf_{t \to \tau_H} \left(\frac{m}{1 - H(t)} \right)^{1/(1-p)} \int_t^{\tau_H} (1 - G)\, dF \\ &\quad \to \infty \quad \text{when } m \to \infty \end{aligned}$$

by (3.5), and that

$$\begin{aligned}
\limsup_n nP&\left(\frac{\delta_1 I_{\{Z_1>\tau_n\}}}{1-H(Z_1-)} > \frac{n^{1-p}}{m}\right)\\
&\le \limsup_n nP(\delta_1 = 1, Z_1 > \tau_n)\\
&= \limsup_n n\int_{\tau_n}^{\tau_H}(1-G)\,dF\\
&\le O(1)\limsup_n n(1-H(\tau_n))^{1/(1-p)} < \infty
\end{aligned}$$

uniformly for all m. We have

$$\liminf_n nP\left(\frac{\delta_1 I_{\{Z_1\le\tau_n\}}}{1-H(Z_1-)} > \frac{n^{1-p}}{m}\right) \to \infty \quad \text{when } m\to\infty.$$

Therefore

$$\begin{aligned}
\liminf_n P&\left(\sup_{i\le n}\frac{\delta_i I_{\{Z_i\le\tau_n\}}}{1-H(Z_i-)} > \frac{n^{1-p}}{m}\right)\\
&\ge \liminf_n\left(1-P\left(\frac{\delta_1 I_{\{Z_1\le\tau_n\}}}{1-H(Z_1-)} \le \frac{n^{1-p}}{m}\right)^n\right)\\
&\ge 1-\exp\left(-\liminf_n nP\left(\frac{\delta_1 I_{\{Z_1\le\tau_n\}}}{1-H(Z_1-)} > \frac{n^{1-p}}{m}\right)\right)\\
&\to 1 \quad \text{when } m\to\infty.
\end{aligned}$$

It then follows from (A.3) that

$$\liminf_n P\left(\sup_{i\le n}\frac{\delta_i I_{\{Z_i\le\tau_n\}}}{Y_n(Z_i)} \ge \frac{n^{-p}}{m}\right) \to 1 \quad \text{when } m\to\infty.$$

Similarly to (2.16), we also have

$$\sup_{i\le n}\frac{\delta_i I_{\{Z_i\le\tau_n\}}}{Y_n(Z_i)} \le 2\sup_{t\le\tau_n}|\hat{\Lambda}_n(t)-\Lambda(t)|.$$

Then (3.31) follows. □

PROOF OF PROPOSITION 3.3. By (3.30), we can replace $\tilde{\Lambda}_n$ by $\hat{\Lambda}_n$ in (2.24) and the order is $O_P(n^{-2p}(\log n)^2)$. Hence (3.8) and (3.9) follow from part (i) of Proposition 3.7. □

PROOF OF PROPOSITION 3.4. We first see that (3.10) is equivalent to

$$0 < \lim_{x\to\infty} x^{1/(1-p)} P\left(\frac{\delta_1}{1-H(Z_1-)} - \Lambda(\tau_H) > x\right) < \infty. \tag{3.32}$$

Observe that the random variable $\delta_1/(1 - H(Z_1 -)) - \Lambda(\tau_H)$ is unbounded above but bounded below. So we know that (3.32) implies

$$n^{p-1} \sum_{i=1}^{n} \left(\frac{\delta_i}{1 - H(Z_i -)} - \Lambda(\tau_H) \right) \to \mathscr{D}_1 \quad \text{in distribution,}$$

where $\mathscr{D}_1$ is a certain stable distribution [see, e.g., Feller (1966)]. Observe that (3.11) implies (3.2). So representation (3.9) is valid. Now (3.11) holds with $\mathscr{D} = (1 - F(\tau_H))\mathscr{D}_1$, which is also stable. The proof is complete. □

PROOF OF PROPOSITION 3.5. Observe (3.12) and the definition of t_n in (3.13). We know $n\int_{t_n}^{\tau_H}(1 - G)\,dF \to 0$. Therefore $nP(Z_1 \geq t_n,\ \delta_1 = 1) \to 0$, which implies $P(N_n(\tau_H) - N_n(t_n) \neq 0) \to 0$. We then conclude $P(\hat{F}_n(t_n) \neq \hat{F}_n(\tau_H)) \to 0$. Thus, if $t_n' \geq t_n$, we have $P(\hat{F}_n(t_n') \neq \hat{F}_n(t_n)) \to 0$, which clearly implies

$$\hat{F}_n(t_n') - \hat{F}_n(t_n) = o_P\left(\left(\frac{1}{n} \int_0^{t_n} \frac{dF}{1 - G} \right)^{1/2} \right).$$

We next show that the above equality still holds for $t_n' \leq t_n$. If $t_n' < t_n$,

$$E\left(\int_{t_n'}^{t_n} dN_n/(n(1 - H)) - \int_{t_n'}^{t_n} dH_1/(1 - H) \right)^2 \leq \frac{1}{n} \int_{t_n'}^{t_n} \frac{dH_1}{(1 - H)^2}$$

$$= o\left(\frac{1}{n} \int_0^{t_n} \frac{dF}{1 - G} \right)$$

by (3.15). So

$$\int_{t_n'}^{t_n} dN_n/(n(1 - H)) = o_P\left(\left(\frac{1}{n} \int_0^{t_n} \frac{dF}{1 - G} \right)^{1/2} \right),$$

since

$$\int_{t_n'}^{t_n} \frac{dH_1}{1 - H} = \Lambda(t_n) - \Lambda(t_n') = O(F(t_n) - F(t_n')) = o\left(\left(\frac{1}{n} \int_0^{t_n} \frac{dF}{1 - G} \right)^{1/2} \right)$$

by (3.16). Now observe that (3.13) implies $1 - H(t_n -) \geq n^{-\lambda}$ for some $0 < \lambda < 1$, so we can use the empirical approximation in Lemma A.1 to show

$$\left| \hat{\Lambda}_n(t_n) - \hat{\Lambda}_n(t_n') \right| = O\left(\int_{t_n'}^{t_n} dN_n/(n(1 - H)) \right) = o_P\left(\frac{1}{n} \int_0^{t_n} \frac{dF}{1 - G} \right)^{1/2}.$$

Hence, by the Taylor expansion, we have

$$\left| \hat{F}_n(t_n) - \hat{F}_n(t_n') \right| = o_P\left(\frac{1}{n} \int_0^{t_n} \frac{dF}{1 - G} \right)^{1/2} \quad \text{for } t_n' \leq t_n.$$

Therefore, to show (3.14), it suffices to show

$$\left(\frac{n}{\int_0^{t_n} dF/(1-G)}\right)^{1/2}\left(\hat{F}_n(t_n)-F(t_n)\right)\to N(0,1).$$

Again by employing the Taylor expansion in an analogous fashion to (2.24), we know that it suffices to show

$$\left(\frac{n}{\int_0^{t_n} dH_1/(1-H)^2}\right)^{1/2}\left(\hat{\Lambda}_n(t_n)-\Lambda(t_n)\right)\to N(0,1), \tag{3.33}$$

since

$$\int_0^{t_n} dH_1/(1-H)^2\Big/\int_0^{t_n} dF/(1-G)\to 1/(1-F(\tau_H))^2.$$

Recall the expression of $\hat{\Lambda}_n(t)-\Lambda(t)$ in (2.14). For the last two terms in (2.14), we notice that $N_n(t)$ is a counting process with compensator

$$\int_0^t Y_n\,d\Lambda=\int_0^t Y_n\,dH_1/(1-H)$$

and write

$$\begin{aligned}
&E\Bigg(\frac{1}{n}\int_0^{t_n}\frac{(Y_n-n(1-H))I_{\{Y_n>n(1-H)/2\}}}{Y_n(1-H)}\,dN_n\\
&\qquad-\frac{1}{n}\int_0^{t_n}\frac{(Y_n-n(1-H))I_{\{Y_n>n(1-H)/2\}}}{(1-H)^2}\,dH_1\Bigg)^2\\
&\quad=E\left(\frac{1}{n}\int_0^{t_n}\frac{(Y_n-n(1-H))I_{\{Y_n>n(1-H)/2\}}}{Y_n(1-H)}(dN_n-Y_n\,d\Lambda)\right)^2\\
&\quad=\frac{1}{n^2}\int_0^{t_n}E\left(\frac{(Y_n-n(1-H))^2I_{\{Y_n>n(1-H)/2\}}}{Y_n(1-H)^3}\right)dH_1\\
&\quad=\frac{O(1)}{n^2}\int_0^{t_n}\frac{dH_1}{(1-H)^3}=\frac{O(1)}{n^2}\int_0^{t_n}\frac{dF}{(1-G)^2}\\
&\quad=\frac{o(1)}{n^2(1-G(t_n))^2}=o(1)\frac{1}{n}\int_0^{t_n}\frac{dF}{1-G},
\end{aligned} \tag{3.34}$$

where the last equality is due to the definition of t_n in (3.13). So we know

$$\frac{1}{n}\int_0^{t_n}\frac{(Y_n-n(1-H))I_{\{Y_n>n(1-H)/2\}}}{Y_n(1-H)}\,dN_n - \frac{1}{n}\int_0^{t_n}\frac{(Y_n-n(1-H))I_{\{Y_n>n(1-H)/2\}}}{(1-H)^2}\,dH_1 = o_P\left(\left(\frac{1}{n}\int_0^{t_n}\frac{dF}{1-G}\right)^{1/2}\right).$$

From condition (3.13), we have $1-H(t_n-)>n^{-\lambda}$ for some $0<\lambda<1$. Therefore, by (A.3), we know that, with probability 1,

$$\inf_{t\le t_n}\frac{Y_n(t)}{n(1-H(t-))}>\frac{1}{2}$$

for all large n. Therefore, with probability 1,

$$\frac{1}{n}\int_0^{t_n}\frac{(Y_n-n(1-H))I_{\{Y_n>n(1-H)/2\}}}{Y_n(1-H)}\,dN_n - \frac{1}{n}\int_0^{t_n}\frac{(Y_n-n(1-H))I_{\{Y_n>(1-H)/2\}}}{(1-H)^2}\,dH_1$$
$$=\frac{1}{n}\int_0^{t_n}\frac{(Y_n-n(1-H))}{Y_n(1-H)}\,dN_n-\frac{1}{n}\int_0^{t_n}\frac{Y_n-n(1-H)}{(1-H)^2}\,dH_1$$

for all large n. The right-hand side is exactly the last two terms of (2.14). Hence, to show (3.33), it suffices to show

$$\left(n\int_0^{t_n}\frac{dH_1}{(1-H)^2}\right)^{-1/2}\sum_{i=1}^{n}\eta_i(t_n)\to N(0,1) \tag{3.35}$$

by the expression of $\hat{\Lambda}_n-\Lambda$ in (2.14). To this end, we check the Lindeberg condition. Set

$$s_n^2=n\int_0^{t_n}\frac{dH_1}{(1-H)^2}$$

and write

$$\frac{1}{s_n}\int_0^{t_n}\frac{dH_1}{(1-H)^2}=O(1)\left(\frac{1}{n}\int_0^{t_n}\frac{dF}{1-G}\right)^{1/2}$$
$$=O(1)(1-G(t_n))\int_0^{t_n}\frac{dF}{1-G}=o(1).$$

Hence, for any $\varepsilon > 0$,

$$P\left(\int_0^{Z_1\wedge t_n} \frac{dH_1}{(1-H)^2} > \varepsilon s_n\right) = 0 \tag{3.36}$$

for all large n. Now observe that $s_n(1-G(t_n)) \to 1/(1-F(\tau_H))$ and define $\tilde{t}_n = \inf\{t\colon\ \varepsilon(1-G(t))/2 \le 1-G(t_n)\}$. Then $1/(1-H(Z_1)) > \varepsilon s_n$ implies $Z_1 > \tilde{t}_n$ for large n. So we can write, for large n,

$$\begin{aligned}
&\frac{1}{s_n^2}\sum_{i=1}^{n} E\Big(\eta_i^2(t_n) I_{\{|\eta_i(t_n)|>\varepsilon s_n\}}\Big)\\
&\quad\le \left(\int_0^{t_n}\frac{dH_1}{(1-H)^2}\right)^{-1} E\Big(\eta_1^2(t_n) I_{\{\delta_1/[1-H(Z_1)]>\varepsilon s_n\}}\Big) \quad \text{by (3.36)}\\
&\quad\le O(1)\left(\int_0^{t_n}\frac{dH_1}{(1-H)^2}\right)^{-1} E\left(\left(\frac{\delta_1 I_{\{Z_1\le t_n\}}}{1-H(Z_1)}\right)^2 I_{\{\delta_1/[1-H(Z_1)]>\varepsilon s_n\}}\right)\\
&\quad\le O(1)\left(\int_0^{t_n}\frac{dH_1}{(1-H)^2}\right)^{-1} E\left(\left(\frac{\delta_1 I_{\{\tilde{t}_n<Z_1\le t_n\}}}{1-H(Z_1)}\right)^2\right)\\
&\quad\le O(1)\frac{\int_{\tilde{t}_n}^{t_n} dH_1/(1-H)^2}{\int_0^{t_n} dH_1/(1-H)^2} \to 0
\end{aligned}$$

by the definition of $\tilde{t}_n$ and the slow variation of $\int_0^t dF/(1-G)$ with respect to $(1-G(t))^{-1}$ [see (3.17)]. Thus the Lindeberg condition holds and (3.33) follows from the central limit theorem. □

Proof of Theorem 3.6. Observe

$$\int_t^{\tau_H}(1-G)\,dF \sim f(\tau_H)g(\tau_H)(\tau_H-t)^2/2$$

and

$$\int_0^t \frac{dF}{1-G} \sim -f(\tau_H)\log(\tau_H-t)/g(\tau_H)$$

when $t\to\tau_H$. To check condition (3.12), we write

$$\frac{\int_t^{\tau_H}(1-G)\,dF}{(1-G(t))^2\int_0^t dF/(1-G)} = O(1)\frac{1}{-\log(\tau_H-t)} \to 0,$$

when $t\to\tau_H$. Let t_n satisfy (3.13). It is easy to show

$$\tau_H - t_n \sim \left(\tfrac{1}{2}f(\tau_H)g(\tau_H)n\log n\right)^{-1/2}. \tag{3.37}$$

Clearly,

$$n(F(\tau_H)-F(t_n))^2\Big/\int_0^{t_n}\frac{dF}{1-G} \to 0.$$

Hence Proposition 3.5 can be applied with t_n satisfying (3.37). It can be easily checked that a t'_n satisfying (3.19) also satisfies (3.15) and (3.16). The proof is complete. □

APPENDIX A

The empirical approximation in the following lemma is the key to the proofs of the main results. We show a complete proof for explicitness. More delicate results may be found in Shorack and Wellner (1986).

LEMMA A.1. *Let t_n be such that $(1 - H(t_n -))n^{\lambda} \to \infty$ for some $0 < \lambda < 1$ and define*

$$\tau_n = \sup\{t: 1 - H(t-) \geq Mn^{p-1}\}, \tag{A.1}$$

where $0 < p < 1$ and $M > 0$. Then

$$\begin{aligned}\sup_{t \leq t_n}\left|\frac{Y_n(t)/n - (1 - H(t-))}{(1 - H(t-))^{1/2}}\right| &= O\left(n^{-1/2}(\log n)^{1/2}\right) \qquad \text{(A.2)}\\ &= \sup_{t \leq t_n}\left|\frac{Y_n(t)/n - (1 - H(t-))}{(Y_n(t)/n)^{1/2}}\right| \quad a.s.\end{aligned}$$

In particular, (A.2) *holds with t_n replaced by τ_n.*

PROOF. Applying the Bernstein inequality, we have

$$P\left(|Y_n(t)/n - H(t-)| > x\right) \leq 2\exp\left(-\frac{nx^2}{2(x + H(t-)(1 - H(t-)))}\right)$$

for any $x > 0$. Let $x = (rH(t-)(1 - H(t-))\log n/n)^{1/2}$ for some large but fixed $r > 0$. Since $(\log n/n) = o(n^{-\lambda})$ for $0 < \lambda < 1$, we know $x < H(t-)(1 - H(t-))$ for all $t \leq t_n$ and large n. So for $t \leq t_n$ and large n,

$$\begin{aligned}&2\exp\left(-\frac{nx^2}{2(x + H(t-)(1 - H(t-)))}\right)\\ &\qquad \leq 2\exp\left(-\frac{nrH(t-)(1 - H(t-))\log n/n}{4H(t-)(1 - H(t-))}\right) = 2n^{-r/4}.\end{aligned}$$

So we have, for large n,

$$P\left(\left|\frac{Y_n(t)/n - (1 - H(t-))}{(1 - H(t-))^{1/2}}\right| > \left(\frac{r\log n}{n}\right)^{1/2}\right) \leq 2n^{-r/4}$$

for all $t \le t_n$. Similarly,

$$P\left(\left|\frac{Y_n(t+)/n - (1-H(t))}{(1-H(t))^{1/2}}\right| > \left(\frac{r\log n}{n}\right)^{1/2}\right) \le 2n^{-r/4}$$

for all $t < t_n$ and large n. Choose $0 = s_0 \le s_1 \le s_2 \le \cdots \le s_n = t_n$ such that $H(s_i -) - H(s_{i-1}) \le 1/n$ for all $1 \le i \le t_n$. By a Bonferroni type of inequality and the Borel–Cantelli lemma, we can show

$$\begin{aligned}&\sup_{0\le i\le n}\left|\frac{Y_n(s_i)/n - (1-H(s_i -))}{(1-H(s_i -))^{1/2}}\right|\\ &\quad = O\left(\left(\frac{\log n}{n}\right)^{1/2}\right) = \sup_{0\le i<n}\left|\frac{Y_n(s_i +)/n - (1-H(s_i))}{(1-H(s_i))^{1/2}}\right| \quad \text{a.s.}\end{aligned}$$

This can be extended to $[0, t_n]$ by the monotonicity of the $Y_n(t)$ and $1 - H(t-)$. The proof is complete. □

We note that (A.2) implies

$$\limsup_n \sup_{t\le t_n}\left|\frac{Y_n(t)/n}{1-H(t-)} - 1\right| = 0 \quad \text{a.s.} \tag{A.3}$$

for all $0 < p < 1$ and t_n given in the above lemma. Hence $n^{-p}Y_n(t_n) \to 0$ a.s. and $n^{1-p}(1 - H(t_n -)) \to 0$ imply each other.

LEMMA A.2. *Suppose* $F(\tau_H) < 1$ *and* $0 < p < 1/2$. *If* (2.2) *holds*, *then*

$$P(N_n(\tau_H) - N_n(\tau_n) \ne 0 \ i.o.) = 0; \tag{A.4}$$

otherwise,

$$\limsup_n \frac{n^p\delta_n}{Y_n(Z_n)} = \infty \quad a.s. \tag{A.5}$$

PROOF. Let $0 < p < 1/2$. Recall the fact that, for $1 < r < 2$ and iid random variables $\{\xi_n,\ n \ge 1\}$, $E(|\xi_1|)^r < \infty$ is equivalent to $|\xi_n|/n^{1/r} \to 0$ a.s., which is also equivalent to $\max\{|\xi_i|,\ i \le n\}/n^{1/r} \to 0$ a.s., and that $E|\xi_1|^r = \infty$ is equivalent to $\limsup_n |\xi_n|/n^{1/r} = \infty$ a.s. [see, e.g., Chow and Teicher (1988)]. Assume (2.2) holds. Since

$$E\left(\frac{\delta_1}{1-H(Z_1)}\right)^{1/(1-p)} = \int_0^{\tau_H}\frac{1}{(1-F)^{1/(1-p)}(1-G)^{p/(1-p)}}\,dF < \infty,$$

we know

$$\lim_n \max\left\{ \frac{\delta_i}{(1 - H(Z_i -))}, i \le n \right\} \Big/ n^{1-p} = 0 \quad \text{a.s.}$$

Therefore, with τ_n defined in (A.1), we can write

$$\frac{1}{M} \lim_n \max\{\delta_i I_{\{Z_i \ge \tau_n\}}, i \le n\}$$
$$\le \lim_n \max\left\{ \frac{\delta_i I_{\{Z_i \ge \tau_n\}}}{(1 - H(Z_i -))}, i \le n \right\} \Big/ n^{1-p} = 0 \quad \text{a.s.}$$

Since $N_n(\tau_H) - N_n(\tau_n)$ is the number of uncensored observations that are larger than τ_n among the first n observations, (A.4) holds by the above equality. On the other hand, suppose (2.2) does not hold, then

$$\limsup_n \frac{\delta_n}{n^{1-p}(1 - H(Z_n -))} = \infty \quad \text{a.s.}$$

Again by the definition of τ_n, we have

$$P(\delta_n = 1, Z_n \ge \tau_n \text{ i.o.}) = 1. \tag{A.6}$$

It follows from (A.3) that $Y_n(\tau_n)/(n(1 - H(\tau_n -))) \to 1$ a.s. Therefore we can write

$$\begin{aligned}
\limsup_n \frac{n^p \delta_n}{Y_n(Z_n)} &\ge \limsup_n \frac{n^p \delta_n}{Y_n(\tau_n)} I_{\{Z_n \ge \tau_n\}} \\
&\ge \limsup_n \frac{\delta_n}{n^{1-p}(1 - H(\tau_n -))} I_{\{Z_n \ge \tau_n\}} \\
&\ge M \limsup_n \delta_n I_{\{Z_n \ge \tau_n\}} \\
&= M \quad \text{a.s.}
\end{aligned}$$

by (A.6). Letting $M \to \infty$, (A.5) follows. The proof is complete. □

The following lemma may be viewed as an adapted Feller–Chung lemma [see Chow and Teicher (1988)].

LEMMA A.3. *Let B_n, A_n, $n \ge 1$, be two sequences of events adapted to an increasing filtration $\{\mathscr{F}_n\}$. If $P(A_n \cap B_n | \mathscr{F}_{n-1}) \ge cP(A_n | \mathscr{F}_{n-1})$ a.s. for some $c > 0$, then*

$$P(A_n \cap B_n \ i.o.) \ge cP(A_n \ i.o.). \tag{A.7}$$

PROOF. For $n \geq 1$, we can write

$$
\begin{aligned}
P\left(\bigcup_{k=n}^{\infty}(A_k \cap B_k)\right)
&= \sum_{k=n+1}^{\infty} P\left((A_k \cap B_k) \bigcap_{j=n}^{k-1} (A_j \cap B_j)^c\right) + P(A_n \cap B_n) \\
&\geq \sum_{k=n+1}^{\infty} P\left(B_k \cap A_k \bigcap_{j=n}^{k-1} A_j^c\right) + P(B_n \cap A_n) \\
&= \sum_{k=n+1}^{\infty} \frac{P\left(B_k \cap A_k \cap_{j=n}^{k-1} A_j^c\right)}{P\left(A_k \cap_{j=n}^{k-1} A_j^c\right)} P\left(A_k \bigcap_{j=n}^{k-1} A_j^c\right) + \frac{P(A_n \cap B_n)}{P(A_n)} P(A_n) \\
&\geq \inf_{k \geq n+1} \frac{E\left(P(A_k \cap B_k \mid \mathscr{F}_{k-1}) I_{\cap_{j=n}^{k-1} A_j^c}\right)}{E\left(P(A_k \mid \mathscr{F}_{k-1}) I_{\cap_{j=n}^{k-1} A_j^c}\right)} \sum_{k=n+1}^{\infty} P\left(A_k \bigcap_{j=n}^{k-1} A_j^c\right) \\
&\quad + \frac{P(A_n \cap B_n)}{P(A_n)} P(A_n) \\
&\geq c \sum_{k=n+1}^{\infty} P\left(A_k \bigcap_{j=n}^{k-1} A_j^c\right) + cP(A_n) = cP\left(\bigcup_{j=n}^{\infty} A_j\right).
\end{aligned}
$$

Letting $n \to \infty$, $P(\cup_{k=n}^{\infty}(A_k \cap B_k)) \to P(A_n \cap B_n \text{ i.o.})$ and $P(\cup_{k=n}^{\infty} A_n) \to P(A_n \text{ i.o.})$, and (A.7) follows. □

We state the Kronecker lemma with a slight extension in the following lemma. The proof is in spirit the same as the proof of the Kronecker lemma [see Chow and Teicher (1988)]. We omit the details.

LEMMA A.4 (Kronecker lemma). *Let b_n be a sequence of increasing positive numbers such that $b_n \to \infty$ as $n \to \infty$. Let $a_n(t)$ be a sequence of functions defined on A such that $\sup_{t \in A} |\sum_{i=n}^{\infty} a_i(t)/b_i| \to 0$ as $n \to \infty$. Then $\sup_{t \in A} |\sum_{i=1}^{n} a_i(t)|/b_n \to 0$ as $n \to \infty$.*

LEMMA A.5. *Suppose $F(\tau_H) < 1$ and* (2.2) *holds for some $0 < p < 1/2$. Then*

$$
n^{p-1} \sup_{t \leq \tau_H} \left| \sum_{j=1}^{n} \eta_j(t) \right| \to 0 \quad a.s. \tag{A.8}
$$

where η_j is defined in (1.4).

PROOF. Recall the definition of $\eta_j(t)$. We shall first prove

$$E\left(\sum_{n=1}^{\infty} n^{p-1}\eta_n(\tau_n)\right)^2 < \infty.$$

Since $E\eta_j(t) = 0$ and $E(\eta_j(t))^2 = \int_0^t (1-H)^{-2}\,dH_1$ for all $j \ge 1$ and $t \le \tau_H$, for $0 < p < 1/2$ we can write

$$\begin{aligned}
E\left(\sum_{n=1}^{\infty} n^{p-1}\eta_n(\tau_n)\right)^2 &= \sum_{n=1}^{\infty} n^{2p-2} E\eta_n^2(\tau_n) \\
&= \sum_{n=1}^{\infty} n^{2p-2}\int_0^{\tau_n} \frac{dH_1}{(1-H)^2} \\
&= O(1)\sum_{n=1}^{\infty} n^{2p-2}\sum_{j=1}^{n}\left(\frac{1}{1-H(\tau_j-)}\right)^{(1-2p)/(1-p)} \\
&\qquad\times\int_{\tau_{j-1}}^{\tau_j}\frac{dH_1}{(1-H)^{1/(1-p)}} \qquad \text{(A.9)}\\
&= O(1)\sum_{n=1}^{\infty}\int_{\tau_{n-1}}^{\tau_n}\frac{dH_1}{(1-H)^{1/(1-p)}} \\
&= O(1)\int_0^{\tau_H}\frac{dH_1}{(1-H)^{1/(1-p)}} \\
&= O(1)(1-F(\tau_H))\int_0^{\tau_H}(1-G)^{-p/(1-p)}\,dF < \infty.
\end{aligned}$$

Since, for any $n \ge 1$, $\{\eta_n(t), \mathscr{F}_t;\ 0 \le t < \tau_H\}$ is a martingale, where

$$\mathscr{F}_t = \sigma\left\{Z_j I_{\{Z_j \le t\}},\ \delta_j I_{\{Z_j \le t\}},\ j = 1, 2, \dots\right\},$$

therefore $\{\sum_n^{\infty} j^{p-1}\,\eta_j(\tau_j \wedge t), \mathscr{F}_t;\ 0 \le t \le \tau_H\}$ is also a martingale for any $n \ge 1$. By Doob's inequality and (A.9), we have

$$\begin{aligned}
E\left(\sup_{t\le\tau_H}\left|\sum_{j=n}^{\infty} j^{p-1}\eta_j(\tau_j\wedge t)\right|\right)^2 &\le 4E\left(\sum_{j=n}^{\infty} j^{p-1}\eta_j(\tau_j)\right)^2 \qquad \text{(A.10)}\\
&\le 4E\left(\sum_{j=1}^{\infty} j^{p-1}\eta_j(\tau_j)\right)^2 < \infty
\end{aligned}$$

for all $n \ge 1$. Clearly, the second term in (A.10) converges to 0 when $n \to \infty$. Therefore $\sup_{t\le\tau_H}|\sum_{j=n}^{\infty} j^{p-1}\eta_j(\tau_j \wedge t)| \to 0$ in probability when $n \to \infty$. Since $\eta_j(\tau_j \wedge t)$, $j \ge 1$, are independent mean 0 random variables, we can see that $\{\sup_{t\le\tau_H}|\sum_{j=n}^{\infty} j^{p-1}\eta_j(\tau_j \wedge t)|, \tilde{\mathscr{F}}_n;\ n \ge 1\}$ is an L^2 reverse submartingale, where

$$\tilde{\mathscr{F}}_n = \sigma\left\{(Z_j, \delta_j);\ j \ge n\right\}.$$

It then converges almost surely by the martingale convergence theorem and the limit is 0. So by Lemma A.4 [with $b_n = n^{1-p}$ and $a_n(t) = \eta_n(\tau_n \wedge t)$], we have

$$\sup_{t \le \tau_H} n^{p-1} \left| \sum_{j=1}^{n} \eta_j(\tau_j \wedge t) \right| = o(1) \quad \text{a.s.}$$

A calculation similar to (A.9) shows

$$\sum_{n=1}^{\infty} n^{p-1} E\left(\int_{\tau_n}^{Z_n \vee \tau_n} \frac{dH_1}{(1-H)^2} + \frac{\delta_n}{1 - H(Z_n -)} I_{\{Z_n > \tau_n\}} \right) < \infty.$$

Again using the Kronecker lemma, we have

$$n^{p-1} \sum_{j=1}^{n} \left(\frac{\delta_j}{1 - H(Z_j -)} I_{\{\tau_j < Z_j\}} + \int_{\tau_j}^{\tau_j \vee Z_j} \frac{dH_1}{(1-H)^2} \right) = o(1) \quad \text{a.s.}$$

Now write

$$\begin{aligned} n^{p-1} & \sup_{t \le \tau_H} \left| \sum_{j=1}^{n} \eta_j(t) \right| \\ & \le n^{p-1} \sup_{t \le \tau_H} \left| \sum_{j=1}^{n} \eta_j(\tau_j \wedge t) \right| + n^{p-1} \sup_{t \le \tau_H} \left| \sum_{j=1}^{n} \big(\eta_j(t) - \eta_j(\tau_j \wedge t) \big) \right| \\ & \le o(1) + n^{p-1} \sum_{j=1}^{n} \left(\frac{\delta_j}{1 - H(Z_j -)} I_{\{\tau_j < Z_j\}} + \int_{\tau_j}^{\tau_j \vee Z_j} \frac{dH_1}{(1-H)^2} \right) \quad \text{a.s.} \\ & = o(1) \quad \text{a.s.} \end{aligned}$$

The proof is complete. □

APPENDIX B

LEMMA B.1. *Suppose $F(\tau_H) < 1$ and (3.2) holds for some $0 < p < 1/2$. Then*

(B.1) $$\lim_{x \to \infty} xP\left(\int_0^{Z_1} \frac{dH_1}{(1-H)^2} > x^{1-p-\varepsilon} \right) = 0,$$

where $0 < \varepsilon < p^2/(1-p)$.

PROOF. Let $0 < \varepsilon < p^2/(1-p)$. Set

$$x = \left(\int_0^t \frac{dH_1}{(1-H)^2} \right)^{1/1-p-\varepsilon}$$

and let $t \to \tau_H$. It is enough to show $x(1 - H(t-)) \to 0$. Observe that

$$\begin{aligned}(1 - H(t-))^{1-p-\varepsilon} \int_0^t \frac{dH_1}{(1-H)^2} &= o(1) \int_0^t \frac{dH_1}{(1-H)^{1-p+\varepsilon}} \\ &= o(1) \int_0^t \frac{dH_1}{(H_1(\tau_H) - H_1(t))^{(1-p)(1+p+\varepsilon)}} \\ &= o(1)\left(p^2 - (1-p)\varepsilon\right)^{-1} / H_1(\tau_H) \\ &= o(1).\end{aligned}$$

Then (B.1) is proved. □

LEMMA B.2. *Suppose* $F(\tau_H) < 1$ *and* $0 < p < 1/2$. *Then* (3.2) *is equivalent to each of the following three assertions*:

$$\limsup_{x \to \infty} xP\left(\frac{\delta_1}{1 - H(Z_1 -)} > x^{1-p}\right) < \infty, \tag{B.2}$$

$$\limsup_{x \to \infty} xP\left(|\eta_1(\tau_H)| > x^{1-p}\right) < \infty \tag{B.3}$$

and

$$\limsup_n P\left(N(\tau_n) \neq N_n(\tau_H)\right) \to 0 \quad \textit{when } M \to \infty, \tag{B.4}$$

where M *is given in the definition of* τ_n *in* (A.1). *Similarly*, (3.4) *is equivalent to each of the following three assertions*:

$$\lim_{x \to \infty} xP\left(\frac{\delta_1}{1 - H(Z_j -)} > x^{1-p}\right) = 0, \tag{B.5}$$

$$\lim_{x \to \infty} xP\left(|\eta_1(\tau_N)| > x^{1-p}\right) = 0 \tag{B.6}$$

and

$$\lim_n P\left(N_n(\tau_n) \neq N_n(\tau_H)\right) = 0. \tag{B.7}$$

PROOF. Set $x = (1 - H(t-))^{-1/(1-p)}$ or $(1 - H(t))^{-1/(1-p)}$ in (B.2) and let $t \to \tau_H$. It is easy to see (3.2) and (B.2) are equivalent. Recall the definition of η_1 in (1.4). We know (B.3) is equivalent to

$$\limsup_{x \to \infty} xP\left(\delta_1\left(\frac{1}{1 - H(Z_1 -)} - \int_0^{Z_1} \frac{dH_1}{(1-H)^2}\right) > x^{1-p}\right) < \infty \tag{B.8}$$

and

$$\limsup_{x \to \infty} xP\left((1 - \delta_1)\int_0^{Z_1} \frac{dH_1}{(1-H)^2} > x^{1-p}\right) < \infty. \tag{B.9}$$

Since

$$\int_0^t \frac{dH_1}{(1-H)^2} \bigg/ (1-H(t))^{-1} \to 0 \quad \text{as } t \to \tau_H,$$

the equivalence between (B.2) and (B.8) is established. On the other hand, (3.2) implies (B.9) by Lemma B.1. Therefore (B.3) is equivalent to each of (B.2) and (3.2). It remains to show the equivalence between (B.4) and (B.2). Notice that (B.2) is equivalent to

$$\limsup_n nP\left(\frac{\delta_1}{1-H(Z_1-)} > Mn^{1-p}\right) \to 0 \quad \text{when } M \to \infty,$$

which, by the definition of τ_n in (A.1), is equivalent to (B.4).

The equivalence between (3.4), (B.5), (B.6) and (B.7) can be shown in an analogous fashion. We omit the details. □

LEMMA B.3. *Suppose* $F(\tau_H) < 1$ *and* $0 < p < 1/2$. *Then* (3.2) *implies*

$$E\left|\frac{1}{n}\sum_{i=1}^n \eta_i(\tau_H)\right| = O(n^{-p}); \tag{B.10}$$

and, likewise, (3.4) *implies*

$$E\left|\frac{1}{n}\sum_{i=1}^n \eta_i(\tau_H)\right| = o(n^{-p}). \tag{B.11}$$

PROOF. Suppose (3.2) holds. By Lemma B.2, we know (B.3) holds. Thus we can adopt a standard approach to calculate

$$\begin{aligned}
&E\left|\frac{1}{n}\sum_{i=1}^n \eta_i(\tau_H)\right| \\
&\quad\le \frac{1}{n}E\left|\sum_{i=1}^n \left(\eta_i(\tau_H) I_{\{|\eta_i(\tau_H)| \le n^{1-p}\}} - E\left(\eta_i(\tau_H) I_{\{|\eta_i(\tau_H)| \le n^{1-p}\}}\right)\right)\right| \\
&\qquad + \frac{2}{n}E\left|\sum_{i=1}^n \eta_i(\tau_H) I_{\{|\eta_i(\tau_H)| > n^{1-p}\}}\right| \\
&\quad\le \left(\frac{1}{n}E\left(\eta_1^2(\tau_H) I_{\{|\eta_1(\tau_H)| \le n^{1-p}\}}\right)\right)^{1/2} + 2E|\eta_1(\tau_H) I_{\{|\eta_1(\tau_H)| > n^{1-p}\}}| \\
&\quad\le \left(\frac{2}{n}\int_0^{n^{1-p}} tP(|\eta_1(\tau_H)| > t)\,dt\right)^{1/2} + 2\int_{n^{1-p}}^\infty P(|\eta_1(\tau_H)| > t)\,dt \\
&\quad\le O(1)\left(\frac{1}{n}\int_1^{n^{1-p}} t^{1-1/(1-p)}\,dt\right)^{1/2} + O(1)\int_{n^{1-p}}^\infty t^{-1/(1-p)}\,dt \\
&\quad\le O(n^{-p}).
\end{aligned}$$

So (B.10) follows. The rest of the proof can be done similarly. We omit the details. □

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